Shortest Single Axioms for SQS-Skeins and Mendelsohn Ternary Quasigroups

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Abstract

In this note, we exhibit shortest single axioms for SQS-skeins and Mendelsohn ternary quasigroups that were found with the aid of the automated theorem-prover Prover9 and the finite model-finder Mace4.

1 Introduction

A *Steiner triple (quadruple) system*, or STS (SQS), is a pair $(X, \mathcal{B})$ where X is a set, the elements of which will be called *points*, and $\mathcal{B}$ is a set of 3-subsets (4-subsets) of X, the elements of which will be called *blocks*, such that every 2-subset (3-subset) of points is contained in exactly one block.

A *Mendelsohn triple (quadruple) system*, or MTS (MQS), is a pair $(X, \mathcal{B})$ where X is a set, the elements of which will be called *points*, and $\mathcal{B}$ is a set of ordered triples (quadruples) of distinct points, the elements of which will be called *cyclic triples (quadruples)*, such that every ordered pair (triple) of distinct points is contained in exactly one cyclic triple (quadruple), where the cyclic triple (quadruple) (a, b, c) $((a, b, c, d))$ contains only the pairs (triples) (a, b), (b, c), and (c, a) $((a, b, c)$, (b, c, d), (c, d, a), and $(d, a, b))$.

Given an SQS (MQS) $(X, \mathcal{B})$, we can construct an algebra $(X; q)$ of type (3) as follows: define $q(y, x, x) = q(x, y, x) = q(x, x, y) = y$ for all $x, y \in X$ and also define $q(x, y, z)$ to be the fourth point in the unique block (cyclic quadruple) containing $\{x, y, z\}$ $((x, y, z))$ for all distinct $x, y, z \in X$.

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Clearly, this algebra satisfies the four identities

$$q(x,x,y) = y \tag{1}$$

$$q(x,y,z) = q(x,z,y) \tag{2}$$

$$q(x,y,z) = q(y,z,x) \tag{3}$$

$$q(x,y,q(x,y,z)) = z \tag{4}$$

in the SQS case and the three identities

$$q(x,x,y) = y \tag{5}$$

$$q(x,y,x) = y \tag{6}$$

$$q(q(x,y,z),x,y) = z \tag{7}$$

in the MQS case.

Conversely, given an algebra $(X;q)$ of type (3) that satisfies (1), (2), (3), and (4) ((5), (6), and (7)), we can construct an SQS (MQS) whose points are the elements of X and whose blocks are the 4-subsets (4-tuples) of points of the form $\{x,y,z,q(x,y,z)\}$ ($(x,y,z,q(x,y,z))$), x, y, and z distinct. Therefore, there is a one-to-one correspondence between SQS's (MQS's) and type (3) algebras satisfying (1), (2), (3), and (4) ((5), (6), and (7)), just as there is a one-to-one correspondence between STS's (MTS's) and idempotent totally symmetric quasigroups, also known as *squags* (idempotent semisymmetric quasigroups, also known as *Mendelsohn quasigroups*). For this reason, type (3) algebras $(X;q)$ satisfying (1), (2), (3), and (4) ((5), (6), and (7)) are known as SQS-*skeins* (*Mendelsohn ternary quasigroups* (MTQ*'s*)).

We will call an identity in a single ternary operation q a *single axiom for* SQS-*skeins* (MTQ*'s*) if and only if the identity is valid in all SQS-skeins (MTQ's) and all models of the identity are SQS-skeins (MTQ's). Single axioms for several varieties of algebras arising from combinatorial designs are known. For instance, in [1], a single axiom for squags was found. Therefore, it is natural to ask if there exists a single axiom for SQS-skeins (MTQ's). If so, what is the length of a shortest such identity (in terms of the number of variable occurrences) and what is the smallest number of distinct variables among such identities? In fact, in [5], the identity

$$q(q(x,y,q(z,q(u,u,q(v,w,q(q(s,t,t),w,v))),z)),x,y) = s$$

was shown to be a single axiom for SQS-skeins and the identity

$$q(q(x,y,q(z,q(u,u,q(v,w,q(q(s,t,t),v,w))),z)),x,y) = s$$

was shown to be a single axiom for MTQ's. Notice that both of these identities have eight distinct variables and 16 variable occurrences.

In this note, we exhibit shortest single axioms for SQS-skeins and MTQ's with fewer distinct variables. Our single axioms were found with the aid of the automated theorem-prover Prover9 and the finite model-finder Mace4 [3]. Prover9 searches for proofs by contradiction of first-order statements while Mace4 searches for finite models of first-order statements. Their combination can be a powerful tool in investigations of this kind. We also used the scripting language Perl to further automate our search. For another example of an automated theorem-prover being used to find a single axiom for a variety of ternary algebras that was shorter than the shortest one previously arrived at by algebraic means, see [4]. For an example of an automated theorem-prover being used to find all of the shortest single axioms for a variety of algebras, see [2].

