Arithmetic Labelings and Geometric Labelings of Finite Graphs

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Abstract

An injective map from the vertex set of a graph G to the set of all natural numbers is called an/a *arithmetic/geometric labeling* of G if the set of all numbers, each of which is the sum/product of the integers assigned to the ends of some edge, form an/a arithmetic/geometric progression. A graph is called *arithmetic/geometric* if it admits an/a arithmetic/geometric labeling. In this note, we show that the two notions just mentioned are equivalent—i.e., a graph is arithmetic if and only if it is geometric.

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All graphs considered in this note are finite and simple. The set of all positive integers is denoted by $\mathbb{N}$. For basic information about graph theory we rely on [6]. Let $G = (V, E)$ be a graph and f be any map from V to $\mathbb{N}$; we associate with f two maps from E to $\mathbb{N}$, denoted by f^+ and $f^\times$: for all $uv \in E$, $f^+(uv) = f(u) + f(v)$ and $f^\times(uv) = f(u) \times f(v)$. If f is injective and the elements of $f^+(E)$ form an arithmetic progression—i.e., if this set can be written as $\{k, k + d, k + 2d, \ldots, k + (|E| - 1)d\}$—then f is called an *arithmetic labeling* of G. (For information about this labeling, see [1, 2].) If f is injective and the elements of $f^\times(E)$ form a geometric progression—i.e., if this set can be written as $\{a, ar, ar^2, \ldots, ar^{|E|-1}\}$ where r need not be an integer—then f is called a *geometric labeling* of G. (For details about this labeling, we refer the reader to [4, 5]; in this connection, see [3] also.) If a graph admits an/a arithmetic/geometric labeling then it is called *arithmetic/geometric*. If a graph is arithmetic then it is easy to see that it is geometric also. The objective of this note is to prove the converse.

Remark. In the definition of arithmetic/geometric labeling, there is no need to have the condition that the induced function on the edge set is also injective. (In this connection, see [1, 5].) Suppose that f is an/a arithmetic/geometric labeling of a graph G. If $\Delta(G) > 1$, then $f^+/f^\times$ is obviously injective. If $\Delta(G) = 1$, then it is easy to construct an/a arithmetic/geometric labeling g of G such that $g^+/g^\times$ is also injective.

Theorem. *A graph $G = (V, E)$ is arithmetic if and only if it is geometric.*

Proof. If σ is an arithmetic labeling of G, then the map: $v \mapsto 2^{\sigma(v)}$ where $v \in V$ is a geometric labeling of G. Now suppose that f is a geometric labeling of G. Then there are distinct primes $P_1, P_2, \ldots, P_n$ such that for all $v \in V$, $f(v)$ can be written as $P_1^{\alpha_1} P_2^{\alpha_2} \ldots P_n^{\alpha_n}$ where for each $i \in \{1, 2, \ldots, n\}$, $\alpha_i \in \mathbb{N} \cup \{0\}$—let us denote the maximum of $\alpha_1, \alpha_2, \ldots, \alpha_n$ by $\mu(v)$.

Let $M = \max\{\mu(v) : v \in V\}$. Let $\mathfrak{P}$ be the collection of all nonzero polynomials of the form $\alpha_1 x + \alpha_2 x^2 + \cdots + \alpha_n x^n$ where for each $i \in \{1, 2, \ldots, n\}$, $|\alpha_i| \in \{0, 1, 2, \ldots, M\}$. Since $\mathfrak{P}$ is finite, we can choose a positive integer t so that for all $F \in \mathfrak{P}$, $F(t) \neq 0$. Now define a map $g : V \mapsto \mathbb{N}$ as follows: Let $v \in V$; then there are nonnegative integers $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that $f(v) = P_1^{\alpha_1} P_2^{\alpha_2} \ldots P_n^{\alpha_n}$; set $g(v) = \alpha_1 t + \alpha_2 t^2 + \cdots + \alpha_n t^n$. The relation between $f^{\times}$ and g^{+} given below can be easily verified:

(**) For any $e \in E$, if $f^{\times}(e) = P_1^{\alpha_1} P_2^{\alpha_2} \ldots P_n^{\alpha_n}$ where $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{N} \cup \{0\}$, then $g^{+}(e) = \alpha_1 t + \alpha_2 t^2 + \cdots + \alpha_n t^n$.

Suppose that $u, v \in V$ such that $g(u) = g(v)$. Then there are nonnegative integers $\alpha_1, \alpha_2, \ldots, \alpha_n, \beta_1, \beta_2, \ldots, \beta_n$ such that $f(u) = P_1^{\alpha_1} P_2^{\alpha_2} \ldots P_n^{\alpha_n}$ and $f(v) = P_1^{\beta_1} P_2^{\beta_2} \ldots P_n^{\beta_n}$. Therefore $g(u) = \alpha_1 t + \alpha_2 t^2 + \cdots + \alpha_n t^n$ and $g(v) = \beta_1 t + \beta_2 t^2 + \cdots + \beta_n t^n$. Thus $(\alpha_1 - \beta_1)t + (\alpha_2 - \beta_2)t^2 + \cdots + (\alpha_n - \beta_n)t^n = 0$. Since for each $i \in \{1, 2, \ldots, n\}$, $|\alpha_i - \beta_i| \leqslant M$, by the choice of t, for all $i \in \{1, 2, \ldots, n\}$, $\alpha_i = \beta_i$. Therefore $f(u) = f(v)$; since f is injective, $u = v$; thus it follows that g is also injective. Let $m = |E|$. Since f is a geometric labeling, the edges of G can be ordered as $e_1, e_2, \ldots, e_m$ so that $f^{\times}(e_1), f^{\times}(e_2), \ldots, f^{\times}(e_m)$ is a geometric progression. For each $i \in \{1, 2, \ldots, m\}$, let $f^{\times}(e_i) = P_1^{\alpha_{i1}} P_2^{\alpha_{i2}} \ldots P_n^{\alpha_{in}}$. Then for each $j \in \{1, 2, \ldots, n\}$, since $(P_j^{\alpha_{ij}})_{i=1}^{m}$ is a geometric progression, $(\alpha_{ij})_{i=1}^{m}$ is an arithmetic progression; it is easy to verify that $(\sum_{j=1}^{n} \alpha_{ij} t^j)_{i=1}^{m}$ too is an arithmetic progression; i.e., $(g^{+}(e_i))_{i=1}^{m}$ is arithmetic because, by (**) for each $i \in \{1, 2, \ldots, m\}$, $g^{+}(e_i) = \sum_{j=1}^{n} \alpha_{ij} t^j$. Therefore g is an arithmetic labeling of G. □

Corollary. *If a graph is geometric and its edge set is E, then it has a geometric labeling f such that $f^{\times}(E) = \{a, ar, ar^2, \ldots, ar^{|E|-1}\}$ where $r \in \mathbb{N}$.*

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