Computation of Some Generalized Ramsey Numbers†

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Abstract. For given graphs G_1, G_2, the 2-color Ramsey number $R(G_1, G_2)$ is defined to be the least positive integer n such that every 2-coloring of the edges of complete graph K_n contains a copy of G_1 colored with the first color or a copy of G_2 colored with the second color. In this note, we obtained some new exact values of generalized Ramsey numbers such as cycle versus book, book versus book, complete bipartite graph versus complete bipartite graph.

1 Introduction

In this note, we shall only consider graphs without multiple edges or loops. For a graph G, the set of vertices of G is denoted by $V(G)$, the set of edges of G is denoted by $E(G)$, the cardinality of $V(G)$ is denoted by $|V(G)|$, the complementary graph of G is denoted by $\overline{G}$. A *cycle* on i vertices is denoted by C_i. A *path* on i vertices is denoted by P_i. A *star* graph, denoted by S_i, is a bipartite graph of order i with one partite set consisting of a single vertex, i.e. $S_i \cong K_{1,i-1}$. A *book* graph, denoted by B_i, has $i+2$ vertices

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and is the result of a single vertex being connected to every vertex of a star S_{i+1}. A *clique* of order i is denoted by K_i. Please refer to [1] for more notion and notation of graph theory.

For given graphs G_1, G_2, the 2-color Ramsey number $R(G_1, G_2)$ is defined to be the least positive integer n such that every 2-coloring of the edges of complete graph K_n contains a copy of G_1 with the first color or a copy of G_2 with the second color. A complete graph G with edges colored with 2 colors red and blue is called a (G_1, G_2)-graph if G does not contain a subgraph isomorphic to G_1 with red color, or a subgraph isomorphic to G_2 with blue color. A (G_1, G_2)-graph on n vertices is denoted by $(G_1, G_2; n)$-graph. The set of all nonisomorphic $(G_1, G_2; n)$-graphs is denoted by $\mathcal{R}(G_1, G_2; n)$, and the cardinality of $\mathcal{R}(G_1, G_2; n)$ is denoted by $|\mathcal{R}(G_1, G_2; n)|$.

Recently, many Ramsey numbers for general graphs such as paths, wheels, books, cycles, stars, trees and fans were researched. In 2005, Chen and Zhang et al. [2] researched the Ramsey numbers of paths versus wheels. They proved that $R(P_n, W_m) = 2n - 1$ for m even and $n \geq m - 1 \geq 3$; $R(P_n, W_m) = 3n - 2$ for m odd and $n \geq m - 1 \geq 2$. In 2006, Surahmat and Baskoro et al. [18] determined the Ramsey numbers of large cycles C_n versus wheels W_m, and they showed that $R(C_n, W_m) = 2n - 1$ for even m and $n \geq 5m/2 - 1$. In 2006, Salman and Broersma [17] investigated the the Ramsey numbers of paths versus fans. In 2007, Cheng and Chen et al. [3] investigated the the Ramsey numbers of cycles versus complete graphs, and they showed that $R(C_m, K_7) = 6m - 5$ for $m \geq 7$. Nevertheless, it is still difficult to determine the generalized Ramsey numbers. In the dynamic survey of small Ramsey numbers [14], some exact values of generalized Ramsey numbers are still open.

By the dynamic survey of small Ramsey numbers [14], for $R(C_m, B_3)$ and $m \geq 3$, the exact value of $R(C_7, B_3)$ remains unsolved; for the small Ramsey numbers $R(K_{m,n}, K_{s,t})$, some exact values are still unknown. Table 1 shows the the known and new exact values and bounds of these numbers, in which the values in bold fonts are new and obtained in this note. In the home page of Radziszowski [19], $R(B_3, B_4)$ remains open. In this note, we use a branch and cut technique and modify some programs used in [11] to compute some unsolved generalized Ramsey numbers. By a large mount of computations, we obtain some unsolved Ramsey numbers $R(K_{m,n}, K_{s,t})$, which are listed as follows. $R(K_{1,4}, K_{2,6}) = 11$, $R(K_{1,4}, K_{2,7}) = 13$, $R(K_{1,4}, K_{3,5}) = 13$, $R(K_{1,4}, K_{4,4}) = 13$, $R(K_{1,5}, K_{2,6}) = 14$, $R(K_{1,5}, K_{3,5}) = 15$, $R(K_{1,5}, K_{2,7}) = 15$, $R(K_{1,5}, K_{4,4}) = 13$, $R(K_{2,4}, K_{3,4}) = 17$ and $R(B_3, B_4) = 15$.

