# Degree Equitable Chromatic Number of a Graph

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**Abstract**

Let $G = (V, E)$ be a connected graph. A subset $S$ of $V$ is called a *degree equitable set* if the degrees of any two vertices in $S$ differ by at most one. The minimum order of a partition of $V$ into independent degree equitable sets is called the *degree equitable chromatic number* of $G$ and is denoted by $\chi_{de}(G)$. In this paper we initiate a study of this new coloring parameter.

**Key Words.** degree equitable set, degree equitable chromatic number
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## 1 Introduction

By a graph $G = (V, E)$ we mean a finite, undirected and connected graph with neither loops nor multiple edges. The order and size of $G$ are denoted by $n$ and $m$ respectively. For graph theoretic terminology we refer to Chartrand and Lesniak [1].

Graph coloring theory has a central position in discrete mathematics and is of interest for its applications in several areas. The fundamental parameter in the theory of graph coloring is the chromatic number $\chi(G)$ of a graph $G$ which is defined to be the minimum number of colors required to color the vertices of $G$ in such a way that no two adjacent vertices of $G$ receive the same color.

Several variations of graph colorings such as edge coloring, total coloring, acyclic coloring, list coloring, star chromatic number, achromatic number, subchromatic number, $T$-colorings, equitable coloring and game chromatic number have been investigated by several authors and for a survey of graph coloring problems one may refer to the book by Jensen and Toft [3].

In this paper we introduce another type of coloring which is based on the fundamental concept of the degree of a vertex, called degree equitable coloring and initiate a study of the corresponding parameter. We need the following definitions and results.

**Definition 1.1.** *The degree set of a graph $G$ is defined to be the set of all distinct degrees of the vertices of $G$ and is denoted by $D(G)$.*

**Definition 1.2.** *A caterpillar is a tree $T$ with the property that the removal of the leaves of $T$ results in a path. This path is referred to as the spine of the caterpillar.*

**Definition 1.3.** *The clique number of a graph $G$, denoted by $\omega(G)$, is the maximum number of vertices in a complete subgraph of $G$.*

**Definition 1.4.** *A graph $G$ is called chordal if every cycle of $G$ of length greater than three has a chord.*

**Theorem 1.5.** ([1], Page 202) *For any chordal graph $G$, $\chi(G) = \omega(G)$.*

**Definition 1.6.** *The independence number of a graph $G$, is the maximum cardinality of an independent set of vertices and is denoted by $\beta_0(G)$.*

The following result is due to Nordhaus and Gaddum [4].

**Theorem 1.7.** *If $G$ is a graph of order $n$, then*

*(i)* $2\sqrt{n} \leq \chi(G) + \chi(\overline{G}) \leq n + 1$

*(ii)* $n \leq \chi(G)\chi(\overline{G}) \leq \left(\frac{n+1}{2}\right)^2$.

**Theorem 1.8.** ([2], Page 191) *For any fixed integer $k \geq 3$, $k$-colorability is $NP$-complete.*

## 2 Main Results

**Definition 2.1.** *Let $G = (V, E)$ be a graph. A subset $S$ of $V$ is called a degree equitable set if the degrees of any two vertices in $S$ differ by at most one. A degree equitable $k$-coloring of $G$ is a partition of $V$ into subsets $V_1, V_2, \ldots, V_k$ such that each $V_i$ is an independent set and also a degree equitable set in $G$. Each $V_i$ is called a de-color class. The degree equitable chromatic number $\chi_{de}(G)$ of a graph $G$ is defined to be the minimum $k$ such that $G$ admits a degree equitable $k$-coloring.*

**Remark 2.2.** *Let $G$ be a graph with $\chi_{de}(G) = k$. Let $\{V_1, V_2, \ldots, V_k\}$ be a degree equitable $k$-coloring of $G$. Then for any two distinct de-color classes $V_i$ and $V_j$, there exist vertices $u \in V_i$ and $v \in V_j$ such that either $u$ and $v$ are adjacent or $|deg\ u - deg\ v| \geq 2$.*

**Remark 2.3.** *For any graph $G$, we have $1 \leq \chi_{de}(G) \leq n$. Further $\chi_{de}(G) = 1$ if and only if $G = \overline{K_n}$ and $\chi_{de}(G) = n$ if and only if for any two vertices $u$ and $v$, either $u$ and $v$ are adjacent or $|deg\ u - deg\ v| \geq 2$. This observation motivates the following definition.*

**Definition 2.4.** *A graph is called de-complete if for any two distinct vertices $u, v$ either $u$ and $v$ are adjacent or $|deg\ u - deg\ v| \geq 2$.*

