Neighborhood Connected Domatic Number of a Graph

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Abstract

Let $G = (V, E)$ be a connected graph. A dominating set S of G is called a *neighborhood connected dominating set* (*ncd-set*) if the induced subgraph $\langle N(S) \rangle$ is connected, where $N(S)$ is the open neighborhood of S. A partition $\{V_1, V_2, \ldots, V_k\}$ of $V(G)$, in which each V_i is a ncd-set in G is called a *neighborhood connected domatic partition* or simply *nc-domatic partition* of G. The maximum order of a nc-domatic partition of G is called the neighborhood connected domatic number (nc-domatic number) of G and is denoted by $d_{nc}(G)$. In this paper we initiate a study of this parameter.

Keywords : dominating set, neighborhood connected dominating set, neighborhood connected domatic number
2000 Mathematics Subject Classification: 05C69

1 Introduction

By a graph $G = (V, E)$ we mean a finite, undirected and connected graph with neither loops nor multiple edges. The order and size of G are denoted by n and m respectively. For graph theoretic terminology we refer to Chartrand and Lesniak [2].

There are many variations of domination in graphs. In the book [5] it is proposed that a type of domination is "fundamental" if (i) every connected nontrivial graph has a dominating set of this type and (ii) this type of dominating set S is defined in terms of some "natural" property of the subgraph induced by S. Examples include total domination, independent domination, connected domination and paired domination.

In [1] we have introduced the concept of neighborhood connected domination, which is a fundamental concept in the above sense.

Definition 1.1. *A dominating set S of a graph G is called a neighborhood connected dominating set (ncd-set) if the induced subgraph $\langle N(S) \rangle$ is connected. The minimum cardinality of a ncd-set of G is called the neighborhood connected domination number of G and is denoted by $\gamma_{nc}(G)$.*

The concepts of domatic number, total domatic number and connected domatic number were introduced respectively by Cockayne and Hedetniemi [4], Cockayne et al. [3] and Laskar et al. [7].

Definition 1.2. *A domatic partition of G is a partition $\{V_1, V_2, \ldots, V_k\}$ of $V(G)$ in which each V_i is a dominating set of G. The maximum order of a domatic partition of G is called the domatic number of G and is denoted by $d(G)$.*

Definition 1.3. *Let G be a graph without isolated vertices. A total domatic partition of G is a partition $\{V_1, V_2, \ldots, V_k\}$ of $V(G)$ in which each V_i is a total dominating set of G. The maximum order of a total domatic partition of G is called the total domatic number of G and is denoted by $d_t(G)$.*

Definition 1.4. *Let G be a connected graph. A connected domatic partition of G is a partition $\{V_1, V_2, \ldots, V_k\}$ of $V(G)$ in which each V_i is a connected dominating set of G. The maximum order of a connected domatic partition of G is called the connected domatic number of G and is denoted by $d_c(G)$.*

A survey of results on domatic numbers of graphs and their variants is given by Zelinka [10] in Chapter 13 of Haynes et al. [6].

In this paper we introduce the concept of neighborhood connected domatic number and initiate a study of this parameter.

We need the following definition and theorems.

Definition 1.5. *The graph G obtained from the stars $K_{1,r}$ and $K_{1,s}$ by joining their centers by an edge is called a bistar and is denoted by $B(r,s)$.*

Theorem 1.6. [9] *Let G be a connected graph which is not complete. Then $d_c(G) \leq \kappa(G)$, where $\kappa(G)$ is the connectivity of G.*

Theorem 1.7. [1] *If P_n is the path on n vertices, then $\gamma_{nc}(P_n) = \lceil \frac{n}{2} \rceil$.*

Theorem 1.8. [1] *If C_n is the cycle on n vertices, then*

$$\gamma_{nc}(C_n) = \begin{cases} \lceil \frac{n}{2} \rceil & \text{if } n \not\equiv 3 (mod\ 4) \\ \lfloor \frac{n}{2} \rfloor & \text{if } n \equiv 3 (mod\ 4). \end{cases}$$

Theorem 1.9. [1] *For any graph $G, \gamma_{nc}(G) \leq \lceil \frac{n}{2} \rceil$.*

2 Main Results

Definition 2.1. *A neighborhood connected domatic partition (nc-domatic partition) of a connected graph G is a partition $\{V_1, V_2, \ldots, V_k\}$ of $V(G)$ in which each V_i is a ncd-set of G. The neighborhood connected domatic number (nc-domatic number) $d_{nc}(G)$ of G is the maximum order of a neighborhood connected domatic partition of G.*