Table 1: Known and new values and bounds of Ramsey numbers $R(K_{m,n}, K_{s,t})$

m,n \ s,t	1,2	1,3	1,4	1,5	1,6	2,2	2,3	2,4	2,5	3,3
2,2	4	6	7	8	9	6				
2,3	5	7	9	10	11	8	10			
2,4	6	8	9	11	13	9	12	14		
2,5	7	9	11	13	14	11	13	16	18	
2,6	8	10	**11**	**14**						
2,7	9	11	**13**	**15**						
3,3	7	8	11	12	13	11	13	16	18	18
3,4	7	9	11	13	14	11	14	**17**	≤ 21	≤ 25
3,5	9	10	**13**	**15**		14				≤ 28
4,4	8	10	**13**	**13**		14				

2 Computation of the Ramsey Numbers

Firstly, we define some notations.

Definition 1 (One-vertex extension) *Suppose G is a $(G_1, G_2; n)$-graph, the operation to find all manners in which a new vertex is joined to G to make a $(G_1, G_2; n+1)$-graph is called one-vertex extension method.*

In this section, we developed a naive backtrack program to obtain $R(G_1, G_2; n+1)$-graphs from $R(G_1, G_2; n)$-graphs. The detailed technique for one-vertex extension is described as follows. For a graph G and a vertex v_{n+1}, where $V(G) = \{v_1, v_2, \cdots, v_n\}$. Let $e_i = (v_i, v_{n+1})$, where $1 \leq i \leq n$, $E = \{e_i | 1 \leq i \leq n\}$. We define a new graph G' as follows. $V(G') = V(G) \cup \{v_{n+1}\}$, $E(G') = E(G) \cup E$. Now, we will assign red or blue color to each edge of E in G' with all manners. The isomorphic rejection was done by using the program *shortg* [10].

2.1 Computation of $R(C_m, B_n)$

Faudree and Rousseau [7] obtained the values of $R(C_m, B_n)$, for some integer m and n. The results are as follows.

Theorem 1 *[7]* $R(C_3, B_n) = \begin{cases} 6, & \text{for } n = 1, \\ 2n+3, & \text{for } n > 1. \end{cases}$

Theorem 2 *[7]* $R(C_5, B_n) = \begin{cases} 9, & \text{for } n = 1, 2, \\ 10, & \text{for } n = 3, \\ 2n+3, & \text{for } n > 3. \end{cases}$

Theorem 3 *[7] If m be an odd integer ≥ 7 and suppose that $n \geq 4m-13$, then $R(C_m, B_n) = 2n+3$; for all $n \geq 1$ and $m \geq 2n+2$, then $R(C_m, B_n) = 2m-1$.*

We summarize the Ramsey numbers $R(C_m, B_n)$ for $m \geq 3, 2 \leq n \leq 6$ in Table 2. In this note, we obtained some other results on these Ramsey numbers, which is shown in Table 2 in bold font.

Table 2: Known and new Ramsey numbers $R(C_m, B_n)$ for $m \geq 3, 2 \leq n \leq 6$

m / n	3	4	5	6	7	8	9	10	11	12	$m \geq 2n+2$
2	7	7	9	11	13	15	17	19	21	23	$2m-1$
3	9	9	10	11	**13**	15	17	19	21	23	$2m-1$
4	11	11	11	**12**	**13**	**15**		19	21	23	$2m-1$
5	13	12	13	**14**	**15**					23	$2m-1$
6	15	13	15								$2m-1$

For some graphs G_1 and G_2, by the one-vertex extension method, we obtain the statistics of the number of nonisomorphic $(G_1, G_2; n)$-graphs, which are shown in Table 3.

Theorem 4 $R(C_7, B_3) = 13$, $R(C_6, B_4) = 12$, $R(C_7, B_4) = 13$, $R(C_8, B_4) = 15$, $R(C_6, B_5) = 14$, $R(C_7, B_5) = 15$.