**Example 2.5.** *Any block graph $G$ with at least two blocks in which the order of any two blocks differ by at least two is de-complete and is not complete. The graph $G$ consisting of a copy of $K_{n_1}$ and a copy of $K_{n_2}$ with $|n_1 - n_2| \geq 2$ and $V(K_{n_1}) \cap V(K_{n_2}) = \{x, y\}$ is a block which is de-complete and is not complete.*

**Proposition 2.6.** *For any graph $G$, let $S_i = \{v \in V : deg\ v = i$ or $i+1\}$, where $\delta \leq i \leq \Delta - 1$. Then $G$ is de-complete if and only if for each $i$ with $S_i \neq \emptyset$, the induced subgraph $\langle S_i \rangle$ is complete.*

*Proof.* Suppose $G$ is de-complete. For any two vertices $u, v \in S_i$, we have $|deg\ u - deg\ v| \leq 1$, and hence it follows that $u$ and $v$ are adjacent in $G$. Thus $\langle S_i \rangle$ is complete. Conversely, suppose $\langle S_i \rangle$ is complete for each $i$ with $S_i \neq \emptyset$. Let $u, v \in V(G)$. If $u, v \in S_i$ for some $i$, then $u$ and $v$ are adjacent in $G$. Otherwise $|deg\ u - deg\ v| \geq 2$. Hence $G$ is de-complete. □

In the following propositions we determine all graphs of order 5 and 6 which are de-complete but not complete.

**Proposition 2.7.** *Let $G$ be a connected de-complete graph which is not complete. Then $|V(G)| \geq 5$. Further the graph $G_1$ given in Figure 1 is the only graph of order 5 which is de-complete but not complete.*

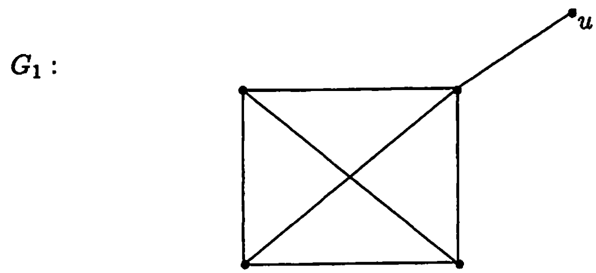


Figure 1

*Proof.* Let $G$ be a de-complete graph which is not complete. Then there exist two non-adjacent vertices $u$ and $v$ such that $|deg\ u - deg\ v| \geq 2$ and hence $|V(G)| \geq 5$. Now, suppose $|V(G)| = 5$. Then $\delta = 1$, $G$ has a unique pendant vertex $u$ and all the three vertices which are non-adjacent to $u$ have degree 3. Hence it follows that $\langle V(G) - \{u\} \rangle$ is complete and hence $G$ is isomorphic to $G_1$. □

**Proposition 2.8.** *Let $G$ be a graph of order 6 which is de-complete but not complete. Then $G$ is isomorphic to the graph $G_1$ or $G_2$ given in Figure 2.*

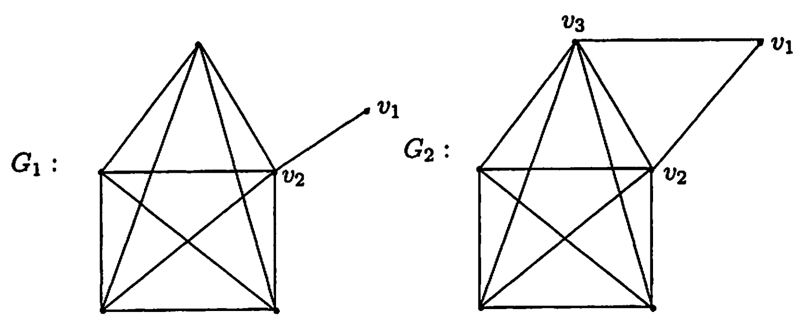


Figure 2

*Proof.* Let $V(G) = \{v_1, v_2, v_3, v_4, v_5, v_6\}$. Since there exist two non-adjacent vertices in $G$ whose degrees differ by at least two we have $\delta = 1$ or 2.

**Case i.** $\delta = 1$.

Let $deg\ v_1 = 1$ and $N(v_1) = \{v_2\}$. Then $deg\ v_i \geq 3$ for $3 \leq i \leq 6$, $\langle v_3, v_4, v_5, v_6 \rangle$ is complete and $deg\ v_2 \geq 2$. If $v_2$ is not adjacent to $v_i$ for some $i$, $3 \leq i \leq 6$, then $v_2$ and $v_i$ can be assigned the same color in a degree equitable coloring of $G$, so that $\chi_{de}(G) < 6$, which is a contradiction. Hence $\langle G - \{v_1\} \rangle$ is complete and $G$ is isomorphic to $G_1$.

**Case ii.** $\delta = 2$.

Let $deg\ v_1 = 2$ and $N(v_1) = \{v_2, v_3\}$. Then the vertices $v_4, v_5$ and $v_6$ all have degree 4 and hence $\langle G - \{v_1\}\rangle$ is complete. Thus $G$ is isomorphic to $G_2$. □

The following problems naturally arise.

