$(m-1)$-Regular Antichains on $[m]$

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Abstract

Let $\mathcal{B} \subseteq 2^{[m]}$ be an antichain of size $|\mathcal{B}| =: n$. $2^{[m]}$ is ordered by inclusion. An antichain $\mathcal{B}$ is called k-regular ($k \in \mathbb{N}$), if for each $i \in [m]$ there are exactly k sets $B_1, B_2, \ldots, B_k \in \mathcal{B}$ containing i. In this case we say that $\mathcal{B}$ is a (k, m, n)-antichain.
Let $m \geq 2$ be an arbitrary natural number. In this note we show that an $(m-1, m, n)$-antichain exists if and only if $n \in [m+2, \binom{m}{2} - 2] \cup \{m, \binom{m}{2}\}$.

Keywords: (Regular) antichain; Completely separating system; Extremal set theory

1 Introduction

1.1 Notations

For nonnegative integers $k \leq m$, the sets $[k, m]$ and $[m]$ are defined by $[k, m] := \{k, k+1, \ldots, m-1, m\}$ and $[m] := [1, m]$. Let $\mathcal{B}$ be a subset of $2^{[m]}$, the power set of $[m]$. The size of $\mathcal{B}$ is $n := |\mathcal{B}|$. We call $\mathcal{B}$ an **antichain (AC)** if there are no two sets in $\mathcal{B}$ which are comparable under set inclusion. An antichain $\mathcal{B}$ is called **k-regular** ($k \in \mathbb{N}$), if for each $i \in [m]$ there are exactly k sets $B_1, B_2, \ldots, B_k \in \mathcal{B}$ containing i. In this case we say that $\mathcal{B}$ is a **(k, m, n)-antichain.**
A **Completely Separating System (CSS)** $\mathcal{C}$ on $[n]$ is a collection of blocks of $[n]$ such that for any pair of points $x, y \in [n]$, there exist blocks $A, B \in \mathcal{C}$ such that $x \in A$, $y \notin A$ and $y \in B$, $x \notin B$. A CSS on $[n]$ without restrictions on the size of the blocks in the collection is said to be an (n)CSS. Let $k < n$. An (n, k)**Completely Separating System** (**(n, k)CSS**) is an (n)CSS in which each

block is of size k.

The **volume** of a collection of sets $\mathcal{B}$ is $v(\mathcal{B}) := \sum_{A\in\mathcal{B}} |A|$. For a (k,m,n)-antichain $\mathcal{B}$, $v(\mathcal{B}) = km$ and for an (n,k)CSS $\mathcal{C}$, $v(\mathcal{C}) = k|\mathcal{C}|$. Let $\mathcal{B} := \{B_1, B_2, \ldots, B_m\}$ be a subset of $2^{[n]}$. The **complement** $\overline{\mathcal{B}}$ of $\mathcal{B}$ is defined by $\overline{\mathcal{B}} := \{\overline{B}_1, \overline{B}_2, \ldots, \overline{B}_m\}$ where $\overline{B}_i := [n]\backslash B_i$. Often we omit brackets and commas in our notation for sets. For example we write 1345 instead of $\{1, 3, 4, 5\}$. A set is called a t-set, if it contains t elements.

1.2 Motivation

Regular antichains have a strong connection to CSS's. In 1961, Rényi [8] found minimum Separating Systems in the context of solving certain problems in information theory, and in 1969, Dickson [3] introduced the notion of a Completely Separating System. Spencer [6] showed that antichains are the duals of *Completely Separating Systems.*

Definition (dual). Let $\mathcal{M} = \{M_1, M_2, \ldots, M_m\}$ be a collection of subsets of $[n]$. We define the dual $\mathcal{M}^*$ of $\mathcal{M}$ to be the collection $\mathcal{M}^* := \{M_1^*, M_2^*, \ldots, M_n^*\}$ of subsets of $[m]$ given by $M_i^* := \{j \in [m] : i \in M_j\}$ $(i = 1, \ldots, n)$.

Lemma 1 ([6]). *If $\mathcal{M}$ is a CSS then its dual $\mathcal{M}^*$ is an antichain and vice versa.*

Lemma 2. *If $\mathcal{M}$ is a (n,k)CSS of size m then its dual $\mathcal{M}^*$ is a (k,m,n)-antichain and vice versa.*

Completely Separating Systems are studied in a lot of papers: In 1973, Katona [4] studied combinatorial search problems in general; Cai [2], Ramsay et al. [7] and Roberts et al. [10] studied Completely Separating Systems with restrictions. The relevant CSSs for the motivation of this paper are the (n,k)CSSs. Much of the recent CSS work is on the minimum size of these.

