Two Upper Bounds of Prime Cordial Graphs

M. A. Seoud, and M. A. Salim*

Department of Mathematics, Faculty of Science, Ain Shams University

Abbassia, Cairo, Egypt

Abstract

We give an upper bound of the number of the edges of a graph with n vertices to be prime cordial graph, and we improve this upper bound to fit bipartite graphs. Also, we determine all prime cordial graphs of order ≤ 6.

Introduction

Sundaram, Ponraj, and Somasundaram [3] have introduced the notion of prime cordial labelings.

A prime cordial labeling of a graph $G = (V(G), E(G))$ with vertex set $V(G)$ is a bijection f from V to $\{1, 2, \ldots, |V(G)|\}$, such that if each edge uv is assigned the label 1 if $gcd(f(u), f(v)) = 1$ and 0 if $gcd(f(u), f(v)) > 1$, then the number of edges labeled with 0 and the number of edges labeled with 1 differ by at most 1.

In [3] Sundaram, Ponraj and Somasundram prove the following graphs are prime cordial: C_n if and only if $n \geq 6$; P_n if and only if $n = 3$ or 5; bistars; dragons; crowns; triangular snakes if and only if the snake has at least three triangles; ladders; $K_{1,n}$ (n *odd*); the graph obtained by subdividing each edge of $K_{1,n}$ if and only if $n \geq 3$; $K_{1,n}$ if n is even and there exists a prime p such that $2p < n + 1 < 3p$; and $K_{3,n}$ if n is odd and if there exists a prime p such that $5p < n + 3 < 6p$. They also prove that if G is a prime cordial graph of even size, then the graph obtained by identifying the central vertex of $K_{1,n}$ with the vertex of G labeled with 2 is prime cordial, and if G is a prime cordial graph of odd size, then the graph obtained by identifying the central vertex of $K_{1,2n}$ with the vertex of G labeled with 2 is prime cordial.

They further prove that K_n is not prime cordial for $4 < n < 181$ and $K_{m,n}$ is not prime cordial for a number of special cases of m and n.

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* m.a.seoud@hotmail.com

Here we present an upper bound of the number of the edges of a graph with n vertices to be a prime cordial graph depending on Euler's function, so we can cover a large range of graphs to be not prime cordial graphs. It is shown directly that the graphs K_n , $2 < n < 500$ are not prime cordial graphs.

We improve another upper bound fitting bipartite graphs.

So we prove that $K_{i,i}$ is not prime cordial graph, $2 < 2i \leq 100$, $K_{i,i+1}$ is not prime cordial graph, $3 < 2i + 1 \leq 99$,..., similarly we continue until we reach to $K_{i,i+11}$, which is not a prime cordial graph, $21 < 2i + 11 \leq 99$.

1) General upper bound

In the following theorem we give an upper bound for the number of edges of a graph G to be a prime cordial graph.

Theorem 1.1: a necessary condition for a graph G of order n to be a prime cordial graph, is that its number of edges $|E(G)| \leq u_1$, where
$u_1 = n(n-1) - 2\Phi(n) + 1, \Phi(n) = \sum_{i=2}^{n} \phi(i)$,

where ϕ is Euler's function: $\phi: \mathbb{N} \to \mathbb{N}\,; \phi(t) = |\{ s \in \mathbb{N}; s < t, \gcd(s,t) = 1\}|$

Proof: we count the all edges could be labeled 1 (so we can count those which could be labeled 0 since the edge which is not labeled 1 is labeled 0)

To count the edges labeled 1 we are going to use Euler's function.

Depending on it, we define another function Φ as follows:

$$\Phi : \mathbb{N} \to \mathbb{N}\,; \Phi(n) = \sum_{i=2}^{n} \phi(i)$$

Now, when G is a graph of order n, then $\Phi(n)$ is exactly the number of all possible edges that could be labeled 1.

On the other hand, the number of possible edges could be labeled 0 is:
$$\frac{n(n-1)}{2} - \Phi(n).$$

By using mathematica e.g., we could make a table representing both of the previous numbers $\Phi(n)$ and $n(n-1)/2 - \Phi(n)$.

We realize that $\Phi(n) > n(n-1)/2 - \Phi(n)$ till the number of vertices is 10000, so a prime cordial graph could maximally contain the following number of edges:

$$n(n-1) - 2\Phi(n) + 1$$

Results 1.2:

1) In *Diagram 1* we present this upper bound of the number of edges until we reach graphs of order 500 (*we can extend it to more than 10000*). As a special case all complete graphs K_n are not prime cordial graphs for $2 < n \leq 500$.
2) In *Table 1* (where, $E_n^i = |E(P_n^i)|$) we present the number of edges of the graphs P_n^i (*the* i^{th} *power of the path* P_n*),* $i = 2,3,\dots,12$, to be compared with the upper bounds.
   Here we can deduce that P_n^i ,$i = 2,3,\dots,12$ are not prime cordial graphs for the numbers of vertices that correspond to the underlined bold numbers.