Observation 2.2. *Since any domatic partition of K_n is also a nc-domatic partition, we have $d_{nc}(K_n) = d(K_n) = n$. Similarly $d_{nc}(K_{m,n}) = d(K_{m,n}) = min\{m, n\}$. Also for the wheel W_n,*

$$d_{nc}(W_n) = d(W_n) = \begin{cases} 4 & \text{if } n \equiv 1 (mod\ 3) \\ 3 & \text{otherwise.} \end{cases}$$

Observation 2.3. *Since any total domatic partition of G is a nc-domatic partition, we have $d_t(G) \leq d_{nc}(G) \leq d(G)$.*

Observation 2.4. *Let $v \in V(G)$ and deg $v = \delta$. Since any ncd-set of G must contain either v or a neighbor of v, it follows that $d_{nc}(G) \leq \delta(G) + 1$.*

Observation 2.5. *Let $\{V_1, V_2, \ldots, V_{d_{nc}}\}$ be a nc-domatic partition of G. Since $|V_i| \geq \gamma_{nc}$ for each i, it follows that $\gamma_{nc}(G) d_{nc}(G) \leq n$.*

Observation 2.6. *Given two positive integers n and k with $n \geq 4$ and $1 \leq k \leq n$, there exists a graph G with n vertices such that $d_{nc}(G) = k$. We take*

$$G = \begin{cases} K_n & \text{if } k = n \\ K_{1,n-1} & \text{if } k = 1 \\ B(n_1, n-2-n_1) & \text{if } k = 2 \\ K_{k-1} + \overline{K_{n-k+1}} & \text{otherwise.} \end{cases}$$

Theorem 2.7. *For any connected graph G, $d_c(G) \leq d_{nc}(G)$. Also the difference $d_{nc}(G) - d_c(G)$ can be made arbitrarily large.*

Proof. If $\Delta(G) < n-1$, then any connected domatic partition of G is a nc-domatic partition of G. If $\Delta(G) = n-1$ and G has a cut vertex, it follows from Theorem 1.6, that $d_c(G) = 1$. Thus $d_c(G) \leq d_{nc}(G)$. Also if k is any positive integer, then for the graph G having exactly two blocks, each isomorphic to K_{k+2}, we have $d_c(G) = 1$ and $d_{nc}(G) = k+1$. Thus $d_{nc}(G) - d_c(G) = k$. □

Theorem 2.8. *For any graph G, $\left\lfloor \frac{d(G)}{2} \right\rfloor \leq d_{nc}(G) \leq d(G)$ and the bounds are sharp.*

Proof. Since every ncd-set is a dominating set, we have $d_{nc}(G) \leq d(G)$. Further, since the union of two disjoint dominating sets is a ncd-set, we have $\left\lfloor \frac{d(G)}{2} \right\rfloor \leq d_{nc}(G)$. Also for the graph $G = K_{1,n-1}$, $d_{nc}(G) = \frac{d(G)}{2}$. For the graph $G = K_n$, $d_{nc}(G) = d(G) = n$. □

Theorem 2.9. *For any non trivial path P_n, we have*

$$d_{nc}(P_n) = \begin{cases} 1 & \text{if } n \text{ is odd} \\ 2 & \text{if } n \text{ is even.} \end{cases}$$

Proof. Let $P_n = (v_1, v_2, \ldots, v_n)$. It follows from Theorem 1.7 that if n is odd then $d_{nc}(P_n) = 1$ and if n is even, then $d_{nc}(P_n) \leq 2$. Further if n is even, then $\{V_1, V - V_1\}$ where $V_1 = \{v_i : i \equiv 2 \text{ or } 3(mod\ 4)\}$ is a nc-domatic partition of P_n and hence $d_{nc}(P_n) = 2$. □

Theorem 2.10. *For any cycle C_n with $n \geq 4$, we have*

$$d_{nc}(C_n) = \begin{cases} 1 & \text{if } n \equiv 1(mod\ 4) \\ 2 & \text{otherwise} \end{cases}$$

Proof. Let $C_n = (v_1, v_2, \ldots, v_n, v_1)$. It follows from Theorem 1.8 that if $n \equiv 1(mod\ 4)$, then $d_{nc}(C_n) = 1$ and $d_{nc}(C_n) \leq 2$ otherwise. Further if $n \not\equiv 1(mod\ 4)$, then $\{V_1, V - V_1\}$ where $V_1 = \{v_i : i \equiv 0 \text{ or } 1(mod\ 4)\}$ is a nc-domatic partition of G and hence $d_{nc}(G) = 2$. □

In the following theorem we obtain a bound for d_{nc} and characterize the class of graphs attaining the bound.