1.3 Facts

Before we start with our results, we want to list some basic results, which are already known or easy to prove.

Lemma 3 ([5]). *A collection $\mathcal{B} = \{B_1, B_2, \ldots, B_n\}$ of sets is an antichain if and only if $\overline{\mathcal{B}}$ is an antichain.*

Lemma 4. *A (k,m,n)-AC $\mathcal{B}$ exists if and only if an $(n-k,m,n)$-AC $\mathcal{C}$ exists.*

Lemma 5. *Let $1 < k < n-1$ and $\mathcal{B}$ be a (k,m,n)-AC. Then there exists no set B in $\mathcal{B}$ with $|B| = 1$, $|B| = m-1$ or $|B| = m$.*

Lemma 6. *Let $\mathcal{B}$ be a (k,m,n)-AC. Then there exists a $(k,m+1,n)$-AC $\mathcal{B}'$.*

m	n	example for an $(m-1, m, n)$-AC
1	0	$\emptyset$
2	1	12
	2	1, 2
3	3	12, 13, 23
4	4	123, 124, 134, 234
	6	12, 13, 14, 23, 24, 34
5	5	1234, 1235, 1245, 1345, 2345
	7	12, 134, 135, 145, 234, 235, 245
	8	12, 13, 14, 15, 234, 235, 245, 345
	10	12, 13, 14, 15, 23, 24, 25, 34, 35, 45
6	6	12345, 12346, 12356, 12456, 13456, 23456
	8	123, 124, 1256, 1356, 1456, 2345, 2346, 3456
	9	123, 126, 156, 234, 345, 456, 1245, 1346, 2356
	10	123, 124, 135, 136, 146, 245, 246, 256, 345, 356
	11	12, 34, 56, 135, 136, 145, 146, 235, 236, 245, 246
	12	12, 16, 26, 36, 46, 56, 134, 135, 145, 234, 235, 245
	13	12, 13, 14, 15, 16, 26, 36, 46, 56, 234, 235, 245, 345
	15	12, 13, 14, 15, 16, 23, 24, 25, 26, 34, 35, 36, 45, 46, 56
7	7	123456, 123457, 123467, 123567, 124567, 134567, 234567
	9	1234, 1256, 3456, 12357, 12467, 13457, 13567, 23467, 24567
	10	1237, 1256, 1346, 1457, 1467, 2347, 2356, 2567, 12345, 34567
	11	123, 456, 1245, 1247, 1346, 1357, 1457, 2356, 2367, 2567, 3467
	12	123, 234, 345, 456, 567, 167, 1256, 1257, 1346, 1347, 2367, 2457
	13	126, 127, 157, 236, 346, 347, 367, 456, 457, 567, 1234, 1235, 1245
	14	123, 124, 127, 137, 156, 167, 234, 235, 267, 345, 346, 456, 457, 567
	15	12, 13, 23, 145, 147, 156, 167, 245, 247, 256, 267, 345, 347, 356, 367
	16	12, 13, 14, 15, 16, 17, 234, 235, 236, 237, 247, 356, 456, 457, 467, 567
	17	12, 17, 27, 34, 37, 47, 56, 57, 67, 135, 136, 145, 146, 235, 236, 245, 246
	18	12, 16, 17, 26, 27, 36, 37, 46, 47, 56, 57, 67, 134, 135, 145, 234, 235, 245
	19	12, 13, 14, 15, 16, 17, 26, 27, 36, 37, 46, 47, 56, 57, 67, 234, 235, 245, 345
	21	12, 13, 14, 15, 16, 17, 23, 24, 25, 26, 27, 34, 35, 36, 37, 45, 46, 47, 56, 57, 67

Table 1: Examples of $(m-1, m, n)$-antichains with $m \leq 7$.

Proof. We fix any $i \in [m]$ and define $\mathcal{B}_i := \{B \in \mathcal{B} : i \in B\}$ and $\mathcal{C} := \{B \cup \{m+1\} : B \in \mathcal{B}_i\}$.

$$\mathcal{B}' := \mathcal{C} \cup (\mathcal{B} \backslash \mathcal{B}_i).$$

Obviously $\mathcal{B}'$ is a $(k, m+1, n)$-antichain. ∎

1.4 Examples of $(m-1)$-regular antichains on $[m]$

First we start with some examples which are shown in Table 1 and which we need to start the induction in the proof of our main theorem.