Proof. From Table 3, the nonexistence of $R(C_7, B_3, 13)$-graph implies that $R(C_7, B_3) \leq 13$. Two $R(C_7, B_3, 12)$-graphs show that $R(C_7, B_3) \geq 13$. Thus $R(C_7, B_3) = 13$. Similarly, we have $R(C_6, B_4) = 12$, $R(C_7, B_4) = 13$, $R(C_8, B_4)$ =15, $R(C_6, B_5) = 14$, $R(C_7, B_5) = 15$.

2.2 Computation of some Ramsey numbers for complete bipartite graphs

By the dynamic survey of small Ramsey numbers [14], for the Ramsey numbers of $K_{m,n}$ graphs, the following results are already known. $R(K_{1,4}, K_{2,2}) = 7$, $R(K_{1,4}, K_{2,3}) = 9$, $R(K_{1,4}, K_{2,4}) = 9$, $R(K_{1,4}, K_{2,5}) = 11$, $R(K_{1,4}, K_{3,3}) = 11$, $R(K_{1,4}, K_{3,4}) = 11$. In this note, with the help of computer computation, we obtain $R(K_{1,4}, K_{2,6}) = 11$, $R(K_{1,4}, K_{2,7}) = 13$, $R(K_{1,4}, K_{3,5}) = 13$, $R(K_{1,4}, K_{4,4}) = 13$, $R(K_{1,5}, K_{2,6}) = 14$, $R(K_{1,5}, K_{3,5}) = 15$, $R(K_{1,5}, K_{2,7}) = 15$, $R(K_{1,5}, K_{4,4}) = 13$ and $R(K_{2,4}, K_{3,4}) = 17$.

By the programm *nauty* [10], 12005168 non-isomorphic graphs with 10 vertices are generated on personal computer. From the 12005168 non-isomorphic graphs with 10 vertices, we find 85 $(K_{1,4}, K_{2,6}; 10)$-graphs, 1188

Table 3: The statistics of the number of nonisomorphic$(G_1, G_2; n)$-graphs

n	9	10	11	12	13	14	15
$\|\mathcal{R}(C_7, B_3; n)\|$	19	14	9	2	0	0	0
$\|\mathcal{R}(C_6, B_4; n)\|$	59	16	2	0	0	0	0
$\|\mathcal{R}(C_7, B_4; n)\|$	372	40	14	2	0	0	0
$\|\mathcal{R}(C_8, B_4; n)\|$	2142	1010	32	21	13	2	0
$\|\mathcal{R}(C_6, B_5; n)\|$	874	236	122	55	11	0	0
$\|\mathcal{R}(C_7, B_5; n)\|$	2486	1151	193	94	17	2	0

$(K_{1,4}, K_{2,7}; 10)$-graphs, 268 $(K_{1,4}, K_{3,5}; 10)$-graphs, 486 $(K_{1,4}, K_{4,4}; 10)$-graphs, 29336 $(K_{1,5},\ K_{2,6};\ 10)$-graphs, 51847 $(K_{1,5}, K_{3,5}; 10)$-graphs, 79076 $(K_{1,5}, K_{2,7}; 10)$-graphs, 69200 $(K_{1,5}, K_{4,4}; 10)$-graphs and 666964 $(K_{2,4}, K_{3,4}; 10)$-graphs. Let $\mathcal{M}(G, G_1,\ G_2; n+1)$ be all $(G_1, G_2; n+1)$-graphs with $n+1$ vertices obtained from a $(G_1, G_2; n)$-graph G by joining one vertex. It is clear that $\mathcal{R}(G_1, G_2; n+1) = \cup_{G\in\mathcal{R}(G_1,G_2;n)} \mathcal{M}(G, G_1, G_2; n+1)$. By the above one-vertex extension algorithm, we complete the extension $G \to \mathcal{M}(G, G_1, G_2; n+1)$, for all $G \in \mathcal{R}(G_1, G_2; n)$ and $10 \le n \le 14$. Table 4 gives the statistics of their values Fig. 1 shows two $(K_{1,4}, K_{3,5}; 12)$-graphs, Fig. 2 shows three $(K_{1,4}, K_{4,4}; 12)$-graphs, Fig. 3 shows the unique $(K_{1,5}, K_{2,6}; 13)$-graph and Fig. 4 shows the unique $(K_{1,5}, K_{3,5}; 14)$-graph.