Theorem 2.11. *Let G be a graph with $\Delta = n-1$ and let k denote the number of vertices of degree $n-1$. Then $d_{nc}(G) \leq \frac{1}{2}(n+k)$. Further $d_{nc}(G) = \frac{1}{2}(n+k)$ if and only if one of the following holds.*

1. *$G = K_k + H$ where $k \geq 2$ and H is isomorphic to $2K_{\frac{n-k}{2}}$.*

2. $G = K_k + H$ *where* H *is a connected graph with* $V(H) = X_1 \cup X_2 \cup \cdots \cup X_r$ *where* $r = \frac{n-k}{2}$, $|X_i| = 2$ *and for all* $i \neq j$, $X_i \cap X_j = \emptyset$ *and the subgraph induced by the edges of* H *with one end in* X_i *and the other end in* X_j *has a perfect matching.*

Proof. Let $\{V_1, V_2, \ldots, V_s\}$ be any nc-domatic partition of G with $|V_i| = 1, 1 \leq i \leq k$. Since $|V_j| \geq 2$ for all j with $k+1 \leq j \leq s$, it follows that $s \leq k + \frac{n-k}{2} = \frac{n+k}{2}$. Hence $d_{nc}(G) \leq \frac{1}{2}(n+k)$.

Now, let G be a graph with $d_{nc}(G) = \frac{1}{2}(n+k)$. Then there exists a nc-domatic partition $\{V_1, V_2, \ldots, V_k, V_{k+1}, \ldots, V_{\frac{n+k}{2}}\}$ such that $|V_i| = 1$ if $1 \leq i \leq k$ and $|V_j| = 2$ if $k+1 \leq j \leq \frac{n+k}{2}$. Clearly, $\langle V_1 \cup V_2 \cup \cdots \cup V_k \rangle \cong K_k$. Let $H = \left\langle V_{k+1} \cup \cdots \cup V_{\frac{n+k}{2}} \right\rangle$.

Case (i). H is disconnected.

If $k = 1$, then $\gamma_{nc}(G) = 2$ and hence $d_{nc}(G) \leq \frac{n}{2}$ which is a contradiction. Hence $k \geq 2$. Since $|V_j| = 2$ for all j with $k+1 \leq j \leq \frac{n+k}{2}$, it follows that H has exactly two components. Let H_1 and H_2 be the components of H. Then each V_j contains one vertex from H_1 and one vertex from H_2 and since V_j is a ncd-set of G, it follows that H_1 and H_2 are complete graphs and $|V(H_1)| = |V(H_2)| = \frac{n-k}{2}$. Hence H is isomorphic to $2K_{\frac{n-k}{2}}$.

Case (ii). H is connected.

Let $X_i = V_{k+i}, 1 \leq i \leq r = \frac{n-k}{2}$. Then $V(H) = X_1 \cup X_2 \cup \cdots \cup X_r$ and $X_i \cap X_j = \emptyset$ when $i \neq j$. Now, since each X_i is a dominating set of G, it follows that the subgraph induced by the edges of H with one end in X_i and the other end in X_j has a perfect matching.

Conversely, suppose G is of the form (1) or (2) given in the theorem. Let $u_1, u_2, \ldots, u_k$ be the vertices of G with $deg\ u_i = n-1$, $1 \leq i \leq k$.

Case (i). $G = K_k + H$ where $k \geq 2$ and H is isomorphic to $2K_{\frac{n-k}{2}}$.

Let H_1 and H_2 be the two components of H with $V(H_1) = \{x_i : k+1 \leq i \leq \frac{n+k}{2}\}$ and $V(H_2) = \{y_i : k+1 \leq i \leq \frac{n+k}{2}\}$.

$$\text{Let } V_i = \begin{cases} \{u_i\} & 1 \leq i \leq k \\ \{x_i, y_i\} & x_i \in H_1, y_i \in H_2, k+1 \leq i \leq \frac{n+k}{2}. \end{cases}$$

Then $\{V_1, V_2, \ldots, V_{\frac{n+k}{2}}\}$ is a nc-domatic partition of G. Hence $d_{nc}(G) \geq \frac{n+k}{2}$, so that $d_{nc}(G) = \frac{1}{2}(n+k)$.

Case (ii). G is of the form given in (2).