We present our results for small m in Table 2. The sets M_m and $M_{m,t}$ are defined by $M_m := \{n \in \mathbb{N} : \exists (m-1, m, n)\text{-AC}\}$ and by $M_{m,t} := \{n \in \mathbb{N} :$

m	$\lvert M_m \rvert$	M_m	$\lvert M_{m,6} \rvert$	$\lvert M_{m,5} \rvert$	$\lvert M_{m,4} \rvert$	$\lvert M_{m,3} \rvert$	$\lvert M_{m,2} \rvert$
1	1	1	-	-	-	-	-
2	2	1, 2	-	-	-	-	1
3	1	3	-	-	-	-	1
4	2	4, 6	-	-	-	1	2
5	4	5, 7, 8, 10	-	-	1	1	4
6	8	6, 8 − 13, 15	-	1	1	4	8
7	13	7, 9 − 19, 21	1	1	3	7	13

Table 2: Overview of the existence of $(m-1, m, n)$-antichains with $m \leq 7$ and possible sizes n.

$\exists(m-1, m, n)$-AC $\mathcal{B}, \forall B \in \mathcal{B} : |B| \geq t\}$.

2 Results

Lemma 7 (sufficient condition). *For all $m \in \mathbb{N}$ with $m \geq 3$ there exists an*

(a) $(m-1, m, m)$*-AC,*

(b) $(m-1, m, \binom{m}{2})$*-AC.*

Proof. We define $\mathcal{B}$ for $m \in \mathbb{N}$ with $m \geq 3$:

(a) $\mathcal{B} := \{B \subseteq [m] : |B| = m-1\}$,

(b) $\mathcal{B} := \{B \subseteq [m] : |B| = 2\}$.

Obviously these collections of sets satisfy all conditions. ■

Lemma 8 (necessary condition i). *For all $m \in \mathbb{N}$ with $m \geq 3$ there does not exist any*

(a) $(m-1, m, m+1)$*-AC,*

(b) $(m-1, m, \binom{m}{2} - 1)$*-AC.*

Proof.

(a) We assume there is an m, such that an $(m-1, m, m+1)$-AC $\mathcal{B}$ exists. Then there is also a $(2, m, m+1)$-AC $\overline{\mathcal{B}}$ (Lemma 4). Because of $\frac{v(\mathcal{B})}{m+1} = \frac{2m}{m+1} < 2$ the collection $\overline{\mathcal{B}}$ of sets must contain a 1-set. This is a contradiction to Lemma 5.

(b) Again we assume that an $(m-1, m, \binom{m}{2}-1)$-AC $\mathcal{B}$ exists for an $m \geq 3$. This contains either two 3-sets B_1', B_2' and $\binom{m}{2}-3$ 2-sets or exactly one 4-sets B_1'' and $\binom{m}{2}-2$ 2-sets. In the first case we look at an arbitrary $i \in B_1' \backslash B_2'$. This one is in at most $(m-3)$ 2-sets and in one 3-set. This is a contradiction. Similarly, in the second case we look at an arbitrary $j \in B_1''$. This one is in at most $(m-4)$ 2-sets and in one 4-set. ■

Lemma 9 (necessary condition ii). *Let $m > 2$ and n be arbitrary natural numbers. If $n < m$ or $n > \binom{m}{2}$ there does not exist any $(m-1, m, n)$-AC.*

Proof. First let $n > \binom{m}{2} \geq 3$. We assume there is an $(m-1, m, n)$-AC $\mathcal{B}$. Because of $m^2 - m = v(\mathcal{B}) = \sum_{B \in \mathcal{B}} |B|$ there is at least one 1-set. This is a contradiction to Lemma 5. Now let $n < m$. Again we assume that such an $(m-1, m, n)$-AC $\mathcal{B}$ exists. With the same argument we know that there must be one m-set. This is also a contradiction. ■

Lemma 10 (recursive construction). *Let $\mathcal{B}$ be an $(m-1, m, n)$-AC with $m \geq 2$. Then there exists an $(m, m+1, n+m)$-AC $\mathcal{D}$.*

Proof. We construct the regular antichain $\mathcal{D}$ with the help of $\mathcal{B}$:

$$\mathcal{D} := \mathcal{B} \cup \{\{i, m+1\} : i \in [m]\}.$$

Obviously $\mathcal{D}$ is an antichain and satisfies all conditions. ■

Theorem 1 (main result). *Let $m \geq 3$ be an arbitrary natural number. An $(m-1, m, n)$-antichain exists if and only if $n \in [m+2, \binom{m}{2}-2] \cup \{m, \binom{m}{2}\}$.*

Before we prove this Theorem we want to mention a result from Böhm [1].

Theorem 2 ([1]). *Let $m, n \in \mathbb{N}$ arbitrary with $m \geq 6$ and with*

$$\begin{cases} m+3 \leq n \leq \lfloor \binom{m}{2} - \frac{2}{5}m \rfloor & \text{if } m \equiv 0,1,3,4,6,8,9 \mod 10, \\ m+3 \leq n \leq \lfloor \binom{m}{2} - \frac{2}{5}m \rfloor - 1 & \text{if } m \equiv 2,5,7 \mod 10. \end{cases}$$

Then there exists an (m, m, n)-AC $\mathcal{B}$.