Algorithm 1.3: Not prime cordial test (NPCT)
Input: A graph G with n vertices, m edges
Output: An answer "Not a prime cordial graph" or "Probably a prime cordial graph". To question "Is G a prime cordial graph"
Compute u_1 (*Theorem 1.1*)
If $u_1 < m$ then
Return "Not a prime cordial graph"
Else
Return " Probably a prime cordial graph"
If algorithm NPCT declares "Not a prime cordial graph", then G is certainly not a prime cordial graph. On the other hand, if the algorithm NPCT declares "probably a prime cordial graph", then no proof is provided that G is indeed a prime cordial graph.
Theorem 1.4: *Algorithm 1.3* NPCT takes time $O(n^2 logn)$
Proof: Since computing of the $g.c.d$ of two integers $1 \leq a \leq b \leq n$ takes $O(logn)$ thus computing u_1 takes $O(n^2 logn)$. Therefore, NPCT takes time $O(n^2 logn)$.
Note: In [1] the authors state that the problem of **deciding whether a graph G admits a cordial labelling** is NP-complete.

Conjecture 1.5: All complete graphs K_n, $n > 2$ are not prime cordial graphs.

This conjecture is supported by the increased difference between the number of edges of K_n and the upper bound u_1, when n increases. (*see Diagram 1*).

2) An upper bound for bipartite Graphs.

In the following we introduce improved numbers of edges of bipartite graphs of order n:

Theorem2.1: a necessary condition for a bipartite graph of order n to be a prime cordial graph is $|E(G)| \leq u_2$, where $u_2 = 2\sum_{p_i \leq \lfloor \frac{n}{2} \rfloor} m(p_i) + 1$ where for a prime number p, m is a function which is defined as follows:

$m : \mathbb{N} \to \mathbb{N}, m(p) = \left\lfloor \frac{n}{2p} \right\rfloor \times \left\lceil \frac{n}{2p} \right\rceil, n = |V(G)|.$

Proof: We count all edges could be labeled 0 as follows:

Let the number of vertices be n. The number of the multiples of each prime number $p \leq \left\lfloor \frac{n}{2} \right\rfloor$, which lie between 1 and n, is $\left\lfloor \frac{n}{p} \right\rfloor$. The best way in which we can get maximum number of edges labeled 0 is obtained by distributing those labels equally on each of the partite sets, to get the number: $\left\lfloor \frac{n}{2p} \right\rfloor \times \left\lceil \frac{n}{2p} \right\rceil$,which is the maximum number of edges could be labeled 0 using the multiples of the prime $p \leq \left\lfloor \frac{n}{2} \right\rfloor$, which lie between 1 and n

So, let us define the following function:

$$m: \mathbb{N} \to \mathbb{N}$$

$$m(p) = \left\lfloor \frac{n}{2p} \right\rfloor \times \left\lceil \frac{n}{2p} \right\rceil, n = |V(G)|.$$

Summing the values of this function applied on all primes $\leq \left\lfloor \frac{n}{2} \right\rfloor$ we get the maximum number of edges could be labeled 0 in a bipartite graph of n vertices.

So, a prime cordial bipartite graph could contain at most $2\sum_{p_i \leq \lfloor \frac{n}{2} \rfloor} m(p_i) + 1$, edges, where $n = |V(G)|$

We calculate those numbers for bipartite graphs containing 100 vertices, (*we can extend it for 10000*), so we get the following results:

Results 2.2:

$K_{i,i}$ is not prime cordial graph, $2 < 2i \leq 100$,
$K_{i,i+1}$ is not prime cordial graph, $3 < 2i + 1 \leq 99$,
$K_{i,i+2}$ is not prime cordial graph, $4 < 2i + 2 \leq 100$,
$K_{i,i+3}$ is not prime cordial graph, $5 \leq 2i + 3 \leq 99$,
$K_{i,i+4}$ is not prime cordial graph, $6 < 2i + 4 \leq 100$,

$K_{i,i+5}$ is not prime cordial graph, $7 < 2i + 5 \leq 99$,
$K_{i,i+6}$ is not prime cordial graph, $12 < 2i + 6 \leq 100$,
$K_{i,i+7}$ is not prime cordial graph, $11 < 2i + 7 \leq 99$,
$K_{i,i+8}$ is not prime cordial graph, $18 < 2i + 8 \leq 100$,
$K_{i,i+9}$ is not prime cordial graph, $15 < 2i + 9 \leq 99$,
$K_{i,i+10}$ is not prime cordial graph, $24 < 2i + 10 \leq 100$,
$K_{i,i+11}$ is not prime cordial graph, $21 < 2i + 11 \leq 99$.