Then each X_i is a ncd-set of G and $\{\{u_1\}, \{u_2\}, \ldots, \{u_k\}, X_1, X_2, \ldots, X_r\}$ is a nc-domatic partition of G. Thus $d_{nc}(G) \geq k + r = \frac{n+k}{2}$ and hence $d_{nc}(G) = \frac{n+k}{2}$. □

In the following theorem we obtain bounds on the size of a graph with $\Delta = n-1$ and $d_{nc} = \frac{1}{2}(n+k)$, where k is the number of vertices of degree $n-1$ and characterize the class of graphs which attain the bounds.

Theorem 2.12. *Let G be a (n,m)-graph with $\Delta = n-1$ and $d_{nc} = \frac{1}{2}(n+k)$ where k is the number of vertices of degree $n-1$. Then $\frac{1}{4}[(n^2-k^2)+2n(k-1)] \leq m \leq \frac{1}{2}[k(n-1)+(n-k)(n-2)]$. Further, $m = \frac{1}{4}[(n^2-k^2)+2n(k-1)]$ if and only if $G = K_k + H$ with $V(H) = X_1 \cup X_2 \cup \cdots \cup X_r$ where $r = \frac{1}{2}(n-k)$, $|X_i| = 2$ and for all $i \neq j$, $\langle X_i \cup X_j \rangle$ is a perfect matching. Also $m = \frac{1}{2}[k(n-1)+(n-k)(n-2)]$ if and only if G is isomorphic to $K_n - M$, where M is a matching of cardinality $\frac{1}{2}(n-k)$.*

Proof. Let G be a graph with $\Delta = n-1$ and $d_{nc}(G) = \frac{1}{2}(n+k)$, where k is the number of vertices of degree $n-1$. Then $G = K_k + H$, where H is given in Theorem 2.11. Clearly $|V(H)| = n-k$. If H is isomorphic to $2K_{\frac{n-k}{2}}$, let H_1 and H_2 be the components of H with $V(H_1) = \{x_1, x_2, \ldots, x_{\frac{n-k}{2}}\}$ and $V(H_2) = \{y_1, y_2, \ldots, y_{\frac{n-k}{2}}\}$. Let $X_i = \{x_i, y_i\}, 1 \leq i \leq \frac{n-k}{2}$. If H is connected, let X_i be as given in (2) of Theorem 2.11. Then $V(H) = X_1 \cup X_2 \cup \cdots \cup X_r, r = \frac{n-k}{2}$, where each X_i is a ncd-set of G with $|X_i| = 2$. Since X_i dominates X_j for all $i \neq j$, the total number of edges with one end in X_i and the other end in $X_j, i \neq j$, is at least $2(r-1)$. Thus there are $r(r-1)$ such edges and since G contains k vertices with degree $n-1$, we have $m \geq \frac{k(k-1)}{2} + 2kr + r(r-1) = \frac{1}{4}[(n^2-k^2)+2n(k-1)]$. Now, $m = \frac{1}{4}[(n^2-k^2)+2n(k-1)]$ if and only if $\langle X_i \cup X_j \rangle$ contains exactly 2 edges and since X_i and X_j are dominating sets, it follows that $\langle X_i \cup X_j \rangle$ is a perfect matching.

Since $deg\ v \leq n-2$ for all $v \in V(H)$ and $deg\ v = n-1$ for all $v \in V(G) - V(H)$, we have $m \leq \frac{1}{2}[k(n-1)+(n-k)(n-2)]$. Also $m = \frac{1}{2}[k(n-1)+(n-k)(n-2)]$ if and only if $deg\ v = n-1$ for all $v \in V(G)-V(H)$ and $deg\ v = n-2$ for all $v \in V(H)$ and hence $G = K_n - M$ where M is a matching in K_n with $|M| = \frac{1}{2}(n-k)$. □

In the following theorem we give Nordhaus-Gaddum type result for d_{nc}. We need the following.

Observation 2.13. *Let G be a graph with $\Delta < n-1$. Then $d_{nc}(G) \leq \frac{n}{2}$. Further $d_{nc}(G) = \frac{n}{2}$ if and only if $V = X_1 \cup X_2 \cup \cdots \cup X_{\frac{n}{2}}$, where $|X_i| = 2$ for all i, $X_i \cap X_j = \emptyset$ if $i \neq j$, the subgraph induced by the edges of G with one end in X_i and the other end in X_j has a perfect matching and $\langle V - X_i \rangle$ is connected if X_i is independent.*

Theorem 2.14. *Let G be any graph such that both G and $\overline{G}$ are connected. Then $d_{nc}(G) + d_{nc}(\overline{G}) \leq n$. Further equality holds if and only if $V(G) = X_1 \cup X_2 \cup \cdots \cup X_{\frac{n}{2}}$, where $X_i \cap X_j = \emptyset$ and $\langle X_i \cup X_j \rangle$ is C_4 or P_4 or $2K_2$ for all $i \neq j$.*

Proof. Since both G and $\overline{G}$ are connected, it follows that $\Delta < n-1$. Hence $d_{nc}(G) \leq \frac{n}{2}$ and $d_{nc}(\overline{G}) \leq \frac{n}{2}$, so that $d_{nc}(G) + d_{nc}(\overline{G}) \leq n$.

