Group Divisible Designs With Two Association Classes and With Groups of Sizes 1, 1, and n

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ABSTRACT. We show that the necessary conditions are sufficient for the existence of group divisible designs (PBIBDs of group divisible type) for block size $k = 3$ and with three groups of sizes 1, 1, and n.

1. Introduction

A GDD($v = v_1 + v_2 + \cdots + v_g, g, k, \lambda_1, \lambda_2$), or group divisible design, is an ordered triple (V, k, B), where V is a set of size v whose elements are the points of the design, B is a collection of subsets (called blocks) of V, and k is the size of each block. The set V is partitioned into g subsets called groups, and group G_i has v_i elements. Each pair of points from the same group occurs in λ_1 blocks, and each pair of points from different groups occurs in λ_2 blocks. We consider GDDs such that all groups are of size 1 except for one group of size $n > 1$. Pairs of symbols occurring in the same group are called first associates, and pairs occurring from different groups are called second associates. The existence of such designs is an old topic, and we refer the reader to Section IV Chapter 1, and Section VI Chapter 42, of Colbourn and Dinitz [1] where these designs are called PBIBDs of group divisible type (and GDD is reserved for designs with $\lambda_1 = 0$). The existence question for $k = 3$ has been solved by Sarvate, Fu and Rodger [2], [3] when all groups are the same size. More recently, Punnim and Sarvate [4] determined the existence of GDDs for $k = 3$ with two groups where the group sizes were 1 and n. Here we consider GDDs in which there are three groups of sizes 1, 1, and n (for $n > 1$), and for which the block size $k = 3$,

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and such that the indices satisfy $\lambda_1 = 1$ and $\lambda_2 = \lambda \geq 2$. We describe the designs as graphs. Let λK_v devote the (multi)graph on v vertices in which each pair of vertices is joined by λ edges. For any graphs G_1 and G_2, we define $G_1 \vee_\lambda G_2$ to be the union of graphs G_1 and G_2 in which each vertex of G_1 is joined to each vertex of G_2 by λ edges. A G-decomposition of a graph H is a partition of the edges of H such that each element of the partition induces a copy of G. When $k = 3$, each block is K_3 and the GDD is a K_3-decomposition of the graph G where:

$$G = (G_1 \vee_\lambda G_2) \cup (G_1 \vee_\lambda G_3) \cup (G_2 \vee_\lambda G_3)$$

and G_1 is the graph with isolated vertex α, G_2 is the graph with isolated vertex β, and G_3 is K_n. Suppose S is a graph or a set whose elements are graphs. We frequently make use of the convenient notation $\alpha * S$ to mean the set of triangles (blocks of size 3) obtained by decomposing S (or its elements) into edges and reforming to make triangles so that α is a vertex in each triangle.

2. Necessary Conditions

We first construct an example to show λ is bounded. A one-factor of a graph is a set of pairwise disjoint edges which partition the vertex set. It is well-known that the edges of the complete graph K_{2n} can be put into $2n-1$ classes (one-factors) in which each vertex appears once and only once in the class.

EXAMPLE 1. *For $n = 7$, $\lambda = 2$, a GDD(v=1+1+7,3,3,1,2) can be constructed from K_6 based on the points $\{1, \cdots, 6\}$. Decompose K_6 into five one-factors, say $F_1, \cdots, F_5$. The blocks for the GDD are: $\{\alpha, \beta, 7\}, \{\alpha, \beta, 7\}$, $\alpha * F_1, \alpha * F_2, \beta * F_3, \beta * F_4, 7 * F_5$.*

THEOREM 1. *Suppose $n = 1 + 2t \geq 7$. Then there exists a GDD($v = 1 + 1 + n, 3, 3, 1, t - 1$).*

PROOF. Decompose the graph K_{2t} into $2t-1$ one-factors. Use one with n and $t-1$ with each of α and β to make blocks as in the example. Make λ copies of the block $\{\alpha, \beta, n\}$. This constructs a GDD($v = 1+1+n, 3, 3, 1, \lambda$) for $\lambda = (n-3)/2 = t-1$, and for $n = 2t+1$. □

