Threshold Dimension of Split-permutation Graphs *

Carmen Ortiz
Facultad de Ingeniería y Ciencias
Universidad Adolfo Ibáñez
Santiago, Chile
cortiz@uai.cl

Mónica Villanueva
Ingeniería Informática
Universidad de Santiago de Chile
Santiago, Chile
mvilla@diinf.usach.cl

Abstract

The threshold dimension of a graph is the minimum number of threshold subgraphs needed to cover its edges. In this work we present a new characterization of split-permutation graphs and prove that their threshold dimension is at most two. As a consequence we obtain a structural characterization of threshold graphs.

Keywords: split graph, permutation graph, threshold graph, threshold dimension, certifying algorithms.

1 Introduction

Let $G = (V, E)$ be a finite connected simple graph with vertex set V, edge set E and $|V| = n$. The complement graph of G is denoted $\overline{G}$. The *subgraph induced* by a subset $A \subseteq V$ in G is the graph $G[A] = (A, E_A)$ where $E_A = \{(x, y) \in E/\ x, y \in A\}$. The *neighbourhood* of a vertex v in $X \subseteq V$ is the set $N(v, X)$ of its adjacent vertices in X and its *degree* in X is $d(v, X) = |N(v, X)|$. If $X = V$ we simply write $N(v)$ and $d(v)$,

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respectively. $\overline{N}(v)$ denotes the set of neighbours of v in $\overline{G}$. If two vertices u and v are adjacent we say that u sees v, otherwise u misses v. A subset K of V is a *clique* if it induces a complete subgraph in G. A *stable set* in G is a subset of pairwise non-adjacent vertices. The *stability number* $\alpha(G)$ of a graph G is the size of a maximum stable set of G and the *clique cover number* $\kappa(G)$ is the size of a minimum clique partition of G.

G is a *comparability* graph if it admits a transitive orientation of its edges. G is *co-comparability* if its complement graph is a comparability graph. Graphs that are both comparability and co-comparability are *permutation* graphs. G is a *split* graph if its vertex set can be partitioned into a stable set S and a clique K. A split graph is denoted by $G = (S \cup K, E)$ [5].

G is a *threshold graph* if $N(x) \subseteq N(y) \cup \{y\}$ or $N(y) \subseteq N(x) \cup \{x\}$ for any pair of vertices x and y. They have applications in the aggregation of inequalities in 0-1 programming [1], in the synchronization of parallel processes [11] and in Guttman scales in psychology [2]. Every threshold graph is a split-permutation graph [4]. For a unified approach on threshold graphs see [7].

The *threshold dimension* $t(G)$ of a graph G is the minimum number of threshold subgraphs of G whose edge union is G. Graphs having threshold dimension 2 are called *2-threshold graphs*. Yannakakis [14] shows that for any fixed $k \geq 3$ the problem of deciding whether $t(G) \leq k$ is NP-complete. For $k = 1$ the problem can be solved in $O(n^2)$ time [1] and for $k = 2$ the problem was solved by Raschle and Simon [12] in $O(n^4)$ time. They build a conflict graph G^* which represents the edges of G that induce one of the forbidden configurations for threshold graphs. They prove that if G^* is bipartite then G is a 2-threshold graph. Sterbini and Raschle [13] present an $O(n^3)$ time recognition algorithm of threshold dimension 2 graphs using a geometrical representation.

A related problem is *Min Threshold Coloring* that aims to cover the vertices of a given graph by a minimum number of threshold graphs. Demange et al. [3] apply it to problems in robotics and prove that this problem is NP-hard in permutation graphs.

In this work we prove that a split-permutation graph is either a threshold graph or a 2-threshold graph. In section 2 we present a structural characterization of split-permutation graphs and we show that threshold graphs are precisely those split-permutation graphs that do not contain any induced P_4. In section 3 we prove that the threshold dimension of this class of graphs is one or two. A similar result was presented in [9] using forbidden characterizations. Here we present an $O(n^2)$ time algorithm that identifies the threshold subgraphs that cover the original graph. A consequence of this result is a structural characterization of threshold graphs that can be used to solve difficult problems as it was done in [10].

2 Split-permutation graphs

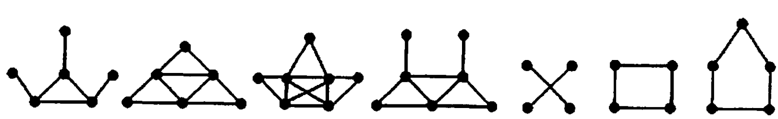
Foldes and Hammer [4] show that G is a split graph if and only if G contains no induced $2K_2, C_4$ or C_5. They additionally prove that a split graph is also a comparability graph if and only if G does not contain an induced subgraph isomorphic to $G_1, \overline{G_1}$ or $\overline{G_3}$, shown in Figure 1. Thus, G is a split-comparability graph if it contains no induced subgraph isomorphic to any of the graphs of the family $F_1 = \{2K_2, C_4, C_5, G_1, \overline{G_1}, \overline{G_3}\}$.