Table 4: The statistic of the number of all nonisomorphic $(G_1, G_2; n)$-graphs

n	10	11	12	13	14	15	16	17
$\|\mathcal{R}(K_{1,4}, K_{2,6}; n)\|$	85	0	0	0	0	0	0	0
$\|\mathcal{R}(K_{1,4}, K_{2,7}; n)\|$	1188	245	18	0	0	0	0	0
$\|\mathcal{R}(K_{1,4}, K_{3,5}; n)\|$	268	29	2	0	0	0	0	0
$\|\mathcal{R}(K_{1,4}, K_{4,4}; n)\|$	486	96	3	0	0	0	0	0
$\|\mathcal{R}(K_{1,5}, K_{2,6}; n)\|$	29336	22916	1150	1	0	0	0	0
$\|\mathcal{R}(K_{1,5}, K_{2,7}; n)\|$	79076	267514	244979	13753	12	0	0	0
$\|\mathcal{R}(K_{1,5}, K_{3,5}; n)\|$	51847	100353	34463	430	1	0	0	0
$\|\mathcal{R}(K_{1,5}, K_{4,4}; n)\|$	69200	201308	44	0	0	0	0	0
$\|\mathcal{R}(K_{2,4}, K_{3,4}; n)\|$	666964	620-7721	287-03347	320-37303	274-3467	4882	2	0

Theorem 5 $R(K_{1,4}, K_{2,6}) = 11$, $R(K_{1,4}, K_{2,7}) = 13$, $R(K_{1,4}, K_{3,5}) = 13$, $R(K_{1,4}, K_{4,4}) = 13$, $R(K_{1,5}, K_{2,6}) = 14$, $R(K_{1,5}, K_{3,5}) = 15$, $R(K_{1,5}, K_{2,7}) = 15$, $R(K_{1,5}, K_{4,4}) = 13$, $R(K_{2,4}, K_{3,4}) = 17$.

Proof: The above computations and results show that there exists no $(K_{1,4},\ K_{2,6}; 11)$-graph, so $R(K_{1,4}, K_{2,6}) \le 11$. On the other

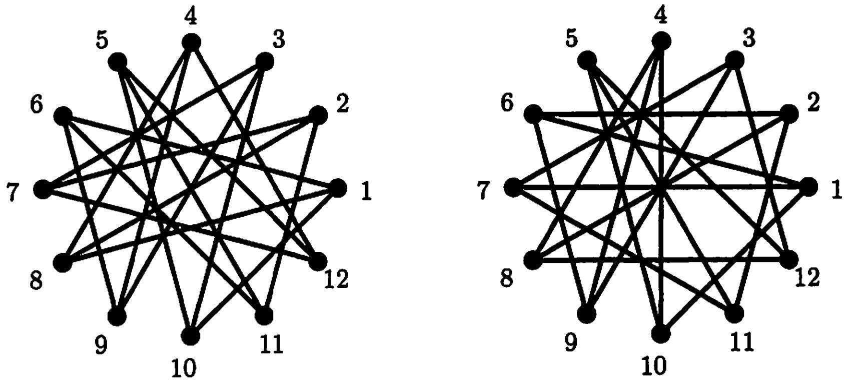

Figure 1: Two $(K_{1,4}, K_{3,5}; 12)$-graphs

hand, there are 85 $(K_{1,4}, K_{2,6}; 10)$-graphs (see Table 1), which indicates that $R(K_{1,4}, K_{2,6}) > 10$. Thus, $R(K_{1,4}, K_{2,6}) = 11$. Similarly, we have $R(K_{1,4}, K_{2,7}) = 13$, $R(K_{1,4}, K_{3,5}) = 13$, $R(K_{1,4}, K_{4,4}) = 13$, $R(K_{1,5}, K_{2,6}) = 14$, $R(K_{1,5}, K_{3,5}) = 15$, $R(K_{1,5}, K_{2,7}) = 15$, $R(K_{1,5}, K_{4,4}) = 13$ and $R(K_{2,4}, K_{3,4}) = 17$.

2.3 Computation of $R(B_m, B_n)$

In 1978, Rousseau and Sheehan [16] obtained $R(B_1, B_n) = 2n + 3$ for integer $n > 1$; in 2005, Nikiforov and Rousseau [12] obtained that for some number c, where $c < 10^6$, if $m \geq cn$, then $R(B_m, B_n) = 2m + 3$; Nikiforov and Rousseau et al. [13] proved that $R(B_n, B_n) = 4n + 2$ when $4n + 1$ is prime. Here, some small Ramsey numbers $R(B_m, B_n)$ are listed in Table 5.