The previous theorem suggests that λ is always bounded below $\frac{(n-3)}{2}$. This is true as we now argue. Let us assume we have constructed a GDD as in the theorem using λ copies of the block $\{\alpha, \beta, n\}$. Let B_α denote the set of blocks which contain α but not β, and define B_β to denote the set of blocks which contain β but not α. Then, $|B_\alpha| = |B_\beta| = \frac{\lambda(n-1)}{2}$. Observe that n is not in any of the blocks of B_α or B_β. We define B_n to be the set of blocks containing neither α nor β. Since each block of B_n which contains x also contains one edge with two points from $N\backslash\{n\}$, $|B_n| \geq \{\frac{n}{2}\}$.. Since each block of B_α, B_β, and B_n has one edge from $N\backslash\{n\}$,

$$\frac{\lambda(n-1)}{2} + \frac{\lambda(n-1)}{2} + \frac{(n-1)}{2} \leq \binom{n-1}{2} = \frac{(n-1)(n-2)}{2}.$$

On simplification this reduces to $\lambda \leq \frac{(n-3)}{2}$. Suppose now an alternate solution exists for construction of the blocks of the GDD so that there are $(\lambda - 1)$ copies of $\{\alpha, \beta, n\}$ and one copy of $\{\alpha, \beta, y\}$ for some y in $N\backslash\{n\}$. Consider the effect on B_α of this change. One occurrence of y in some block of B_α is replaced by an occurrence of n in some block of B_α, and WLOG assume it is the same block. The change in B_β is similar, and therefore $|B_\alpha|$ $= |B_\beta| = \frac{\lambda(n-1)}{2}$ as before. Similarly, for the set B_n, the blocks without α and without β, two pairs with y (one triangle) now replace a triangle with n, but, the $|B_n|$ is unchanged. It follows that λ is bounded exactly as before.

There are (at least) two other important necessary conditions.

THEOREM 2. *If a GDD*$(v = 1 + 1 + n, 3, 3, 1, \lambda)$ *exists, then (a)* n *is odd; (b)* 3 *divides* $n - 1$ *or* 3 *divides* $\lambda + n$*; (c)* $\lambda \leq \frac{(n-3)}{2}$

PROOF. For part (a), suppose x is an element in N, the largest group. There are $2\lambda + n - 1$ edges for x. The edges are consumed two per block. So the number of these edges must be even. Thus $n - 1$ is even. For part (b), there are 3λ edges used in the blocks which contain both α and β. If we subtract these from the total number of edges in G, the remaining must also be a multiple of three. Now, the size, or number of edges in G, is $e = \binom{n}{2} + 2\lambda n + \lambda$. Hence, $2n\lambda - 2\lambda + \frac{n(n-1)}{2} \equiv 0 \pmod 3$. This can be shown to imply $3|(n-1)$ or $3|\lambda + n$. □

We may summarize some consequences of the two previous theorems as follows:

- If $n = 6t + 1$, then $\lambda \in \{2, 3, 4, \cdots, \frac{(n-3)}{2}\}$.
- If $n = 6t + 3$, then $\lambda \in \{3, 6, 9, \cdots, \frac{(n-3)}{2}\}$.
- If $n = 6t + 5$, then $\lambda \in \{4, 7, 10, \cdots, \frac{(n-3)}{2}\}$.

We will show in Section 3 that these designs all exist, and consequently:

THEOREM 3. *A GDD*$(v = 1 + 1 + n, 3, 3, 1, \lambda)$ *exists if and only if the necessary conditions are satisfied.*

3. Existence results

The method used for $n = 7$ in Example 1 does not work for larger n (and $\lambda = 2$) since there are one-factors left over – that is they must be used to make triangles without α or β. To solve this problem we use the partition of indices method introduced by Stanton and Goulden [6] and extended by Sarvate [5]. The edges of K_{2n} can be put into disjoint classes $P_1, P_2, \cdots, P_n$ where edge $\{i, j\}$ is in P_k if and only if $(i - j) \equiv k \pmod{2n}$. We combine known results into the next lemma:

LEMMA 1. *With respect to the complete graph K_{2k} we have:*
(a) The triangles $\{1+i, 2+i, 4+i\}$ for $i = 1, 2, \cdots, 2k$ contain exactly the edges from P_1, P_2 and P_3. Put another way, the graph K_{2k} may be factored into $2k-1$ one-factors, six of which may be combined into $2k$ triangles.
(b) The triangles $\{1+i, 1+x+i, 1+x+y+i\}$ for $i = 1, 2, \cdots, 2k$ contain exactly the edges from P_x, P_y, and P_{x+y} where $x + y < k$.
(c) The pairs in P_{2x+1} (for $2x+1 < k$) split into two one factors and the pairs in P_{2x} form a two-factor.
(d) If $2x + 1 < k$, Then $P_{2x} \cup P_{2x+1}$ splits into four one-factors. P_k is a single one-factor. If k is odd, the set $P_{k-1} \cup P_k$ can be split into three one-factors.
(e) For the complete graph K_{6s}, the set $P_{2s} \cup P_s$ forms $4s$ distinct triangles. The set P_{2s} forms $2s$ distinct triangles in which each point of $1, 2, \cdots, 6s$ appears exactly once. The triangles are $\{i, i+2s, i+4s\}$ for $i = 1, 2, \cdots, 2s$. It may be observed that each triangle consumes 3 edges in which each point appears twice. Thus the set P_{2s} is also a two-factor.

EXAMPLE 2. *Here $n = 13$, and $\lambda = 2, 3, 4, 5$. We apply the Lemma. Decompose the complete graph K_{12} into $P_1, P_2, \cdots, P_6$.*
(a) First, for $\lambda = 2$, construct the blocks by making triangles from P_1, P_2, and P_3 (part (b) of the Lemma) and use the additional blocks $\{\alpha, \beta, 13\}$, $\{\alpha, \beta, 13\}$, $\alpha * P_4$, $\beta * P_5$, $13 * P_6$. It is straightforward to check that $\lambda_1 = 1$ and $\lambda_2 = 2$.
(b) To increase λ from 2 to 5, instead of making triangles (without α and without β) from P_1, P_2, and P_3, decompose P_1 into two one-factors F_3 and F_4 and make the blocks $\alpha * F_3, \alpha * P_2, \beta * F_4$, and $\beta * P_3$.
(c) For $\lambda = 3$, use P_1 and P_2 to make triangles (part (e) of the Lemma with $s = 1$) without α and without β. Decompose P_5 into two one-factors (as 5 is odd, using part (c) of the Lemma), say F_1 and F_2. The remaining blocks are $\{\alpha, \beta, 13\}, \{\alpha, \beta, 13\}, \{\alpha, \beta, 13\}, \alpha * P_4, \alpha * F_1, \beta * F_2, \beta * P_5, 13 * P_6$.
(d) For $\lambda = 4$, use P_4 to make triangles without α and without β (part (e) of the Lemma). Use four copies of the block $\{\alpha, \beta, 13\}$, and use the blocks $\alpha * P_1, \alpha * P_3, \beta * P_2, \beta * P_5$, and $13 * P_{12}$.

If $n = 6t + 1$, then 3 divides $n - 1$ and there are no conditions on λ except that it be greater than one. For $n = 7, 13$ the designs possible according to Theorem 2 are constructed in Examples 1, 2. Suppose now $n = 6t + 1 > 13$.

Case 1: $t = 2j$, and $n = 12j + 1$. We decompose the complete graph K_{12j} into $P_1, P_2, \cdots, P_{6j}$. We (almost completely) partition the integers $1, 2, \cdots, 6j$ into triples $\{a, b, a + b\}$ such that $a + b \leq 6j$. In this way, the triples can be used in conjunction with the Lemma to indicate one-factors and two-factors from which the necessary blocks can be constructed. The partition we will use is the following:

$\{1, 3j - 1, 3j\}, (3, 3j - 2, 3j + 1\}, \cdots \{2j - 1, 2j, 4j - 1\}$, and
$\{2, 5j - 2, 5j\}, \{4, 5j - 3, 5j + 1\}, \cdots, \{2j - 2, 4j, 6j - 2\}$.

Not listed are points $5j - 1, 6j - 1$ and $6j$.