Theorem 1 *G is a split-permutation graph if and only if G does not contain an induced subgraph isomorphic to any of the graphs of $F_2 = F_1 \cup \{G_3\}$.*

Proof: We can obtain the forbidden induced subgraphs for a split-permutation graph as the union of F_1 and the set of their complement graphs. For each graph of F_1 its complement graph is already included in F_1, with the exception of $\overline{G_3}$. Thus, we only need to add G_3 and the result follows. ■

Corollary 1 *A split-comparability graph is a permutation graph if and only if it contains no induced subgraph isomorphic to G_3.*

Chvátal and Hammer [1] show that G is a threshold graph if and only if G has no induced $2K_2, P_4$ or C_4.



G_1 $\overline{G}_1$ G_3 $\overline{G}_3$ $2K_2$ C_4 C_5

Figure 1: Forbidden induced subgraphs of split-permutation graphs.

Lemma 1 *A split graph G is a threshold graph if and only if G contains no induced subgraph isomorphic to P_4.*

In fact, a split graph contains no induced $2K_2, C_4$ or C_5. If it is also a threshold graph, it cannot contain an induced P_4. Thus, C_5 is redundant for a minimal family of forbidden induced subgraphs.

The following characterization of split-comparability graphs, presented in [8], is based on a partition of a maximum clique. For a total order $\prec$ of $v_1, v_2, \ldots, v_r$ let $[v_i, v_j] = \{v_k \;/\; v_i \prec v_k \prec v_j\}$ be the *segment* with v_i and v_j as ends.

Lemma 2 *A split graph $G = (S \cup K, E)$ is a comparability graph iff K can be totally ordered $v_1 \prec v_2 \prec \ldots \prec v_r$ and partitioned into three (possibly empty) segments $K_p = [v_1, v_p], K_q = [v_q, v_r]$ and $K_t = K \setminus (K_p \cup K_q)$ such that $N(s)$ has one of the following forms for every vertex $s \in S$:*

i) $[v_1, v_i]$ *for* $i \leq p$

ii) $[v_j, v_r]$ *for* $q \leq j \leq r$

iii) $[v_1, v_i] \cup [v_j, v_r]$ *for* $i \leq p$ *and* $q \leq j \leq r$.

Note that K_t is the set of vertices of K that see no vertex in S. This partition of K induces a partition of S into three sets: S_p, S_q and S_t. Vertices of S_p see only vertices of K_p and their neighbourhoods have the form (*i*). Similarly, vertices of S_q see only vertices of K_q and their neighbourhoods have the form (*ii*). On the other hand, vertices of S_t see vertices of both K_p and K_q and their neighbourhoods have the form (*iii*).

Given a split graph $G = (S \cup K, E)$, define the graph H_G with vertex set S. There is an edge (x, y) in H_G if in G we have that $N(x) \subseteq N(y)$ or $N(y) \subseteq N(x)$. In this case we say that x and y are *comparable.* It is easy to see that H_G is a comparability graph when G is a split-permutation graph. In fact, we can orient every edge (x, y) of H_G from x to y if $N(x) \subseteq N(y)$, otherwise orient it from y to x. This orientation is transitive.

Theorem 2 *A split-comparability graph $G = (S \cup K, E)$ is a permutation graph iff S can be partitioned into S_1 and S_2, and each subset can be totally ordered such that $s_i < s_j$ if $N(s_i) \subseteq N(s_j)$ for $s_i, s_j \in S_1$ or $s_i, s_j \in S_2$.*

Proof:
(If part) Let G be a split-comparability graph. Order $S_1 = \{s_1, s_2, \ldots, s_a\}$ and $S_2 = \{s_{a+1}, s_{a+2}, \ldots, s_f\}$ such that $s_1 < s_2 < \ldots < s_a$ and $s_f < \ldots < s_{a+2} < s_{a+1}$. Therefore $S_1 = [s_1, s_a]$ and $S_2 = [s_f, s_{a+1}]$.

By the self-complementary property of split graphs, $\overline{G} = (S \cup K, \overline{E})$ is also a split graph. Orient the edges of the clique induced by S in $\overline{G}$ such that (s_i, s_j) is directed from s_i to s_j if $s_i, s_j \in S_1$ and $s_i < s_j$ in G or if $s_i, s_j \in S_2$ and $s_i > s_j$. Edges with one endpoint in S_1 and the other in S_2 are oriented from the vertex of S_1 to the vertex of S_2.