Now, suppose $d_{nc}(G)+d_{nc}(\overline{G})=n$. Then $d_{nc}(G)=\frac{n}{2}$ and $d_{nc}(\overline{G})=\frac{n}{2}$. Since $d_{nc}(G)\le\delta(G)+1$, it follows that $\delta(G)\ge\frac{n}{2}-1$ and $\delta(\overline{G})\ge\frac{n}{2}-1$ and hence $deg\ v=\frac{n}{2}-1$ or $\frac{n}{2}$ for all $v\in V(G)$.

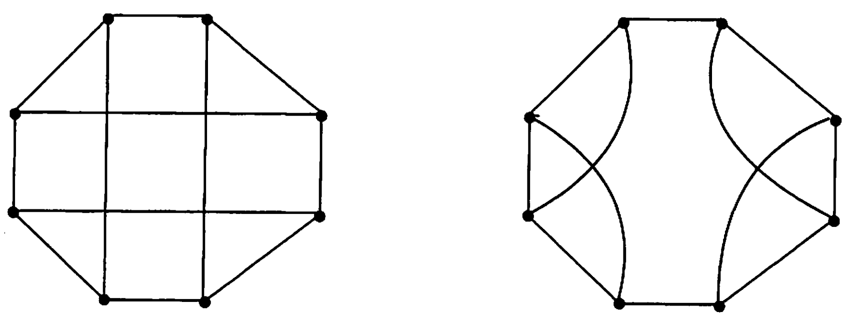
Now, let $V=X_1\cup X_2\cup\cdots\cup X_{\frac{n}{2}}$ be a nc-domatic partition of G. Then the subgraph induced by the edges of G with one end in X_i and the other end in X_j has a perfect matching. Further, if $\langle X_i\cup X_j\rangle$ has more than four edges, then at least one vertex v of $\langle X_i\cup X_j\rangle$ has degree at least 3. Since there are $\frac{n}{2}-2$ ncd-sets other than X_i and X_j, $deg\ v\ge\frac{n}{2}+1$ which is a contradiction. Thus $\langle X_i\cup X_j\rangle$ contains at most four edges and hence is isomorphic to C_4 or P_4 or $2K_2$. The converse is obvious. □

Remark 2.15. *Let G be any graph such that both G and $\overline{G}$ are connected and $d_{nc}(G)+d_{nc}(\overline{G})=n$. Then $\frac{n}{2}(\frac{n}{2}-1)\le m\le\frac{n^2}{4}$. Further $m=\frac{n}{2}(\frac{n}{2}-1)$ if and only if G is $(\frac{n}{2}-1)$-regular and $m=\frac{n^2}{4}$ if and only if G is $\frac{n}{2}$-regular.*

Theorem 2.16. *Let T be a tree such that $\overline{T}$ is connected. Then $d_{nc}(T)+d_{nc}(\overline{T})=n$ if and only if T is isomorphic to P_4.*

Proof. Let T be a tree such that $\overline{T}$ is connected and let $d_{nc}(T)+d_{nc}(\overline{T})=n$. Then $V(T)=X_1\cup X_2\cup\cdots\cup X_{\frac{n}{2}}$ where $|X_i|=2$, $X_i\cap X_j=\emptyset$, and $\langle X_i\cup X_j\rangle$ is either P_4 or $2K_2$. Hence if $\frac{n}{2}\ge3$, then T contains a cycle. Thus $\frac{n}{2}=2$, so that $n=4$ and T is isomorphic to P_4. The converse is obvious. □

Theorem 2.17. *Let G be any cubic graph such that both G and $\overline{G}$ are connected. Then $d_{nc}(G)+d_{nc}(\overline{G})=n$ if and only if G is isomorphic to one of the graphs G_1 or G_2 given in Figure 1.*



G_1 G_2

Figure 1

Proof. Let G be a cubic graph such that both G and $\overline{G}$ are connected and let $d_{nc}(G)+d_{nc}(\overline{G})=n$. Then $d_{nc}(G)=d_{nc}(\overline{G})=\frac{n}{2}$. Let $\{X_1,X_2,\ldots,X_{\frac{n}{2}}\}$

be a nc-domatic partition of G, so that $|X_i| = 2$, each X_i is a ncd-set of G and $\langle X_i \cup X_j \rangle$ is either P_4 or C_4 or $2K_2$.