Proof (main result). If $n \leq m+1$ or $n \geq \binom{m}{2}-1$ then Lemmas 7, 8, 9 show that Theorem 1 is correct. So we only have to analyze the cases $m+2 \leq n \leq \binom{m}{2}-2$. We prove this by induction on m. We start with $m = 2, 3, 4, 5, 6, 7$ (examples and Lemmas 8 and 9). Let $m \geq 8$ be an arbitrary natural number. We assume that the statement is true for $m-1$ such that we can use it to show that is also true for m. If $m+2 \leq n < 2m$ then there is an $(m-1, m-1, n)$-AC (Theorem 2) and with the help of Lemma 6 we know that there is also an $(m-1, m, n)$-AC. If $2m \leq n \leq \binom{m}{2}-2$ we know that an $(m-2, m-1, t)$-AC exists for all $t \in [m+1, \binom{m-1}{2}-2]$ (induction hypothesis) and then with the help of Lemma

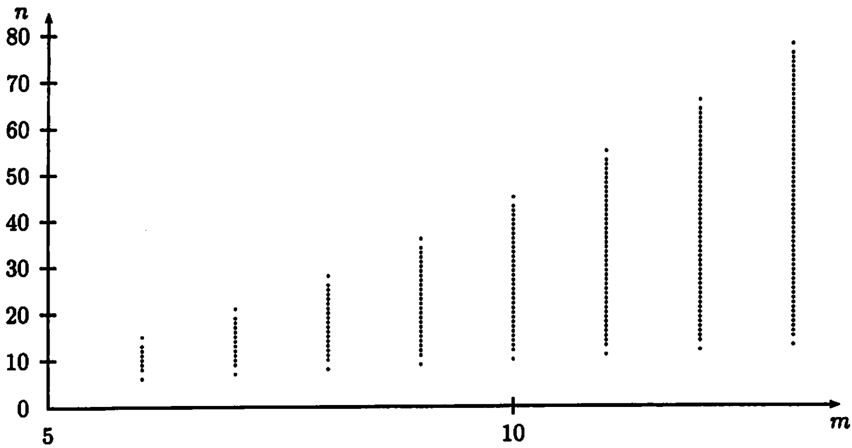

Figure 1: Overview of $(m-1, m, n)$-antichains

10 we get an $(m-1, m, n)$-AC with $n \in [(m+1)+(m-1), \binom{m-1}{2}-2+m-1] = [2m, \binom{m}{2}-2]$. ■

The dual statement to Theorem 1 is that an (n, m)CSS antichain of size $m+1$ exists if and only if $n \in [m+3, \binom{m+1}{2}-2] \cup \{m+1, \binom{m+1}{2}\}$. Our results are illustrated in Figure 1, where the dots indicate the pairs (m, n) such that there exists an $(m-1, m, n)$-AC.

Corollary 1. *Let m be a natural number with $m \geq 5$. Then*

$$|M_m| = \binom{m-1}{2} - 2.$$

Proof. This corollary follows directly from Theorem 1 using the equivalence $(\binom{m}{2} - 2 \geq m+2) \Leftrightarrow (m \geq 5)$ and the equality $((\binom{m}{2}-2)-(m+1)+2 = \binom{m-1}{2}-2$. ■

Using Theorem 1 we know if for given parameter-pair (m, n) an $(m-1, m, n)$-AC exists or not. Now, we could ask how many non isomorphic $(m-1, m, n)$-ACs exist. For example it is easy to check that if $m \leq 5$ there are only the ones which are mentioned in Table 1 and that for every $m > 2$ there is exactly one $(m-1, m, m)$-AC and one $(m-1, m, \binom{m}{2})$-AC. For $m = 6$ and $n = 8$ we get five non isomorphic $(5, 6, 8)$-ACs: $B_1 = \{123, 456, 1245, 1246, 1345, 1356, 2346, 2356\}$, $B_2 = \{123, 145, 1246, 1256, 1346, 2345, 2356, 3456\}$, $B_3 = \{123, 124, 1256, 1356, 1345, 2346, 2456, 3456\}$, $B_4 = \{123, 124, 1256, 1356, 1456, 2345, 2346, 3456\}$ and $B_5 = \{123, 124, 1346, 1356, 1456, 2345, 2356, 2456\}$. For arbitrary (m, n) the exact number of non isomorphic $(m-1, m, n)$-ACs is unknown and the subject of ongoing research.

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