Edges (v, s) of $\overline{G}$ with $v \in K$ are oriented from s to v if $s \in S_1$ and from v to s otherwise. Clearly this orientation is transitive. In fact, we have that in G every vertex $v \in K$ is such that $N(v, S) = [s_{b+1}, s_a] \cup [s_{d-1}, s_{a+1}]$ where s_{b+1} is the first vertex of S_1 that sees v and s_{d-1} is the first vertex of S_2 that is adjacent to v. Thus, $\overline{N}(v) = [s_1, s_b] \cup [s_f, s_d]$ with one of these segments possibly empty. See Figure 2 (for a better understanding it doesn't contain all the edges).

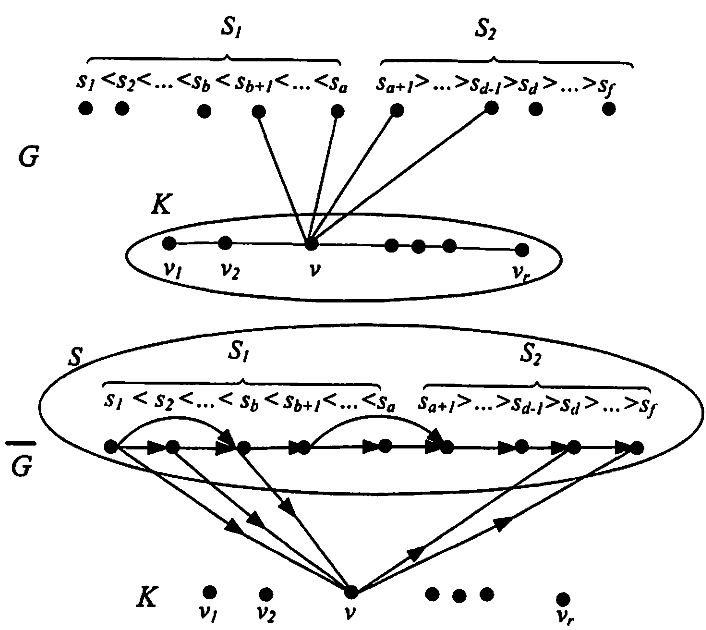


Figure 2: A split-comparability graph G and its complement graph $\overline{G}$. Vertices inside the oval form a clique.

(Only if part) Let $G = (S \cup K, E)$ be a split-permutation graph with K ordered $v_1 \prec v_2 \prec \ldots \prec v_r$, according to Lemma 2. Consider the graph H_G with vertex set S as defined before. H_G is a perfect graph because it is a comparability graph. So the clique cover number $\kappa(H_G)$ equals the stability number $\alpha(H_G)$.

We claim that $\kappa(H_G) \leq 2$. Otherwise, $\alpha(H_G) \geq 3$ and this implies the existence of an induced subgraph isomorphic to G_3 in G. In fact, let $N(s) = [v_1, v_s] \cup [w_s, v_r]$ for $s \in S$. If $s \in S_p$ or $s \in S_q$, one of these segments is empty. If $\alpha(H_G) \geq 3$, there are three incomparable vertices $x, y, z \in S$ such that at most one of them may belong to S_p and at most one of them may be in S_q. Suppose that $v_y \prec v_z \prec v_x$ and $w_y \prec w_z \prec w_x$. Then $w_z, w_y \notin N(x)$, $v_z, v_x \notin N(y)$ and $v_x, w_y \notin N(z)$. Thus, v_z, v_x, w_y, w_z, x, y and z induce a subgraph isomorphic to G_3 in G. A contradiction, because G is a permutation graph.

Since $\kappa(H_G) \leq 2$, let $S_1, S_2 \subseteq S$ be the sets of vertices that induce the cliques that cover H_G. Clearly, S_1 and S_2 are the required partition of S.■

Figure 3 shows a split-permutation graph characterized by the above theorem.

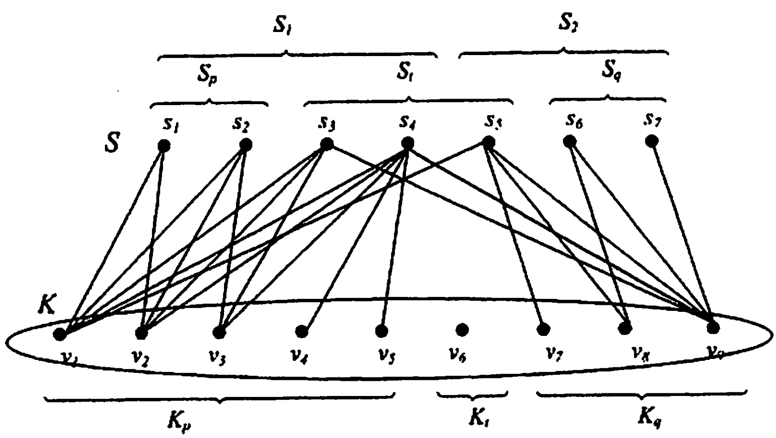


Figure 3: A split-permutation graph. Vertices inside the oval form a clique.