**Problem 2.9.** *Given a positive integer n, find the number of graphs of order n which are de-complete but not complete.*

**Problem 2.10.** *What is the minimum (maximum) size of a graph of order n which is de-complete but not complete?*

**Remark 2.11.** *Obviously for any graph G,* $\chi(G) \leq \chi_{de}(G)$. *Further if G is regular or the degree set of G is given by* $D(G) = \{r, r+1\}$ *for some* $r \geq 1$, *then* $\chi(G) = \chi_{de}(G)$. *However the difference between* $\chi(G)$ *and* $\chi_{de}(G)$ *can be made arbitrarily large. For example consider the caterpillar T whose spine* $P = (v_1, v_2, \ldots, v_n)$ *is such that deg* $v_i = 2i+1$. *Then in any degree equitable coloring of T, each* $\{v_i\}$ *is a de-color class, and hence* $\chi_{de}(T) = n+1$ *whereas* $\chi(T) = 2$.

In fact we have the following proposition.

**Proposition 2.12.** *If a and b are positive integers with* $a \leq b$, *then there exists a graph G with* $\chi(G) = a$ *and* $\chi_{de}(G) = b$.

*Proof.* If $a = b$, then we take $G$ to be the complete graph $K_a$. Suppose $a < b$.

**Case 1.** $a = 2$.

For the caterpillar $T_b$ with spine $P = (v_1, v_2, \ldots, v_{b-1})$ and $deg\ v_i = 2i+1$, we have $\chi(T_b) = 2 = a$ and $\chi_{de}(T_b) = b$.

**Case 2.** $a = 3$ and $b = 4$.

For the graph $G$ obtained by joining a vertex of $K_2$ with a vertex on the rim of the wheel $W_n$, where $n$ is odd, we have $\chi(G) = 3$ and $\chi_{de}(G) = 4$.

**Case 3.** $a = 3$ and $b > a+1$.

Let $T$ be the caterpillar with spine $P = (v_1, v_2, \ldots, v_{b-3})$ and $deg\ v_i = 2i+2$. Then for the graph $G$ obtained by joining a leaf of $T$ with a vertex of $K_3$, we have $\chi(G) = 3$ and $\chi_{de}(G) = b$.

**Case 4.** $a > 3$ and $b = a+1$.

For the graph $G$ obtained by joining the centre of the star $K_{1,a-1}$ with a vertex of $K_a$, we have $\chi(G) = a$ and $\chi_{de}(G) = a+1 = b$.

**Case 5.** $a > 3$ and $b > a+1$.

Let $T$ be the caterpillar with spine $P = (v_1, v_2, \ldots, v_{b-(a+1)})$ and $deg\ v_i = a + 2i + 1$. Then for the graph $G$ obtained by joining a leaf of $T$ with a vertex of $K_a$, we have $\chi(G) = a$ and $\chi_{de}(G) = b$. □

**Definition 2.13.** *A graph $G$ with $\chi_{de}(G) = 2$ is called a de-bipartite graph.*

In the following theorems we characterize trees and unicyclic graphs which are de-bipartite.

**Theorem 2.14.** *A tree $T$ is de-bipartite if and only if $T \in \mathcal{F}$, where $\mathcal{F} = \bigcup_{i=0}^{\infty} \mathcal{F}_i$ and $\mathcal{F}_i$ is the family of trees defined recursively as follows. $\mathcal{F}_0 = \{K_{1,n} : n \geq 3\} \cup \{P_n : n \geq 2\}$. For $i > 0$, $\mathcal{F}_i = \mathcal{F}_{i-1} \cup \mathcal{G}_{i-1}$, where $\mathcal{G}_{i-1}$ is the family of all trees obtained from two trees $T_1, T_2 \in \mathcal{F}_{i-1}$ by identifying a leaf of $T_1$ with a leaf of $T_2$, subject to the following restrictions.*

*(i) $\Delta(T_1) \geq 3$, $\Delta(T_1) \geq \Delta(T_2)$ and $D(T_2) \subseteq D(T_1) \cup \{\Delta(T_1) - 1\}$.*

*(ii) If $T_2$ is a path, then its length is even.*

*Proof.* Let $T \in \mathcal{F}$. Let $i \geq 0$ be the least integer such that $T \in \mathcal{F}_i$. We prove that $T$ is de-bipartite by induction on $i$. If $i = 0$ then $T$ is either a path or a star and trivially $T$ is de-bipartite. Suppose $i > 0$ and the result is true for $i - 1$. Since $T \in \mathcal{F}_i$, there exist trees $T_1, T_2 \in \mathcal{F}_{i-1}$, satisfying the conditions (i) and (ii) of the theorem such that $T$ is obtained from $T_1$ and $T_2$ by identifying a leaf $v_1$ of $T_1$ with a leaf $v_2$ of $T_2$ and let $v$ denote the corresponding vertex of degree 2 in $T$. By induction $T_1$ and $T_2$ are de-bipartite. Let $\mathcal{C}_1 = \{V_1, W_1\}$ and $\mathcal{C}_2 = \{V_2, W_2\}$ be degree equitable colorings of $T_1$ and $T_2$ respectively with $v_1 \in V_1$ and $v_2 \in V_2$.