For $\lambda = 2$, use two copies of the block $\{\alpha, \beta, n\}$ and form the blocks $\alpha * P_{5j-1}, \beta * P_{6j-1}$ and $(12j + 1) * P_{6j}$. By part (d) of the Lemma, P_{6j} is a one-factor. P_{5j-1} and P_{6j-1} are either two-factors or composed of two one-factors. Each triple of remaining indices corresponds to three sets P_x, P_y, P_{x+y} which are used to form triangles (without α or β). It is clear that $\lambda_1 = 1$ and $\lambda_2 = 2$ and that the blocks give the GDD($v = 1 + 1 + (6t + 1), 3, 3, 1, 2$).

For $\lambda = 2 + 3m$, a small bit of caution is necessary in increasing the second index in units of three. To increase λ from 2 to 5, we decompose P_1, P_{3j-1} and P_{3j} to make blocks with α and with β as in Example 2. The index 1 is odd and P_1 can be decomposed into two one-factors. To increase the index from 2 to 8, use two different triples of indices, one with α and one with β, and select these two triples so that all three indices are even. The point is that, in the Lemma, P_{2x} decomposes into a two-factor, not into two one-factors, and for some triples of indices, all are even. With this caveat, it is straightforward to arrange for $\lambda = 2 + 3m$.

For $\lambda = 4 + 3m$, we first extend the solution from $\lambda = 2$ to 4 by using the triple $(2j - 2, 4j, 6j - 2)$ since P_{4j} can be used all by itself to make triangles - part (e) of the Lemma, as $4j = 2s$. The other blocks are $\alpha * P_{2j-2}$ and$\beta * P_{6j-2}$. This increases the index to 4, and further increases in multiples of three can be done as in the previous paragraph.

For $\lambda = 3$, use three copies of the block $\{\alpha, \beta, n\}$. Use the new partition of indices below in order to apply the Lemma as in part (c) of Example 2:

$\{1, 3j - 1, 3j\}, \{3, 3j - 2, 3j + 1\}, \cdots, \{2j - 3, 2j + 1, 4j - 2\}$, and
$\{2, 5j - 1, 5j + 1\}, \{4, 5j - 2, 5j + 2\}, \cdots, \{2j, 4j, 6j\}$.

Not listed are the indices $5j, 2j - 1$, and $4j - 1$. Use the sets P_{5j}, P_{2j-1}, and P_{4j-1} to make blocks with α and β. Use the one-factor P_{6j} to make triangles with n. Use P_{2j} and P_{4j} to make triangles without α and β. Increasing λ from 3 to $3 + 3m$ is done as was increasing λ from 2 to $2 + 3m$.

Case 2: $t = 2j + 1$ and $n = 12j + 6 + 1$. The construction is similar. Decompose K_{12j+6} into $P_1, P_2, \cdots, P_{6j+3}$. Now apply the partition of indices:

$\{1, 3j, 3j+1\}, \{3, 3j-1, 3j+2\}, \cdots, \{2j-1, 2j+1, 4j\}$, and

$\{2, 5j, 5j+2\}, \{4, 5j-1, 5j+3\}, \cdots, \{2j, 4j+1, 6j+1\}$.

Not listed are $5j+1, 6j+2$ and $6j+3$. The necessary blocks are constructed as in Case 1 except for $\lambda = 3 + 3m$, and for that case use the following partition:

$\{1, 5j+2, 5j+3\}, \{3, 5j+1, 5j+4\}, \cdots, \{2j+1, 4j+2, 6j+3\}$, and

$\{2, 3j, 3j+2\}, \{4, 3j-1, 3j+3\}, \cdots \{2j-2, 2j+2, 4j\}$.

