# Chromatic Transversal Domination in Graphs

L. BENEDICT MICHAELRAJ, S.K. AYYASWAMY
Department of Mathematics
St. Joseph's College, Trichy - 620 002
INDIA
e-mail: *benedict.mraj@gmail.com*, *sjcayya@yahoo.co.in*
and
S. ARUMUGAM
Core Group Research Facility (CGRF)
National Centre for Advanced Research in Discrete Mathematics ($n$-CARDMATH)
Kalasalingam University
Anand Nagar, Krishnankoil-626190, INDIA.
e-mail: *s_arumugam_akce@yahoo.com*

**Abstract**

Let $G = (V, E)$ be a graph with chromatic number $k$. A dominating set $D$ of $G$ is called a chromatic-transversal dominating set (ctd-set) if $D$ intersects every color class of any $k$-coloring of $G$. The minimum cardinality of a ctd-set of $G$ is called the *chromatic transversal domination number* of $G$ and is denoted by $\gamma_{ct}(G)$. In this paper we initiate a study of this parameter.

**Keywords :** Domination, Coloring, Chromatic Transversal domination.
**2000 Mathematics Subject Classification:** 05C69

# 1 Introduction

By a graph $G = (V, E)$ we mean a finite, undirected graph with neither loops nor multiple edges. The order and size of $G$ are denoted by $n$ and $m$ respectively. For graph theoretic terminology we refer to Chartrand and Lesniak [2].

One of the fastest growing areas within graph theory is the study of domination and related subset problems such as independence, covering and matching. A comprehensive treatment of fundamentals of domination is given in the book by Haynes et al. [3]. Surveys of several advanced topics in domination can be seen in the book edited by Haynes et al. [4].

Another important area of research within graph theory is graph colourings which deals with the fundamental problem of partitioning a set of objects into classes according to certain rules.

A set $S \subseteq V$ is called a dominating set of $G$ if every vertex in $V - S$ is adjacent to a vertex in $S$. The minimum cardinality of a dominating set in $G$ is called the *domination number* of $G$ and is denoted by $\gamma(G)$. Several types of domination parameters have been studied by different authors by imposing conditions on $S$ and more than seventy five models of domination are listed in the appendix of Haynes et al. [4].

Sampathkumar and Walikar [6] introduced concept of connected domination. A dominating set $S$ of a connected graph $G$ is called a *connected dominating set* if the induced subgraph $\langle S \rangle$ is connected and the connected domination number $\gamma_c(G)$ is the minimum cardinality of a connected dominating set of $G$.

Sampathkumar [5] introduced the concept of global domination in graphs. A subset $S$ of $V$ is called a global dominating set of $G$ if $S$ is a dominating set of $G$ as well as its complement $\overline{G}$. The global domination number $\gamma_g(G)$ is the minimum cardinality of a global dominating set of $G$.

In this paper we introduce a graph theoretic parameter which combines the concept of domination and vertex colouring. A vertex colouring of a graph $G$ is a partition of $V$ into independent sets and the minimum order of such a partition is called the chromatic number of $G$ and is denoted by $\chi(G)$. If $\mathcal{C} = \{V_1, V_2, \ldots, V_k\}$ is a $k$-colouring of $G$ then a subset $D$ of $V$ is called a *transversal* of $\mathcal{C}$ if $D \cap V_i \neq \emptyset$ for all $i = 1, 2, \ldots, k$.

We need the following definitions and theorems.

**Definition 1.1.** *The corona $G_1 \circ G_2$ of two graphs $G_1$ and $G_2$ is defined to be the graph $G$ obtained by taking one copy of $G_1$ and $|V(G_1)|$ copies of $G_2$, and then joining the $i^{th}$ vertex of $G_1$ to every vertex in the $i^{th}$ copy of $G_2$.*

**Definition 1.2.** *A subdivision of an edge $uv$ is obtained by removing edge $uv$, adding a new vertex $w$, and adding edges $uw$ and $vw$. A wounded spider is the graph formed by subdividing at most $t-1$ of the edges of a star $K_{1,t}$.*

**Definition 1.3.** *Let $S \subseteq V$ and let $u \in S$. A vertex $v$ is called a private neighbor of $u$ with respect to $S$ if $N[v] \cap S = \{u\}$. The set of all private neighbors of $u$ with respect to $S$ is denoted by $pn[u, S]$.*