Let $X_1 = (V_1 - \{v_1\}) \cup (V_2 - \{v_2\}) \cup \{v\}$ and $X_2 = W_1 \cup W_2$. Since $\Delta(T_1) \geq 3$, $V_1$ contains all the leaves of $T_1$. If $\Delta(T_2) \geq 3$, then trivially $V_2$ also contains all the leaves of $T_2$. Also if $\Delta(T_2) = 2$, then $T_2$ is a path of even length and since $v_2 \in V_2$ it follows that $V_2$ contains both the leaves of $T_2$. Hence degree of each vertex in $X_1$ is 1 or 2, so that $X_1$ is degree equitable and independent in $T$. Also it follows from (i) that degree of each vertex of $X_2$ is $\Delta(T_1) - 1$ or $\Delta(T_1)$. Hence $X_2$ is also degree equitable and independent in $T$ and $\mathcal{C} = \{X_1, X_2\}$ is a degree equitable coloring of $T$. Thus $T$ is de-bipartite.

Conversely, let $T$ be a de-bipartite tree. We use induction on the order of $T$ to prove that $T \in \mathcal{F}$. This is trivial if $T$ is a path or a star. We now assume that the result is true for all trees of order less than $n$ and let $T$ be a tree of order $n$ such that $T$ is neither a star nor a path and $T$ is de-bipartite. Hence $\Delta(T) \geq 3$ and let $deg\ w = \Delta(T)$. Let $\mathcal{C} = \{V_1, V_2\}$ be a degree equitable coloring of $T$. Since $\Delta(T) \geq 3$, we may assume that

$V_1$ contains all the leaves of $T$, so that $w \in V_2$. Since $T$ is not a star, the vertex $w$ has a neighbour $v_1$ which is not a leaf and $v_1 \in V_1$. Since $V_1$ is a degree equitable set, it follows that *deg* $v_1 = 2$. Let $T_1$ and $T_2$ be the subtrees of $T$ induced by the sets $A \cup \{v_1\}$ and $B \cup \{v_1\}$ where $A$ and $B$ are the vertex sets of the two components $T - \{v_1\}$ and let $\Delta(T_1) = \Delta(T)$, so that $\Delta(T_1) \geq \Delta(T_2)$ and $\Delta(T_1) \geq 3$. Clearly $\mathcal{C}_1 = \{V_1 \cap V(T_1), V_2 \cap V(T_1)\}$ and $\mathcal{C}_2 = \{V_1 \cap V(T_2), V_2 \cap V(T_2)\}$ are degree equitable colorings of $T_1$ and $T_2$ respectively. By induction hypothesis $T_1$ and $T_2$ are in $\mathcal{F}$. Let $i \geq 0$ be the smallest integer such that both $T_1$ and $T_2$ are in $\mathcal{F}_i$. Since $T$ is de-bipartite, it follows that $D(T) - \{1,2\} \subseteq \{\Delta(T), \Delta(T) - 1\}$, hence $D(T_2) \subseteq D(T_1) \cup \{\Delta(T_1) - 1\}$. Also if $T_2$ is a path, then both its end vertices are in $V_1 \cap V(T_2)$, it follows that $T_2$ is a path of even length. Hence $T \in \mathcal{F}_{i+1} \subseteq \mathcal{F}$. □

**Theorem 2.15.** *Let $G$ be a unicyclic graph with even cycle $C$. Then $G$ is de-bipartite if and only if $G = C$ or $G$ can be obtained from a de-bipartite tree $T$ by identifying two of its leaves whose adjacent support vertices are distinct.*

*Proof.* Suppose $G$ is de-bipartite. If $G = C$, there is nothing to prove. Suppose $G \neq C$. Let $\mathcal{C} = \{V_1, V_2\}$ be a degree equitable coloring of $G$ such that degree of every vertex in $V_1$ is 1 or 2. Since $G$ is de-bipartite, there exists at least one vertex $v$ on $C$ with *deg* $v = 2$, such that $v \in V_1$. Let $T$ be the tree obtained from $G$ by replacing $v$ by two new vertices $w_1$ and $w_2$ and joining $w_1$ to one neighbor of $v$ and $w_2$ to another neighbor of $v$. Clearly $\{(V_1 - \{v\}) \cup \{w_1, w_2\}, V_2\}$ is a degree equitable coloring of $T$. Hence $T$ is de-bipartite and $G$ can be obtained from $T$ by identifying the leaves $w_1$ and $w_2$.