Table 5: Some known and new Ramsey numbers $R(B_m, B_n)$

m \ n	2	3	4	5	6
2	10 [4]				
3	**11**	14 [8]			
4	13 [15]	**15**	18 [16]		
5	16 [16]	17 [16]		21 [16]	
6			22 [16]		26 [16]

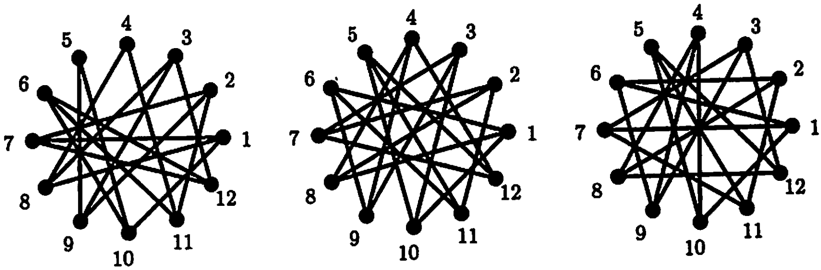

Figure 2: Three $(K_{1,4}, K_{4,4}; 12)$-graphs

The one-vertex extension method above can be use to compute the Ramsey number $R(B_m, B_n)$. In this note, for higher efficiency, the gluing algorithm was modified from [11] to implement another extension method. We define some notations as follows.

Definition 2 (Feasible cone) *For a graph G and a vertex v, where $V(G) = \{v_1, v_2, \cdots, v_{|V(G)|}\}$, if $S \subseteq V(G)$ and the graph induced by the edge set $E(G) \cup \{(v,x)|x \in S\}$, denoted by G', contains no B_m and $\overline{G'}$ contains no B_n, then we call S is a* feasible cone *of G for (B_m, B_n).*

Definition 3 (Feasible interval) *If B and T are feasible cones of G for (B_m, B_n) and $B \subseteq T$, we call $[B, T]$ is a* feasible interval *of G for (B_m, B_n).*

Obviously, a feasible cone is correspondence to a $(B_m, B_n; |V(G)| + 1)$-graph. We can see that the feasible interval $[B, T]$ contains $2^{|T|-|B|}$ feasible cones. The detailed technique for generating feasible intervals of G for (K_m, K_n) was described in [11]. For graph G, B_m and B_n, Algorithm 1 is used to generate feasible intervals of G for (B_m, B_n). The original algorithm was used to generate feasible intervals of G for (K_4, K_5) [11]. In this note, we need to process for book graphs. So X_i are changed from a clique to a star and from an independent set to the complement of a star, which are shown in line 4 and line 17.

Algorithm 1

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1 $\mathfrak{T} = \{[\phi, V_G]\}$