Since $\frac{n}{2} = d_{nc} \leq \delta + 1 = 4$, it follows that $n \leq 8$. If $n = 4$, then $G = K_4$ and $\overline{G}$ is disconnected, which is a contradiction. If $n = 6$ then $\overline{G} = C_6$ or $2K_3$ so that either $d_{nc}(\overline{G}) = 2 \neq \frac{n}{2}$ or $\overline{G}$ is disconnected, which is again a contradiction. Hence $n = 8$. We now claim that for $i \neq j$, the induced subgraph $\langle X_i \cup X_j \rangle$ is $2K_2$.

Suppose $\langle X_1 \cup X_2 \rangle = C_4$ or P_4. Let v be a vertex in $X_1 \cup X_2$ having degree 2 in $\langle X_1 \cup X_2 \rangle$. Since X_3 and X_4 are both dominating sets in G, v is adjacent to a vertex in X_3 and to a vertex in X_4, so that $deg\ v \geq 4$, which is a contradiction. Thus $\langle X_i \cup X_j \rangle = 2K_2$.

Let $X_1 = \{x_1, x_2\}$, $X_2 = \{x_3, x_4\}$, $X_3 = \{x_5, x_6\}$ and $X_4 = \{x_7, x_8\}$. Without loss of generality we assume that $x_1x_3, x_2x_4, x_3x_5, x_4x_6, x_1x_7, x_2x_8 \in E(G)$. Then there are two cases.

i. x_7 is adjacent to x_3 and x_8 is adjacent to x_4

ii. x_7 is adjacent to x_4 and x_8 is adjacent to x_3.

Case (i). x_7 is adjacent to x_3 and x_8 is adjacent to x_4.

Then x_7 is adjacent to x_5 or x_6. If x_7 is adjacent to x_5, then x_8 is adjacent to x_6. Since G is connected, x_1 is adjacent to x_6 and x_2 is adjacent to x_5 and $G \cong G_2$. If x_7 is adjacent to x_6, then x_8 is adjacent to x_5. Then x_1 is adjacent to x_5 and x_2 is adjacent to x_6 or x_1 is adjacent to x_6 and x_2 is adjacent to x_5. Hence $G \cong G_2$.

Case (ii). x_7 is adjacent to x_4 and x_8 is adjacent to x_3.

Then x_7 is adjacent to either x_5 or x_6. If x_7 is adjacent to x_5, then x_8 is adjacent x_6. Also x_1 is adjacent to x_5 or x_6. If x_1 is adjacent to x_5 and x_2 is adjacent to x_6 then $G \cong G_2$. If x_1 is adjacent to x_6 and x_2 is adjacent to x_5 then $G \cong G_1$. Suppose x_7 is adjacent to x_6. Then x_8 is adjacent to x_5. Also x_1 is adjacent x_5 and x_2 is adjacent x_6 or x_1 is adjacent to x_6 and x_2 is adjacent to x_5. In both cases we have $G \cong G_2$. □

Theorem 2.18. *For any connected graph G, $\gamma_{nc}(G) + d_{nc}(G) \leq n + 1$ and equality holds if and only if $G = K_n$.*

Proof. **Case (i).** $\Delta < n - 1$.

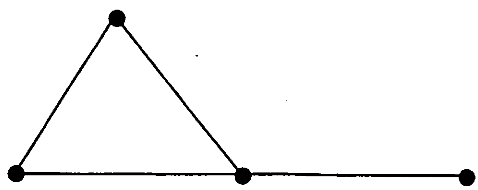
Since $\gamma_{nc} \leq n - \Delta$ and $d_{nc} \leq \delta + 1$, we have $\gamma_{nc} + d_{nc} \leq n - \Delta + \delta + 1 \leq n + 1$.

Case (ii). $\Delta = n - 1$.

Then $\gamma_{nc} = 1$ or 2. If $\gamma_{nc} = 1$, then $d_{nc} \leq n$ and hence $\gamma_{nc} + d_{nc} \leq n + 1$. If $\gamma_{nc} = 2$, then $d_{nc} \leq \frac{n}{2}$ and hence $\gamma_{nc} + d_{nc} \leq n + 1$.