Conversely, suppose $G \neq C$ and $G$ is obtained from a de-bipartite tree $T$ by identifying two of its leaves $w_1, w_2$ whose adjacent support vertices are distinct and let $v$ be the corresponding degree 2 vertex in $G$. Let $\mathcal{C} = \{V_1, V_2\}$ be a degree equitable coloring of $T$. Since $G \neq C$ and $T$ is not a path and all the leaves of $T$ are in the same color class, say $V_1$. Now $\{(V_1 - \{w_1, w_2\}) \cup \{v\}, V_2\}$ is a degree equitable coloring of $G$ and hence $G$ is de-bipartite. □

**Remark 2.16.** *Using repeatedly the construction given in Theorem 2.15, we can obtain several families of de-bipartite graphs. In fact if $G$ is any de-bipartite graph with $\delta = 1, \Delta \geq 3$ and having at least two distinct support vertices, then the graph obtained from $G$ by identifying two leaves at distinct support vertices gives a de-bipartite graph $H$ with $\delta(H) = 1$ or 2. Further any de-bipartite graph $H$ with $\delta = 1$ and $\Delta \geq 3$ can be obtained in this way. In fact, if $\mathcal{C} = \{V_1, V_2\}$ is a de-coloring of $H$ where $V_1$ contains all the leaves of $H$, then $V_1$ contains a vertex $v$ of degree 2. If $G$ is the graph* *obtained from H by replacing v by two new vertices $w_1$ and $w_2$ and joining $w_1$ to one neighbour of v and $w_2$ to the other neighbour of v, then G is a de-bipartite graph and H is obtained from G by identifying the leaves $w_1$ and $w_2$.*

**Remark 2.17.** *Obviously if G is a bipartite graph with bipartition X, Y then $\chi(G) = \chi_{de}(G) = 2$ if and only if there exist two positive integers r and s such that deg $u = r$ or $r+1$, for all $u \in X$ and deg $v = s$ or $s+1$, for all $v \in Y$. In particular if G is any graph which is either regular or $D(G) = \{r, r+1\}$ for some positive integer r, then $\chi(S(G)) = \chi_{de}(S(G)) = 2$ where $S(G)$ is the subdivision of G, obtained by subdividing each edge of G exactly once.*

The problem of characterizing graphs for which $\chi_{de}(G) = \chi(G)$ when $\chi(G) \geq 3$ is a difficult problem in view of the following theorem.

**Theorem 2.18.** *Given a positive integer $k \geq 3$, the problem of deciding whether $\chi_{de}(G) \geq k$ is NP-complete for any graph G with $\chi(G) \geq 3$.*

*Proof.* Let $G$ be a graph with $\chi(G) \geq 3$. Let $G_1$ be the graph obtained from $G$ by attaching suitable number of pendant vertices at each vertex of $G$ so that degree of any vertex of $G$ in $G_1$ is $\Delta(G) + 1$. Clearly $\chi_{de}(G_1) = \chi(G) + 1$. Hence the result follows from Theorem 1.8. □

We now proceed to obtain bounds for $\chi_{de}$ for several classes of graphs.

**Theorem 2.19.** *Let T be a tree with at least three vertices and having k distinct degrees. Then $\lceil \frac{k}{2} \rceil \leq \chi_{de}(T) \leq 2k - 1$. Further given two positive integers k and a such that $k \geq 3$ and $\lceil \frac{k}{2} \rceil \leq a \leq 2k - 1$, there exists a tree T such that $|D(T)| = k$ and $\chi_{de}(T) = a$.*

*Proof.* Let $\{V_1, V_2, \ldots, V_m\}$ be a degree equitable coloring of $T$. Since each $V_i$, where $1 \leq i \leq m$, covers vertices of at most two distinct degrees, we have $m \geq \lceil \frac{k}{2} \rceil$. Hence $\lceil \frac{k}{2} \rceil \leq \chi_{de}(T)$. Now, let $\{d_1, d_2, \ldots, d_k\}$ be the degree set of $G$ with $d_1 = 1$. Let $A_i = \{v \in V : deg\ v = d_i\}$ where $1 \leq i \leq k$. Since $A_1$ is independent, we assign one color to all the vertices of $A_1$. Also, since $\langle A_i \rangle$ is a forest all the vertices of $\langle A_i \rangle$ can be colored with at most two colors, where $2 \leq i \leq k$. Hence $\chi_{de}(T) \leq 2k - 1$.

Now, let $a$ and $k$ be two positive integers such that $\lceil \frac{k}{2} \rceil \leq a \leq 2k - 1$.