**Theorem 1.4.** ([3], Page 41) *If a graph $G$ has no isolated vertices, then $\gamma(G) \leq \frac{n}{2}$.*

**Theorem 1.5.** ([3], Page 42) *For a graph $G$ with even order $n$ and having no isolated vertices, $\gamma(G) = \frac{n}{2}$ if and only if the components of $G$ are the cycle $C_4$ or the corona $H \circ K_1$ for any connected graph $H$.*

**Theorem 1.6.** ( [3], Page 163) *Let $G$ be a connected graph of order $n$ and maximum degree $\Delta$. Then $\gamma_c \leq n - \Delta$.*

**Theorem 1.7.** [3] *If $G$ is a connected graph other than a complete graph or an odd cycle, then $\chi(G) \leq \Delta(G)$.*

**Theorem 1.8.** ( [3], Page 210) *If $G$ is a triangle free graph, then $\gamma \leq \gamma_g \leq \gamma + 1$.*

**Theorem 1.9.** [1] *Let $G$ be a connected bipartite graph with bipartition $(X, Y)$ and $|X| \leq |Y|$. Then $\gamma_g = \gamma + 1$ if and only if either $G$ is isomorphic to $K_2$ or every vertex in $X$ is adjacent to at least two pendant vertices and there exists a vertex in $Y$ which is adjacent to all vertices in $X$.*

# 2 Main Results

**Definition 2.1.** *Let $G = (V, E)$ be a graph. A dominating set $D$ of $G$ is called a chromatic transversal dominating set (ctd-set) if $D$ is a transversal of every chromatic partition of $G$. A ctd-set $D$ is called a minimal ctd-set if no proper subset of $D$ is a ctd-set.*

We observe that $V$ is a ctd-set of any graph. Further if $D$ is a ctd-set and $D \subseteq D_1$, then $D_1$ is also a ctd-set. Hence $D$ is a minimal ctd-set if and only if $D - \{v\}$ is not a ctd-set for all $v \in D$. The following theorem gives a necessary and sufficient condition for a ctd-set to be minimal.

**Theorem 2.2.** *A ctd-set $D$ is minimal if and only if for every vertex $u \in D$ one of the following holds.*

(i) $pn[u, S] \neq \emptyset$.

(ii) *There exists a $\chi$-partition $C = \{V_1, V_2, \ldots, V_\chi\}$ such that $D \cap V_i = \{u\}$ for some $i$.*

*Proof.* Suppose $D$ is a minimal ctd-set of $G$ and let $u \in D$. Then $D - \{u\}$ is not a ctd-set of $G$. Hence either $D - \{u\}$ is not a dominating set or $D - \{u\}$ is not a transversal of some $\chi$-partition of $G$. If $D - \{u\}$ is not a dominating set of $G$, then $pn[u, S] \neq \emptyset$. If there exists a $\chi$-partition $C = \{V_1, V_2, \ldots, V_k\}$ such that $D - \{u\}$ is not a transversal of $C$ then $(D - \{u\}) \cap V_i = \emptyset$ for some $i$. Further $D \cap V_i \neq \emptyset$ and hence it follows that $D \cap V_i = \{u\}$.

Conversely if (i) or (ii) holds, then $D - \{u\}$ is not a ctd-set of $G$ for every $u \in D$ and hence $D$ is a minimal ctd-set of $G$. □

**Definition 2.3.** The minimum (maximum) cardinality of a minimal ctd-set of $G$ is called the *chromatic-transversal domination number* (upper chromatic transversal domination number) of $G$ and is denoted by $\gamma_{ct}(G)$ $(\Gamma_{ct}(G))$.

**Remark 2.4.** *Let $G$ be a graph with $\chi(G) = k$. Let $\{V_1, V_2, \ldots, V_k\}$ be a $k$-colouring of $G$ and let $D$ be a $\gamma_{ct}$-set of $G$. Since $D \cap V_i \neq \emptyset$ for $1 \leq i \leq k$ and each $V_i$ forms a clique in $\overline{G}$, it follows that $D$ is a dominating set of $\overline{G}$. Hence $D$ is a global dominating set of $G$ so that $\gamma_g \leq \gamma_{ct}$.*

**Example 2.5.**

(i) *Obviously $\gamma_{ct}(K_n) = \gamma_{ct}(K_n^c) = n$ and $\gamma_{ct}(K_{m,n}) = 2$.*