Not listed are the triples $2j, 3j+1$, and $4j+1$. The blocks are constructed as in the previous case. These constructions and the necessary conditions prove the following:

THEOREM 4. *There exists a GDD($v = 1 + 1 + n, 3, 3, 1, 2 + 3m$) for $2 \leq 2 + 3m \leq \frac{(n-3)}{2}$ if and only if $n = 6t + 1$ for some $t \geq 1$.*

For $n = 6t + 3$, $\lambda = 3m$ for some m. First, for $n = 9$, $\lambda = 3$ is the only possibility. The construction in Theorem 1 is suitable. For $n = 15$, decompose the graph K_{14} into $P_1, P_2, \cdots, P_7$. Use P_1, P_2, and P_3 to make triangles without α or β. Use P_7 (a one-factor) to make triangles with $n = 15$. Use the other three Ps to make blocks with α or with β as before. The rest is clear. Now for $6t + 3 > 15$, we consider two cases.

Case 1: $t = 2j$ and $n = 12j + 3$. We apply the Lemma and decompose K_{12j+2} into $P_1, \cdots, P_{6j+1}$. The partition of indices we need is the same as for Case 1 for $n = 6t + 1$, except here we have one more index $6j + 1$. This gives four indices not in triples: $5j - 1, 6j - 1, 6j$ and $6j + 1$. Since we want $\lambda = 3$, we use the first three of these to make blocks with α and with β, and as P_{6j+1} is a one-factor, we use it to make blocks with $n = 12j + 3$. The rest follows as before.

Case 2: $t = 2j + 1$ and $n = 12j + 9$. Decompose K_{12j+8} into $P_1, P_2, \cdots, P_{6j+4}$. The partition of indices needed is:

$\{1, 3j+2, 3j+3\}, \{3, 3j+1, 3j+4\}, \cdots, \{2j+1, 2j+2, 4j+3\}$, and

$\{2, 5j+3, 5j+5\}, \{4, 5j+2, 5j+6\}, \cdots \{2j, 4j+4, 6j+4\}$.

The only index not listed in a triple is $5j + 4$. We use P_{5j+4} and the Ps corresponding to the triple $\{2j, 4j+4, 6j+4\}$ to make blocks for α, β, and n (P_{6j+4} is the one-factor). The rest is clear.

THEOREM 5. *There exists a GDD($v = 1 + 1 + n, 3, 3, 1, 3m$) for $3 \leq 3m \leq \frac{(n-3)}{2}$ if and only if $n = 6t + 3$ for some $t \geq 1$.*

Now we consider $n = 6t + 5$. For $n = 5$, there is no design. For $n = 11$, a GDD($13 = 1 + 1 + 11, 3, 3, 1, 4$) may be constructed as in Theorem 1. For $n = 17$, a GDD with $\lambda = 4$ or $\lambda = 7$ may be constructed applying the Lemma. We consider only the general $n = 6t + 5$, and there are two cases.

Case 1: $t = 2j$ and $n = 12j + 5$. Decompose the graph K_{12j+4} into $P_1, \cdots, P_{6j+2}$. Here we may use the triples:

$\{1, 3j, 3j+1\}, \{3, 3j-1, 3j+2\}, \cdots, \{2j-1, 2j+1, 4j\}$, and
$\{2, 5j, 5j+2\}, \{4, 5j-1, 5j+3\}, \cdots, \{2j, 4j+1, 6j+1\}$.

Not listed are $5j+4, 6j+2$. Use $P_{2j}, P_{4j+1}, P_{6j+1}$, and P_{5j+4} to make blocks with α and β, and use P_{6j+2} to make blocks with n. The rest follows as before.

Case 2: $t = 2j+1$ and $n = 12j+11$. Decompose K_{12j+10} into $P_1, \cdots, P_{6j+5}$. Here use the partition of indices given by:

$\{1, 3j+2, 3j+3\}, \{3, 3j+1, 3j+4\}, \cdots, \{2j+1, 2j+2, 4j+3\}$, and
$\{2, 5j+3, 5j+5\}, \{4, 5j+2, 5j+6\}, \cdots \{2j, 4j+4, 6j+4\}$.

Here not listed are $5j+1$ and $6j+5$. The blocks are constructed as in Case 1. This proves:

THEOREM 6. *There exists a GDD($v = 1 + 1 + n, 3, 3, 1, 4 + 3m$) for* $4 \leq 4 + 3m \leq \frac{(n-3)}{2}$ *if and only if* $n = 6t + 5$ *for some* $t \geq 1$.

This theorem completes the proof of Theorem 3 in Section 2.

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