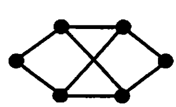
Conjecture 2.3: All these bipartite graphs are not prime cordial if the orders of the graphs are greater than 100.
This conjecture is supported by the increased difference between the number of edges of these graphs and the upper bound u_2, when the orders of theses graphs increase. (*see Diagrams 2 and 3*).

3) Prime cordial graphs of order ≤ 6.

Let $G(n, m)$ denote all graphs with n vertices and m edges.

Theorem 3.1: Among all graphs of order ≤ 6 [2], only the following graphs are not prime cordial:

1) $G(3, |E(G)| \geq 2), G(4 \text{ or } 5, |E(G)| \geq 4), G(6, |E(G)| \geq 10)$,
2) $K_{2,4}, K_{3,3}$,
3) The following graph:



Proof:

1) The graphs mentioned in 1) are not prime cordial by *Theorem 1.1.*
2) The graphs mentioned in 2) are not prime cordial by *Theorem 2.1.*
3) The graph mentioned in 3) contains 8 edges so it should contain 4 of them to be labeled 0, but to achieve this, the vertex labeled 6 must be adjacent to each of the labels: 2,3 and 4, i.e. the vertex labeled 6 is of degree at least 3, also the two vertices which are labeled 2 and 4 must be adjacent. Here we realize that whatever the vertex labeled 6 is, it is not possible for the vertices labeled 2 and 4 to be adjacent.

All the remaining graphs of order ≤6 [2] are prime cordial graph (*it is easy to label them as prime cordial graphs*).

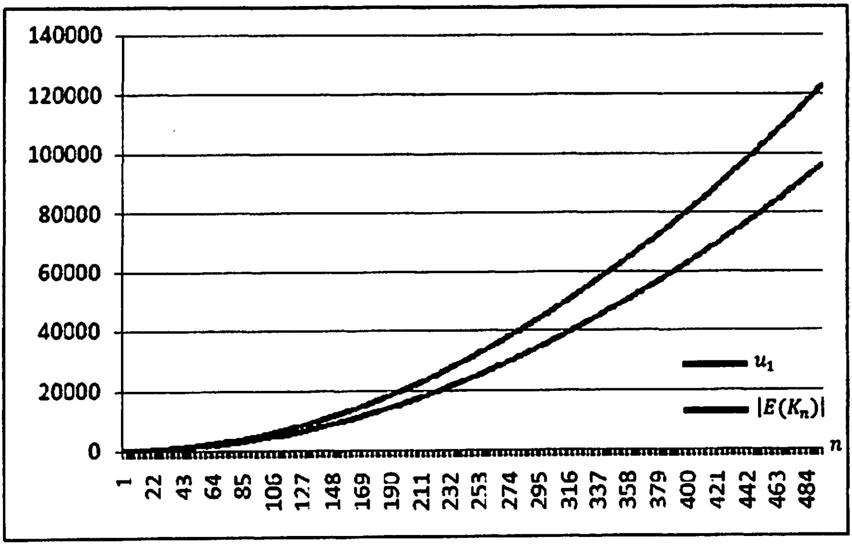


Diagram l: The first upper bound compared with the maximum number of edges in complete graphs until graphs of order 500

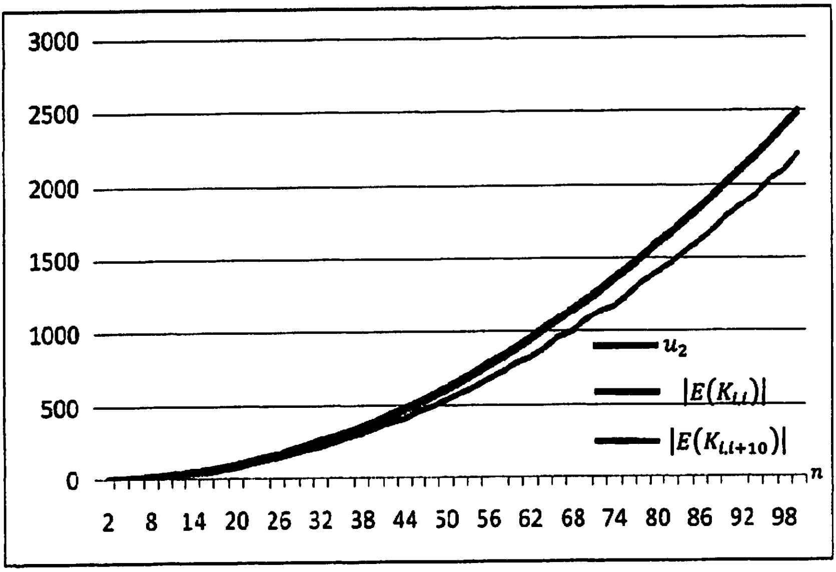


Diagram 2: The second upper bound compared with the maximum number of edges in complete bipartite graphs until graphs of order 100