(ii) *Let $G$ be a connected bipartite graph with bipartition $(X, Y)$. If there exists a $\gamma$-set $D$ of $G$ such that $D \cap X \neq \emptyset$ and $D \cap Y \neq \emptyset$, then $\gamma_{ct}(G) = \gamma$. Otherwise $\gamma_{ct}(G) = \gamma + 1$. In particular $\gamma_{ct}(C_n) = \gamma(C_n) = \lceil \frac{n}{3} \rceil$ if $n$ is even.*

(iii) *If for every $v \in V$, $\{v\}$ is a colour class of a $\chi$-partition of $G$, then $\gamma_{ct}(G) = n$. In particular $\gamma_{ct}(C_n) = n$ if $n$ is odd and $\gamma_{ct}(W_n) = n$ if $n$ is even, where $W_n$ is the wheel on $n$ vertices.*

(iv) *$\gamma_{ct}(P) = 5$ where $P$ is the Petersen graph given in Figure 1.*

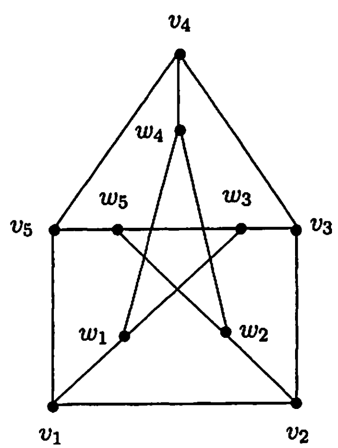


Figure 1

Clearly $\gamma(P) = \chi(P) = 3$. Further if $S \subseteq V$ is independent and $|S| = 3$, then there exists a chromatic partition $\{V_1, V_2, V_3\}$ of $P$ such that $S \subseteq V_1$.

Also if $D$ is any dominating set of $P$ with $|D| = 3$ or 4, then there exists an independent set $S$ with $|S| = 3$ such that $S \subseteq D$. Hence it follows that $\gamma_{ct}(P) \geq 5$. Also $\{v_1, v_2, v_3, v_4, v_5\}$ is a ctd-set of $P$ and hence $\gamma_{ct}(P) = 5$.

In the following theorem we obtain a characterization of graphs for which $\gamma_{ct}(G) = n$.

**Theorem 2.6.** *For a connected graph $G$ of order $n$, $\gamma_{ct}(G) = n$ if and only if $G$ is $\chi$-critical.*

*Proof.* If $G$ is $\chi$-critical, then for every $v \in V(G)$, there exists a $\chi$-colouring of $G$ in which $\{v\}$ is a colour class and hence it follows that $\gamma_{ct}(G) = n$. Conversely, suppose $\gamma_{ct}(G) = n$. If $G$ is not $\chi$-critical, then $G$ contains a proper subgraph $H$ of $G$ such that $\chi(H) = \chi(G) = k$ and $H$ is $\chi$-critical. Now, let $v \in V(G) - V(H)$. Then $V(G) - \{v\}$ is a ctd-set of $G$, which is a contradiction. Hence $G$ is $\chi$-critical. □

**Corollary 2.7.** *Let $G$ be a disconnected graph of order $n$. Then $\gamma_{ct}(G) = n$ if and only if $G$ has at most one nontrivial component $G_1$, which is $\chi$-critical.*

*Proof.* Suppose $\gamma_{ct}(G) = n$. If $G$ has more than one non-trivial component, let $G_1$ be a component of $G$ with $\chi(G_1) = \chi(G)$. Let $D_2$ be a minimum dominating set of another non-trivial component $G_2$. Then $D_2 \cup [V(G) - V(G_2)]$ is a ctd-set of $G$, which is a contradiction. Hence $G$ has at most one non-trivial component, which is critical. The converse is obvious. □

**Lemma 2.8.** *For any graph $G$, $\gamma_{ct}(G) \geq \gamma(G)$. Further given two positive integers $a$ and $b$ with $a \leq b$, there exists a connected graph $G$ with $\gamma(G) = a$ and $\gamma_{ct}(G) = b$.*

*Proof.* The inequality is trivial. Now, let $a$ and $b$ be two positive integers with $a \leq b$.