Now, let G be any graph with $\gamma_{nc} + d_{nc} = n + 1$. We claim that $\gamma_{nc} = 1$. Suppose $\gamma_{nc} \geq 2$, then $d_{nc} \leq \frac{n}{2}$. Also $\gamma_{nc} \leq \lceil \frac{n}{2} \rceil$ and hence $\gamma_{nc} + d_{nc} \leq \frac{n}{2} + \lceil \frac{n}{2} \rceil \leq n + \frac{1}{2}$, which is a contradiction. Thus $\gamma_{nc} = 1$. Hence $d_{nc} = n$ and G is isomorphic to K_n. □

Theorem 2.19. *For any connected graph G, $\gamma_{nc}(G)+d_{nc}(G)=n$ if and only if G is isomorphic to K_n-e or P_4 or C_4 or the graph G_1 given in Figure 2.*



G_1

Figure 2

Proof. It can be easily verified that for all the graphs given in the theorem, $\gamma_{nc}+d_{nc}=n$.

Now, let G be a connected graph with $\gamma_{nc}+d_{nc}=n$. We claim that $\gamma_{nc}\leq 2$. Suppose $\gamma_{nc}=k\geq 3$. Then $d_{nc}=n-k$. Also $d_{nc}\leq \frac{n}{k}$, so that $n-k\leq \frac{n}{k}$. Hence $n\leq k+1$. However, $k=\gamma_{nc}\leq \lceil \frac{n}{2}\rceil \leq \lceil \frac{k+1}{2}\rceil$, so that $k\leq 2$, which is a contradiction. Hence $\gamma_{nc}\leq 2$.

Case (i). $\gamma_{nc}=1$.

In this case $d_{nc}=n-1$. Let $\{V_1,V_2,\ldots,V_{n-1}\}$ be a nc-domatic partition of G, where $|V_i|=1$ if $1\leq i\leq n-2$ and $|V_{n-1}|=2$. Hence G contains $n-2$ vertices with degree $n-1$ and G is isomorphic to K_n-e.

Case (ii). $\gamma_{nc}=2$.

In this case $d_{nc}=n-2$. Also $d_{nc}\leq \frac{n}{2}$, and hence $n=3$ or 4. If $n=3$, G is isomorphic to P_3 and if $n=4$, G is isomorphic to P_4 or C_4 or G_1. □

In the following theorem we use the proof technique given in [8] to improve the bounds for the sum $\gamma_{nc}(G)+d_{nc}(G)$ when G is a graph with $\gamma_{nc}(G)\geq 2$ and $d_{nc}(G)\geq 2$.

Theorem 2.20. *Let G be a graph with $\gamma_{nc}(G)\geq 2$ and $d_{nc}(G)\geq 2$. Then $\gamma_{nc}(G)+d_{nc}(G)\leq \lfloor \frac{n}{2}\rfloor+2$. Further, $\gamma_{nc}(G)+d_{nc}(G)=\lfloor \frac{n}{2}\rfloor+2$ if and only if $\{\gamma_{nc}(G),d_{nc}(G)\}=\{\lfloor \frac{n}{2}\rfloor,2\}$ or $n=9$ with $\gamma_{nc}(G)=d_{nc}(G)=3$.*

Proof. Since $\gamma_{nc}(G)\geq 2$, we have $d_{nc}(G)\leq \lfloor \frac{n}{2}\rfloor$. If either $d_{nc}(G)=2$ or $\gamma_{nc}(G)=2$, then since $\gamma_{nc}d_{nc}\leq n$, we have $d_{nc}\leq \left\lfloor \frac{n}{\gamma_{nc}}\right\rfloor \leq \lfloor \frac{n}{2}\rfloor$ or $\gamma_{nc}\leq \lfloor \frac{n}{2}\rfloor$. Hence the inequality holds.

If either $d_{nc}(G)=3$ or $\gamma_{nc}(G)=3$, then $d_{nc}(G)\leq \lfloor \frac{n}{3}\rfloor$ or $\gamma_{nc}(G)\leq \lfloor \frac{n}{3}\rfloor$. Since $\gamma_{nc}(G)\geq 3$ and $\gamma_{nc}(G)\leq \lceil \frac{n}{2}\rceil$, we have $n\geq 5$. Then $\gamma_{nc}(G)+d_{nc}(G)\leq \lfloor \frac{n}{3}\rfloor+3\leq \lfloor \frac{n}{2}\rfloor+2$.

Suppose $d_{nc}(G)\geq 4$ and $\gamma_{nc}(G)\geq 4$. Then we have $d_{nc}(G)\leq \lfloor \frac{n}{4}\rfloor$ and $\gamma_{nc}(G)\leq \lfloor \frac{n}{4}\rfloor$ which gives $\gamma_{nc}(G)+d_{nc}(G)\leq 2\lfloor \frac{n}{4}\rfloor<\lfloor \frac{n}{2}\rfloor+2$. Thus in all the cases $\gamma_{nc}(G)+d_{nc}(G)\leq \lfloor \frac{n}{2}\rfloor+2$.