2 A Shortest Single Axiom for SQS-Skeins

In this section, we describe our search for a shortest single axiom for SQS-skeins.

We began by generating all identities (up to renaming and symmetry) in a single ternary operation q with at most 10 variable occurrences and with the following properties. One side consists of a single variable (otherwise it would be valid in any model of $q(x,y,z) = q(u,v,w)$) that is not the left-most (right-most) variable on the other side (otherwise it would be valid in any model of $q(x,y,z) = x$ ($q(x,y,z) = z$)) where each occurring variable occurs at least twice (otherwise it would imply $x = y$ in an SQS-skein). This resulted in 571926 identities.

We then sent the negation of each identity (stored in the Perl variable `$negated_identity`) to Mace4 and ran

```
assign(iterate_up_to, 10).  % model of size at most 10
clauses(theory).
q(x,x,y) = y.  q(x,y,z) = q(x,z,y).  q(x,y,z) = q(y,z,x).
q(x,y,q(x,y,z)) = z.  % SQS-skein
$negated_identity.
end_of_list.
```

to search for an SQS-skein that does not satisfy the identity and then removed the identities for which a model was found. This resulted in 137709 identities.

Next, we sent the negation of each of these identities to Prover9 and ran

```
set(auto).  % autonomous mode
clauses(sos).  % set of support
q(x,x,y) = y.  q(x,y,z) = q(x,z,y).  q(x,y,z) = q(y,z,x).
q(x,y,q(x,y,z)) = z.  % SQS-skein
$negated_identity.
end_of_list.
```

to search for a proof that the identity is implied by (1), (2), (3), and (4), and is therefore valid in the variety of SQS-skeins. A proof was found for all of these identities.

We then sent each of these identities (stored in the variable **$identity**) to Mace4 and ran

```
assign(iterate_up_to, 200).  % model of size at most 200
clauses(theory).
$identity.
q(0,0,a) != a | q(0,a,b) != q(0,b,a) | q(0,a,b) != q(a,b,0) |
 q(0,a,q(0,a,b)) != b.  % not SQS-skein
end_of_list.
```

to search for a model of the identity that does not satisfy at least one of (1), (2), (3), and (4), and therefore is not an SQS-skein, and then removed the identities for which a model was found. This eliminated all of these identities.

We continued by generating all identities (up to renaming and symmetry) in a single ternary operation q with one side consisting of a single variable that is not the left-most or right-most variable on the other side and with six distinct variables occurring twice each. This resulted in 2321865 identities.

We then used Prover9 to extract a large number of these identities that are valid in the variety of SQS-skeins by sending each of them to Prover9 and searching for a proof that it is implied by (1), (2), (3), and (4). For example, $q(x,x,q(y,q(q(q(y,z,u),z,u),v,w),v)) = w$ is easily seen to be valid in all SQS-skeins.

Finally, we sent each of these identities to Prover9 to search for a proof that it implies (1), (2), (3), and (4), and is therefore a single axiom for SQS-skeins, until one was found. For example, running Prover9 on the input

```
set(auto).  % autonomous mode
```

```
clauses(sos).  % set of support
q(x,x,q(y,q(q(q(y,z,u),z,u),v,w),v)) = w.  % candidate identity
q(a,a,b) != b | q(a,b,c) != q(a,c,b) | q(a,b,c) != q(b,c,a) |
 q(a,b,q(a,b,c)) != c.  % not SQS-skein
end_of_list.
```

produces a proof that the identity from above is a single axiom for SQS-skeins.

Theorem 1. *The identity*

$$q(x, x, q(y, q(q(q(y, z, u), z, u), v, w), v)) = w$$

is a shortest single axiom for SQS-*skeins.*

3 A Shortest Single Axiom for MTQ's

In this section, we describe our search for a shortest single axiom for MTQ's.

We began with the 571926 identities with at most 10 variable occurrences from Section 2. Using Mace4 and Prover9, we showed that exactly 58870 of these identities are valid in the variety of MTQ's (513056 of them were shown by Mace4 to be violated in an MTQ of size at most eight and the remaining 58870 were all shown by Prover9 to follow from (5), (6), and (7)). However, using Mace4, we showed that each of these identities is valid in some model that is not an MTQ.

We then turned to the 2321865 identities with 12 variable occurrences from Section 2. Using Prover9, we extracted a large number of these identities that are valid in the variety of MTQ's. Finally, using Prover9, we searched among these identities for a single axiom for MTQ's until one was found.

Theorem 2. *The identity*

$$q(x, x, q(y, y, q(q(z, u, q(v, z, u)), w, v))) = w$$

is a shortest single axiom for MTQ*'s.*

Problem 3. Find all shortest single axioms for SQS-skeins (MTQ's).

Problem 4. Find a single axiom for SQS-skeins (MTQ's) with less than six distinct variables or show that none exist.

References

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