**Case i.** $a = b$.

For the graphs $G_1, G_2$ and $G_3$ where $G_1 = K_1$, $G_2$ is the wounded spider obtained from $K_{1,3}$ by subdividing exactly one edge and $G_3$ is the wounded spider obtained from $K_{1,3}$ by subdividing exactly two edges, we have $\gamma(G_i) = \gamma_{ct}(G_i) = i, 1 \leq i \leq 3$. For $a \geq 4$, we have $\gamma_{ct}(P_{3a}) = \gamma(P_{3a}) = a$.

**Case ii.** $a < b$.

Let $G$ be the graph obtained from the path $P_{3a}$ by identifying a vertex of the complete graph $K_{b-a+1}$ with a support of $P_{3a}$. Clearly $\gamma(G) = a$ and $\gamma_{ct}(G) = b$. □

**Theorem 2.9.** *For a non-trivial connected graph $G$, $\gamma_{ct}(G) = 2$ if and only if $G$ is a bipartite graph with bipartition $(X, Y)$ and there exists a set $S = \{x, y\}$ with $|pn(x, S)| = |Y| - 1$ and $|pn(y, S)| = |X| - 1$.*

*Proof.* Let $G$ be a non-trivial connected graph with $\gamma_{ct}(G) = 2$. Then $\chi(G) = 2$, so that $G$ is bipartite graph. Let $(X, Y)$ be a bipartition of $G$. Then for any $\gamma_{ct}$-set $S = \{x, y\}$ we have $x \in X, y \in Y$, $pn(x, S) = Y - \{y\}$ and $pn(y, S) = X - \{x\}$. The converse is obvious. □

In the following theorem we characterize bipartite graphs with $\gamma_{ct} = \gamma + 1$.

**Theorem 2.10.** *Let $G$ be a connected bipartite graph with bipartition $(X, Y)$ where $|X| \leq |Y|$ and $n \geq 3$. Then $\gamma_{ct}(G) = \gamma(G) + 1$ if and only if every vertex in $X$ has at least two pendant neighbours.*

*Proof.* Suppose $\gamma_{ct} = \gamma + 1$. Then no $\gamma$-set of $G$ intersects both $X$ and $Y$ and hence it follows that $X$ is the unique $\gamma$-set of $G$. Now, let $u \in X$ and $v \in N(u)$. If $u$ is not a support vertex or $u$ is a support with exactly one pendant neighbour, then $D = (X - \{u\}) \cup \{v\}$ is a $\gamma$-set, so that $\gamma_{ct} = \gamma$, which is a contradiction. Thus every vertex of $X$ has at least two pendant neighbours. The converse is obvious. □

**Corollary 2.11.**

(i) $\gamma_{ct}(P_n) = \gamma(P_n), n \geq 4$.

(ii) $\gamma_{ct}(P_k \times P_l) = \gamma(P_k \times P_l)$

(iii) $\gamma_{ct}(Q_n) = \gamma(Q_n)$ *where $Q_n$ is the $n$-dimensional hypercube.*

**Remark 2.12.** *Let $G$ be a disconnected graph with $k$ components $G_1, G_2, \ldots, G_k$ and let $\gamma_{ct}(G_k) = \min_i\{\gamma_{ct}(G_i) : \chi(G_i) = \chi(G)\}$. Then $\gamma_{ct}(G) = \sum_{i=1}^{k-1} \gamma(G_i) + \gamma_{ct}(G_k)$.*

**Definition 2.13.** *Let $m, n$ and $r$ be positive integers. Then the graph obtained from $K_{1,m}$ and $K_{1,n}$ by joining the centres of $K_{1,m}$ and $K_{1,n}$ by a path of length $r$ is called a double star and is denoted by $D_{m,n,r}$.*

**Theorem 2.14.** *Let $T$ be a tree. Then $\gamma_{ct}(T) = 2$ if and only if $T$ is isomorphic to one of the graphs $K_{1,n}, D_{m,n,1}, D_{m,1,2}$ or $D_{m,n,3}$ where $m, n \geq 1$.*

*Proof.* Let $T$ be a tree with $\gamma_{ct}(T) = 2$. If $T$ is not star, then $T$ has exactly two supports $u$ and $v$ and $d(u, v) \leq 3$. Further, if $d(u, v) = 2$, then in any 2-colouring of $T$, $u$ and $v$ belong to the same colour class and hence either $deg\ u$ or $deg\ v$ is 2. Hence $T$ is isomorphic to one of the graphs $D_{m,n,1}, D_{m,1,2}$ or $D_{m,n,3}$ where $m, n \geq 1$. The converse is obvious. □

**Theorem 2.15.** *For a tree $T$ of even order $n$ with $n \geq 4$, $\gamma_{ct} \leq \frac{n}{2}$ and equality holds if and only if $T$ is $K_{1,3}$ or $H \circ K_1$ where $H$ is any tree.*

*Proof.* Let $X, Y$ be a bipartition of $T$ with $|X| \leq |Y|$. Then $\gamma_{ct}(T) = \gamma$ or $\gamma + 1$.