**Case i.** $a < k$.

If $k = 3$ and $a = 2$, then let $T$ be the caterpillar whose spine is the path $(v_1, v_2, v_3)$ with degrees 3, 2 and 3 in this order. If $k = 4$ and $a = 2$, let $T$ be the caterpillar whose spine is the path $(v_1, v_2, v_3)$ with degrees 3, 2 and 4 in this order. Now suppose $a \geq 3$. Let $P = (v_1, v_2, \ldots, v_{k-1})$ be a path on $k - 1$ vertices. Let $T$ be the caterpillar with spine $P$ such that the degrees of $v_i$ are $(3, 7, 11, \ldots, 3 + 4(a-2), 4, 8, \ldots, 4 + 4(k-a-1))$ in this order. Clearly $T$ has $k$ distinct degrees and $\chi_{de}(T) = a$.

**Case ii.** $a \geq k$.

Let $a = k+r$, where $r \geq 0$. Let $P = (v_1, v_2, \ldots, v_{a-1})$ be a path on $a-1$ vertices. Let $T$ be the tree obtained from $P$ by attaching suitable number of pendant vertices at each $v_i$ so that the degrees of $v_i$ are $(3, 7, 11, \ldots, 4k-5)$ in this order if $r = 0$ and $(3, 3, 7, 7, \ldots, 3 + 4r - 4, 3 + 4r - 4, 3 + 4r, 3 + 4r + 4, \ldots, 4k-5)$ if $r > 0$. Clearly $T$ has $k$ distinct degrees and any degree equitable coloring of $T$ has $a-1$ de-color classes with one element and one de-color class which consists of all pendant vertices. Therefore $T$ is the required tree with $k$ distinct degrees and $\chi_{de}(T) = a$. □

In the following theorems, we characterize trees with $k$ distinct degrees for which $\chi_{de}(T) = 2k-1$ and $\chi_{de}(T) = \lceil \frac{k}{2} \rceil$.

**Theorem 2.20.** *Let $T$ be a tree with degree set $D(T) = \{1 = d_1, d_2, \ldots, d_k\}$. Let $U_i = \{v \in V : deg\ v = d_i\}$. Then $\chi_{de}(T) = 2k-1$ if and only if the following conditions are satisfied:*

*(a) $U_i$ is not independent for $i = 2, 3, \ldots, k$.*

*(b) $|d_i - d_j| \geq 2$ , if $i \neq j$.*

*Proof.* Suppose (a) and (b) are satisfied. Consider any degree equitable coloring of $T$ with $\chi_{de}$ colors. Clearly $U_1$ is a de-color class. Also if $2 \leq i \leq k$, then $\langle U_i \rangle$ is a forest with at least one edge and hence is a union of two de-color classes. Further, since $|d_i - d_j| \geq 2$, the two colors used for $\langle U_i \rangle$ and the two colors used for $\langle U_j \rangle$ are different. Thus $\chi_{de}(T) = 2k-1$.

Conversely, let $\chi_{de}(T) = 2k-1$. If $U_i$ is an independent set for some $i$, $2 \leq i \leq k$, then there exists a de-coloring of $T$ in which $U_1$ and $U_i$ are color classes and each of the remaining $U_j$ is a union of at most two color classes, so that $\chi_{de}(T) < 2k-1$, which is a contradiction. Also if $|d_i - d_j| \leq 1$ for $i \neq j$, then there exists a de-coloring of $T$ in which $U_1$ is a color class, $U_i \cup U_j$ is a union of two color classes and each of the remaining $k-3$ sets $U_l$ is a union of at most two color classes, so that $\chi_{de}(T) < 2k-1$, which is again contradiction. Hence conditions (a) and (b) are satisfied. □

**Theorem 2.21.** *Let $T$ be a tree with degree set $D(T) = \{1 = d_1, d_2, \ldots, d_k\}$. Then $\chi_{de}(T) = \lceil \frac{k}{2} \rceil$ if and only if the following conditions are satisfied:*

*(a) There exist $\lfloor \frac{k}{2} \rfloor$ disjoint subsets $S_1, S_2, \ldots, S_{\lfloor \frac{k}{2} \rfloor}$ of $D(T)$ such that $S_i = \{r_i, r_i + 1\}$ for some integer $r_i$, $1 \leq i \leq \lfloor \frac{k}{2} \rfloor$.*

*(b) The set $X_i$ of all vertices of $T$ with degree $r_i$ or $r_i+1$ is an independent set, where $1 \leq i \leq \lfloor \frac{k}{2} \rfloor$.*

*(c) If $k$ is odd and* $D(T) - \left(\bigcup_{i=1}^{\lfloor \frac{k}{2} \rfloor} S_i\right) = \{d_j\}$ *for some $j$, then the set $X$ of all vertices of $T$ with degree $d_j$ is independent.*