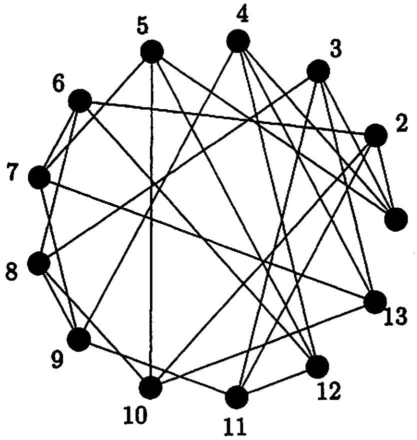

Figure 3: The unique $(K_{1,5}, K_{2,6}; 13)$-graph

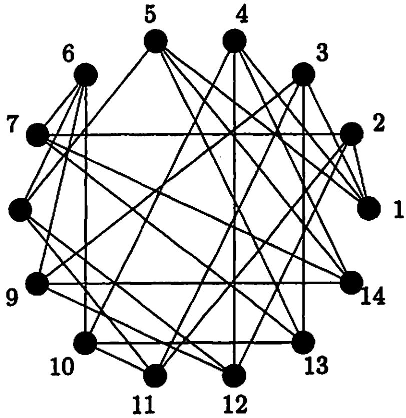

Figure 4: The unique $(K_{1,5}, K_{3,5}; 14)$-graph

```
for i=1 to r do
{
    if X_i is a S_{m+1} then
    {
     for each [B,T] ∈ 𝔗 such that X_i ⊆ T do
     if X_i ⊆ B then
     {
      Delete [B,T] from 𝔗
     }
     else
     {
      Replace [B,T] by [B ∪ {y_1, y_2, ···, y_{j-1}}, T − {y_j}]
        for j = 1, 2, ···, k, where X_i − B = {y_1, y_2, ···, y_k}
     }
    }
    else [if X_i is a \overline{S_{n+1}}]
    {
      for each [B,T] ∈ T such that X_i ∩ B = φ do
      {
       if X_i ∩ T = φ then
       {
```

```
        Delete [B,T] from 𝔗
      }
      else
      {
      Replace [B,T] by [B ∪ {y_j}, T − {y_1, y_2, ···, y_{j−1}}]
       for j = 1,2,···,k, where X_i ∩ T = {y_1, y_2, ···, y_k}
      }
    }
  }
}
```

Let G ba a (B_m, B_n)-graph, we implement one-vertex extension method based on interval cones to obtain $\mathcal{R}(B_m, B_n; |V(G)|+1)$ as follows.
Step 1. Use Algorithm 1 to find the set of feasible intervals $\mathfrak{T}$ of G for (B_m, B_n),
Step 2. Generate all feasible cones according to $\mathfrak{T}$,
Step 3. Convert feasible cones into $(B_m, B_n; |V(G)|+1)$-graphs.
Step 4. Remove isomorphic graphs in (B_m, B_n).

By above algorithm, we obtained some statistics of $|\mathcal{R}(B_m, B_n)|$, which are listed in Table 6. Four $(B_2, B_3; 10)$-graphs are shown in Fig. 5 and the unique $(B_3, B_4; 14)$-graph is shown in Fig. 6.

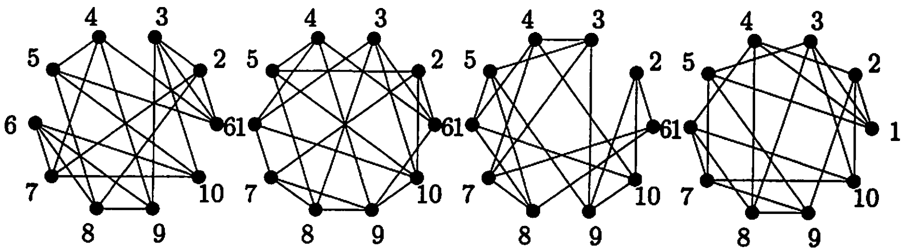

Figure 5: Four $(B_2, B_3; 10)$-graphs

Table 6: The statistics of some nonisomorphic $R(B_m, B_n)$-graphs

n	9	10	11	12	13	14	15
$\|\mathcal{R}(B_2, B_3; n)\|$	30	4	0	0	0	0	0
$\|\mathcal{R}(B_3, B_4; n)\|$	16118	85745	224771	108393	1524	1	0

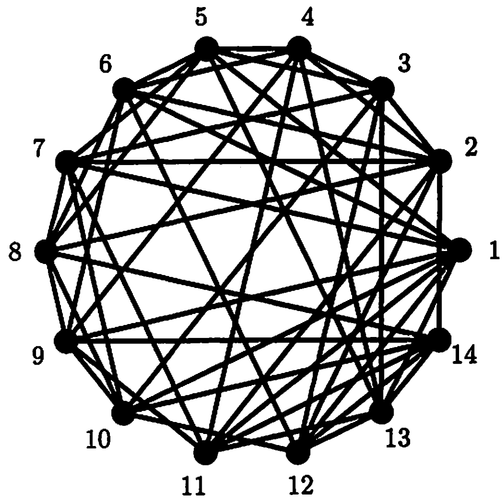

Figure 6: The unique $(B_3, B_4; 14)$-graph

By Table 6, we have

Theorem 6 $R(B_2, B_3) = 11$, $R(B_3, B_4) = 15$.

3 Remarks

In 1991, Faudree and Rousseau obtained some results about cycle-book Ramsey numbers $R(C_m, B_n)$, some of which were not obtained. In this note, we obtained some new results and complete the results for the case $n = 3$.

A general utility program for graph isomorph removal, *nauty*, *shortg*[10], written by Brendan McKay, was used. One-vertex extension method described above was implemented by the first author.

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