If $\gamma_{ct}(T) = \gamma$, then it follows from Theorem 1.4 that $\gamma_{ct}(T) \leq \frac{n}{2}$. Suppose $\gamma_{ct}(T) = \gamma + 1$. Then it follows from Theorem 2.10 that $X$ is the only $\gamma$-set of $T$ and $|X| \leq \frac{n}{2}$ and hence the inequality follows.

Now, suppose $T$ is a tree of even order with $n \geq 4$ and $\gamma_{ct} = \frac{n}{2}$. Then $\gamma = \frac{n}{2}$ or $\frac{n}{2} - 1$. If $\gamma = \frac{n}{2}$ it follows from Theorem 1.5 that $T \simeq H \circ K_1$ where $H$ is any tree.

If $\gamma = \frac{n}{2} - 1$, then it follows from Theorem 2.10 that $|X| = \gamma = \frac{n}{2} - 1$ and $|Y| \geq 2|X| = 2(\frac{n}{2} - 1) = n - 2$. If $|Y| = n - 2$, then $n = 6$ and $\gamma = |X| = 2$ and in this case $\gamma_{ct} = 2$, which is a contradiction. Hence $|Y| = n - 1$. In this case $n = 4$ and $\gamma = 1$, so that $T = K_{1,3}$. □

**Theorem 2.16.** *Let $T$ be a tree. Then $\gamma_{ct}(T) \leq n - \Delta$ and equality holds if and only if $T$ is a wounded spider which is not a star.*

*Proof.* Let $T$ be a wounded spider which is not a star. Let $v$ be the vertex of maximum degree, $N(v) = \{v_1, v_2, \ldots, v_k\}$ and $V - N[v] = \{w_1, w_2, \ldots, w_l\}$ with $w_i$ adjacent to $v_i$. Clearly $n = k + l + 1$, $\Delta = k$ and $\{v, v_1, v_2, \ldots, v_l\}$ a minimum ctd-set of $T$ and hence $\gamma_{ct} = n - \Delta$.

Conversely, let $T$ be a tree with $\gamma_{ct} = n - \Delta$. Let $v$ be a vertex of maximum degree $\Delta$. Let $(X, Y)$ be the 2-colouring of $T$ with $v \in X$, so that $N(v) \subseteq Y$. If there exists $w \in Y - N(v)$, then $D = I \cup \{v\}$ where $I$ is a maximal independent set in $\langle V - N[v] \rangle$ containing $w$, is a ctd-set of $T$ with $|D| < n - \Delta$, which is a contradiction. Hence $Y = N(v)$ and $T$ is a wounded spider. □

The following are some interesting problems for further investigation.

**Problem 2.17.**

(i) *Characterize the class of graphs $G$ for which $\gamma_{ct}(G) = \gamma_g(G)$.*

(ii) *Characterize the class of graphs $G$ for which $\gamma_{ct}(G) = \chi(G)$.*

(iii) *Characterize the class of graphs $G$ for which $\gamma_{ct}(G) = \gamma(G)$.*

# References


[1] S. Arumugam and R. Kala, A Note on Global Domination in Graphs, *Ars Combin.*, (To appear)

[2] G. Chartrand and L. Lesniak, *Graphs and Digraphs*, Chapman and Hall, CRC, 2005.

[3] T.W. Haynes, S.T. Hedetniemi and P.J. Slater, *Fundamentals of Domination in Graphs*, Marcel Dekker Inc., 1998.

[4] T.W. Haynes, S.T. Hedetniemi and P.J. Slater, *Domination in Graphs - Advanced Topics*, Marcel Dekker Inc., 1998.

[5] E. Sampathkumar, The Global Domination Number of a graph, *Jour. Math. Phy. Sci.*, **23**(5) (1989), 377-385.

[6] E. Sampathkumar and H. B. Walikar, The connected domination number of a graph, *J. Math. Phys. Sci.*, **13**(1979), 607-613.