*Proof.* Suppose (a), (b) and (c) are satisfied. Then $C = \left\{X_1, X_2, \ldots, X_{\lfloor \frac{k}{2} \rfloor}\right\}$ if $k$ is even and $\left\{X_1, X_2, \ldots, X_{\lfloor \frac{k}{2} \rfloor}, X\right\}$ if $k$ is odd, is a degree equitable coloring of $T$ using $\lceil \frac{k}{2} \rceil$ colors. Hence it follows that $\chi_{de}(T) = \lceil \frac{k}{2} \rceil$. Conversely, let $T$ be a tree with $\chi_{de}(T) = \lceil \frac{k}{2} \rceil$. Let $C = \left\{X_1, X_2, \ldots, X_{\lceil \frac{k}{2} \rceil}\right\}$ be a degree equitable coloring of $T$. Then any de-color class $X_i$, except possibly one de-color class, when $k$ is odd, must contain all vertices of $T$ whose degrees are two consecutive integers. Hence conditions (a), (b) and (c) of the theorem are satisfied. □

**Proposition 2.22.** *Let $G$ be any connected graph with degree set $\{d_1, d_2, \ldots, d_k\}$. Then $\chi_{de}(G) \leq k\chi(G)$.*

*Proof.* Let $V_i = \{v \in V : deg\ v = d_i\}$. Clearly $\chi_{de}(G) \leq \sum_{i=1}^{k} \chi(\langle V_i \rangle) \leq k\chi(G)$. □

**Corollary 2.23.** *For any connected planar graph $G$, with $k$ distinct degrees $\chi_{de}(G) \leq 4k$.*

**Corollary 2.24.** *For any chordal graph $G$ with $k$ distinct degrees, $\chi_{de}(G) \leq k\omega(G)$, where $\omega(G)$ is the clique number of $G$.*

We now proceed to prove that the bounds given in Proposition 2.22 and Corollary 2.23 are sharp.

**Proposition 2.25.** *Given any positive integer $k$, there exists a connected graph $G$ with $k$ distinct degrees such that $\chi_{de}(G) = k\chi(G)$.*

*Proof.* Let $G_i$ denote the complete $m$-partite graph with each part having $2^{i+1}$ vertices, where $1 \leq i \leq k$. Let $V_{i_1}, V_{i_2}, \ldots, V_{i_m}$ be the partition of the vertex set of $G_i$ into independent sets with $|V_{ij}| = 2^{i+1}$. Let $G$ be the graph obtained from $G_1, G_2, \ldots, G_k$ by joining each vertex of $V_{ij}$ with two vertices of $V_{(i+1)j}$, where $1 \leq i \leq k-1$ and $1 \leq j \leq m$. Clearly $G$ is a $m$-partite graph with $k$ distinct degrees and $\chi(G) = m$ and $\chi_{de}(G) = km$. □

**Proposition 2.26.** *Given any positive integer $k$, there exists a connected planar graph $G$ with $k$ distinct degrees such that $\chi_{de}(G) = 4k$.*

*Proof.* Let $H$ denote the icosahedron given in Figure 3.

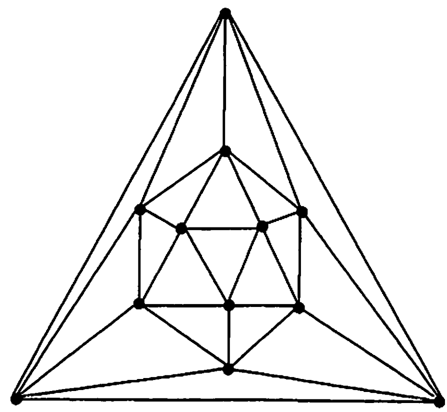

Figure 3

We construct a graph $G$ as follows. Take $k-1$ copies of $K_4$, say $G_1, G_2, \ldots, G_{k-1}$. Choose vertices $v_1, v_2, \ldots, v_{2k-4}$ such that $v_1 \in V(G_1)$, $v_{2k-4} \in V(G_{k-1})$ and $v_{2i-2}, v_{2i-1} \in V(G_i)$, where $2 \leq i \leq k-2$. Take $k-2$ copies of $H$, say $H_1, H_2, \ldots, H_{k-2}$ and identify the vertices $a, b$ of $H_i$ with the vertices $v_{2i-1}, v_{2i}$ respectively. Now attach one copy of $H$ at each vertex in $V(G_1) - \{v_1\}$ for each $i$ with $2 \leq i \leq k-2$, attach $i$ copies of $H$ at each vertex in $V(G_i) - \{v_{2i-2}, v_{2i-1}\}$, and $i-1$ copies of $H$ at $v_{2i-2}, v_{2i-1}$, attach $k-1$ copies of $H$ at each vertex in $V(G_{k-1}) - \{v_{2k-4}\}$ and attach $k-2$ copies of $H$ at $v_{2k-4}$. It can be easily verified that the resulting graph $G$ is a connected planar graph with $\chi_{de}(G) = 4k$. □