Now, let G be a connected graph with $\gamma_{nc}(G) \geq 2$, $d_{nc}(G) \geq 2$ and $\gamma_{nc}(G) + d_{nc}(G) = \lfloor \frac{n}{2} \rfloor + 2$.

Suppose $\gamma_{nc}(G) \geq 4$ and $d_{nc}(G) \geq 4$. Then we have $d_{nc}(G) \leq \lfloor \frac{n}{4} \rfloor$ and $\gamma_{nc}(G) \leq \lfloor \frac{n}{4} \rfloor$ which gives $\gamma_{nc}(G)+d_{nc}(G) \leq 2\lfloor \frac{n}{4} \rfloor < \lfloor \frac{n}{2} \rfloor + 2$. Hence either $\gamma_{nc}(G) \leq 3$ or $d_{nc}(G) \leq 3$. Let $\gamma_{nc}(G) = 3$ or $d_{nc}(G) = 3$. This implies $n \geq 5$ and $\gamma_{nc}(G) + d_{nc}(G) \leq \lfloor \frac{n}{3} \rfloor + 3$. Then $\lfloor \frac{n}{2} \rfloor + 2 \leq \lfloor \frac{n}{3} \rfloor + 3$, which gives $n \leq 9, n \neq 8$. Therefore, $5 \leq n \leq 9$ and $n \neq 8$.

If $n = 9$, then $\gamma_{nc}(G) + d_{nc}(G) = 6$ and hence $\gamma_{nc}(G) = d_{nc}(G) = 3$.

If $n = 6$ or 7 and $\gamma_{nc}(G) = 3$ (or $d_{nc}(G) = 3$), then $d_{nc}(G) = 2$ (or $\gamma_{nc}(G) = 2$). Hence $\{\gamma_{nc}, d_{nc}\} = \{\lfloor \frac{n}{2} \rfloor, 2\}$.

If $n = 5$ and either $\gamma_{nc}(G) = 3$ or $d_{nc}(G) = 3$, then the equality does not hold. Hence if $n = 5$ then either $\gamma_{nc}(G) = 2$ or $d_{nc}(G) = 2$. This proves the result. □

Definition 2.21. *A graph G is called nc-domatically full if $d_{nc}(G) = \delta(G) + 1$.*

Definition 2.22. *A graph G is called nc-domatically critical if $d_{nc}(G - e) < d_{nc}(G)$ for every noncut edge $e \in E(G)$.*

Theorem 2.23. *Every regular nc-domatically full graph is nc-domatically critical.*

Proof. Let G be a regular nc-domatically full graph. Then $d_{nc}(G) = \delta(G) + 1$. Thus $\delta(G) = d_{nc}(G) - 1$. Hence $deg\ v = d_{nc}(G) - 1$ for all $v \in V(G)$. Let $e \in E(G)$ be a noncut edge of G, so that $G' = G - e$ is connected. Then $\delta(G') = \delta(G) - 1$. Now, $d_{nc}(G') \leq \delta(G') + 1 = \delta(G) - 1 + 1 = \delta(G) < d_{nc}(G)$. Hence G is nc-domatically critical. □

Theorem 2.24. *Let G be a regular nc-domatically full graph of order n. Then $d_{nc}(G)$ divides n.*

Proof. Let G be a regular nc-domatically full graph of order n so that $d_{nc}(G) = \delta + 1$. Let $\{V_1, V_2, \ldots, V_{d_{nc}}\}$ be a nc-domatic partition of $V(G)$. Let $v \in V_i$. Since v is a adjacent to at least one vertex in each $V_j, j \neq i$, we have V_i is an independent set and any vertex in V_i is adjacent to exactly one vertex in $V_j, i \neq j$. Hence $|V_i| = |V_j| = \frac{n}{d_{nc}}$ and hence d_{nc} divides n. □

The connection between the nc-domatic number with nc-domination number and other types of domatic numbers suggest the following natural problems for further investigation.

Problem 2.25. *Characterize graphs G for which $\gamma_{nc}(G)d_{nc}(G) = n$.*

Problem 2.26. *Characterize graphs G for which $d_{nc}(G) = d(G)$.*

Problem 2.27. *Characterize graphs G for which $d_{nc}(G) = \frac{d(G)}{2}$.*

Problem 2.28. *Characterize nc-domatically full graphs.*

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