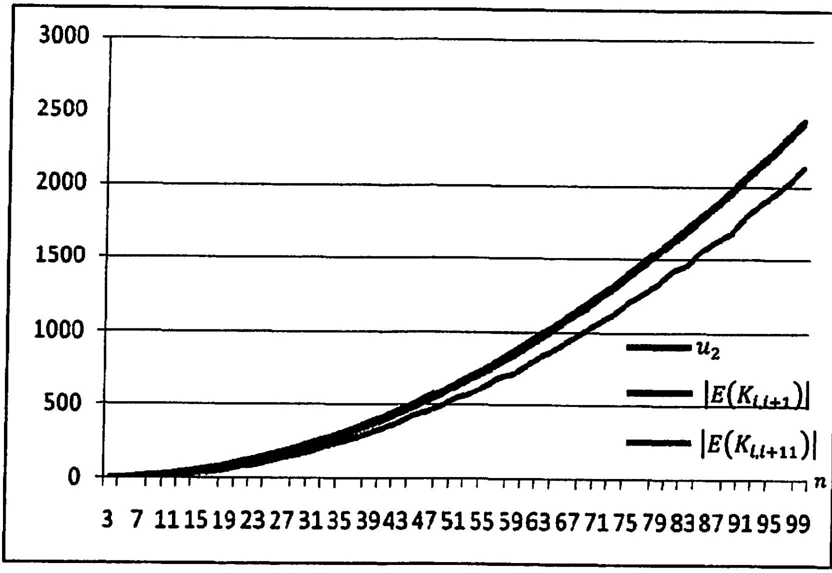


Diagram 3: The second upper bound compared with the maximum number of edges in complete bipartite graphs until graphs of order 99

Table 1: a table giving the number of edges of p_n^i compared with u_1

n	u_1	E_n^2	E_n^3	E_n^4	E_n^5	E_n^6	E_n^7	E_n^8	E_n^9	E_n^{10}	E_n^{11}	E_n^{12}
2	1	1										
3	1	**3**	**3**									
4	3	**5**	**6**	**6**								
5	3	**7**	**9**	**10**	**10**							
6	9	9	**12**	**14**	**15**	**15**						
7	9	**11**	**15**	**18**	**20**	**21**	**21**					
8	15	13	**18**	**22**	**25**	**27**	**28**	**28**				
9	19	15	**21**	**26**	**30**	**33**	**35**	**36**	**36**			
10	29	17	24	**30**	**35**	**39**	**42**	**44**	**45**	**45**		
11	29	19	27	**34**	**40**	**45**	**49**	**52**	**54**	**55**	**55**	
12	43	21	30	38	**45**	**51**	**56**	**60**	**63**	**65**	**66**	**66**
13	43	23	33	42	**50**	**57**	**63**	**68**	**72**	**75**	**77**	**78**
14	57	25	36	46	55	**63**	**70**	**76**	**81**	**85**	**88**	**90**
15	69	27	39	50	60	69	**77**	**84**	**90**	**95**	**99**	**102**
16	83	29	42	54	65	75	**84**	**92**	**99**	**105**	**110**	**114**
17	83	31	45	58	70	81	**91**	**100**	**108**	**115**	**121**	**126**
18	105	33	48	62	75	87	98	**108**	**117**	**125**	**132**	**138**
19	105	35	51	66	80	93	105	**116**	**126**	**135**	**143**	**150**
20	127	37	54	70	85	99	112	124	**135**	**145**	**154**	**162**
21	143	39	57	74	90	105	119	132	**144**	**155**	**165**	**174**
22	165	41	60	78	95	111	126	140	153	165	**176**	**186**
23	165	43	63	82	100	117	133	148	162	**175**	**187**	**198**
24	195	45	66	86	105	123	140	156	171	185	**198**	**210**
25	203	47	69	90	110	129	147	164	180	195	**209**	**222**
26	229	49	72	94	115	135	154	172	189	205	220	**234**
27	245	51	75	98	120	141	161	180	198	215	231	**246**
28	275	53	78	102	125	147	168	188	207	225	242	258
29	275	55	81	106	130	153	175	196	216	235	253	270
30	317	57	84	110	135	159	182	204	225	245	264	282
31	317	59	87	114	140	165	189	212	234	255	275	294
32	347	61	90	118	145	171	196	220	243	265	286	306
33	371	63	93	122	150	177	203	228	252	275	297	318
34	405	65	96	126	155	183	210	236	261	285	308	330
35	425	67	99	130	160	189	217	244	270	295	319	342

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