**Proposition 2.27.** *Given any positive integer $k$, there exists a chordal graph $G$ with $k$ distinct degrees such that $\chi_{de}(G) = k\omega(G) - 1$, where $\omega(G)$ is the clique number of $G$.*

*Proof.* Let $G$ be the graph obtained as in the proof of Proposition 2.26 where we take $H$ to be a copy of $K_\omega$ and $G_1, G_2, \ldots, G_{k-1}$ to be $k-1$ copies of $K_\omega$. The resulting graph $G$ is a connected chordal graph with $\chi_{de}(G) = k\omega(G) - 1$. □

The following problem naturally arises:

**Problem 2.28.** *Does there exist a chordal graph $G$ for which $\chi_{de}(G) = k\omega(G)$, where $k$ is the number of distinct degrees in $G$ and $\omega(G)$ is the clique number of $G$?*

**Definition 2.29.** *A subset $S$ of $V$ is called a maximal independent degree equitable set if $S$ is both independent and degree equitable and for any proper superset $S_1 \supset S$, we have either $S_1$ is not independent or $S_1$ is not degree equitable.*

**Definition 2.30.** *Let $G$ be a graph. The maximum cardinality of a maximal independent degree equitable set in $G$ is called the independent degree equitable number of $G$ and is denoted by $D_{ie}(G)$. The minimum cardinality of a maximal independent degree equitable set in $G$ is called the lower independent degree equitable number of $G$ and is denoted by $d_{ie}(G)$.*

**Example 2.31.** *For the complete bipartite graph $K_{m,n}$, we have $D_{ie}(K_{m,n}) = \max\{m,n\}$ and $d_{ie}(K_{m,n}) = \min\{m,n\}$. Hence the difference between the parameters $D_{ie}(G)$ and $d_{ie}(G)$ can be made arbitrarily large.*

**Proposition 2.32.** *Let $G$ be a graph on $n$-vertices. Then $\frac{n}{D_{ie}(G)} \leq \chi_{de}(G) \leq n - D_{ie}(G) + 1$.*

*Proof.* Let $\chi_{de}(G) = k$. Let $\{V_1, V_2, \ldots, V_k\}$ be a degree equitable coloring of $G$. Then $|V_i| \leq D_{ie}(G)$ and $|V_1| + |V_2| + \cdots + |V_k| = n$. Hence $n \leq kD_{ie}(G)$ so that $\frac{n}{D_{ie}(G)} \leq k = \chi_{de}(G)$. Now, let $V_1$ be an independent degree equitable subset of $V$ with $|V_1| = D_{ie}(G)$. Let $V_1 = \{v_1, v_2, \ldots, v_{D_{ie}}\}$. Then $(V_1, \{v_{D_{ie}+1}\}, \{v_{D_{ie}+2}\},$
$\ldots, \{v_n\})$ is a degree equitable coloring of $G$ and hence $\chi_{de}(G) \leq n - D_{ie}(G) + 1$. □

**Remark 2.33.** *The bounds given in the above proposition are sharp. For the complete graph $K_n$, $D_{ie} = 1$ and $\chi_{de} = \frac{n}{D_{ie}} = n - D_{ie} + 1$.*

**Proposition 2.34.** *For any graph $G$,*

*(i) $2\sqrt{n} \leq \chi_{de}(G) + \chi_{de}(\overline{G}) \leq k(n+1)$ and*

*(ii) $n \leq \chi_{de}(G)\chi_{de}(\overline{G}) \leq k^2 \left(\frac{n+1}{2}\right)^2$, where $k$ is the number of distinct degrees in $G$.*

*Proof.* Since $\chi(G) \leq \chi_{de}(G) \leq k\chi(G)$, the result follows from Theorem 1.7. □

**Remark 2.35.** *The bounds given in the above proposition are sharp.*

*1 For the graphs $P_4$ and $C_4$, $\chi_{de} = \overline{\chi}_{de} = 2$, so that $\chi_{de} + \overline{\chi}_{de} = 4 = 2\sqrt{n}$ and $\chi_{de} \cdot \overline{\chi}_{de} = 4 = n$.*

*2 Also for the graph $C_5$, $k = 1$, $\chi_{de} = \overline{\chi}_{de} = 3$ and hence $\chi_{de} + \overline{\chi}_{de} = 6 = k(n+1)$.*

**Conclusion and scope.** In this paper we have initiated a study of a new coloring parameter which depends just on the fundamental concept of the degree of a vertex. There is abundant scope for further research on this topic. Work on criticality concepts, effect of edge removal on $\chi_{de}$, $\chi_{de}$-perfect graphs and degree equitable edge chromatic number will be reported in subsequent papers.


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