Observe that S_p as well as S_q induces a clique in H_G. Thus, each vertex of smallest degree of S_p is a source in H_G. Analogously for S_q. Also note that in a split-permutation graph, each vertex of S_t must be comparable to all vertices of one of these two subsets but to some vertices of the other one. However, a vertex of S_t may be comparable to all vertices of both S_p and S_q. This implies that in H_G, a vertex of S_t cannot partially see both S_p and S_q.

Corollary 2 *If G is a split-permutation graph then $\overline{H_G}$, the complement graph of H_G, is bipartite with bipartition (S_1, S_2).*

Proof: The result follows directly from Theorem 2.■

3 Threshold dimension of split-permutation graphs

Consider the *vicinal preorder* $\lesssim$ on $V(G)$ given by $x \lesssim y$ iff $N(x) \subseteq N(y) \cup \{y\}$. G is a threshold graph if and only if this relation induces a chain on $V(G)$ [4]. The forbidden subgraph characterization of split-permutation graphs given by Theorem 1 shows that a split graph is also a permutation graph, precisely if there is no antichain of size three with respect to this relation.

Theorem 3 *A split-permutation graph has threshold dimension two.*

Proof: Let G be a split-permutation graph characterized by Theorem 2. Consider $T_i = S_i \cup K$, $i = 1, 2$. Each T_i induces a threshold graph $G[T_i]$, $i = 1, 2$. In the vicinal preorder, vertices of S_i are followed by the vertices of K. Thus, $(G[T_1], G[T_2])$ is a cover of G by threshold graphs, as required.■

For the graph of Figure 3 we have that $T_1 = K \cup \{s_1, s_2, s_3, s_4\}$ and $T_2 = K \cup \{s_5, s_6, s_7\}$.

Corollary 3 *A split-interval graph has threshold dimension two.*

In fact, we only need that S contains no antichain of size three in the order and thus we do not need to forbid $\overline{G_3}$. It follows that any split graph that is the complement of a comparability graph, i.e., any split graph that is also an interval graph, has threshold dimension two.

Given a split permutation graph G, the proof of Theorem 2 yields to the following algorithm that builds a partition of S into S_1 and S_2 such that $G[S_1 \cup K]$ and $G[S_2 \cup K]$ are the threshold graphs that cover G.

The algorithm SP bellow builds the graph $\overline{H_G}$ for a given split-permutation graph G. It applies a Breadth First Search (BFS) procedure to colour its vertices red or blue. S_1 is the set of blue vertices and S_2 is the set of red vertices. BFS uses a queue Q and an array COLOR that contains the colour assigned to each vertex. This array is initialized to white and Q is initialized to be empty. For simplicity, $\overline{H}_G$ will be denoted $\overline{H}$.

Algorithm SP

Input: A split-permutation graph G with its clique K ordered and partitioned into K_p, K_t and K_q, and its stable set S partitioned into S_p, S_q and S_t according to Lemma 2.

Output: The sets S_1 and S_2 that partition S.

```
Step 1: Build the graph H̄
    Set V(H̄) := S; E(H̄) := {(x,y)/ x ∈ S_p and y ∈ S_q}
    For each vertex s ∈ S_t do
        For each vertex x ∈ S_p do
            if N(s,K_p) ⊊ N(x,K_p) then E(H̄) := E(H̄) ∪ {(x,s)}
        For each vertex x ∈ S_q
            if N(s,K_q) ⊊ N(x,K_q) then E(H̄) := E(H̄) ∪ {(x,s)}
        For each vertex x ∈ (S_t − s)
            if [N(s,K_p) ⊊ N(x,K_p) or N(s,K_q) ⊊ N(x,K_q)] and
              [N(x,K_p) ⊊ N(s,K_p) or N(x,K_q) ⊊ N(s,K_q)] then
              E(H̄) := E(H̄) ∪ {(x,s)}

Step 2: Colour the vertices of H̄ red or blue
        COLOR(s_1):= blue; S_1 := {s_1}; S_2 := φ
        Add s_1 to Q                    /The search begins with vertex s_1
        While Q ≠ ∅ do
            x:= the first vertex of Q
            for each w ∈ N(x) do
                if COLOR(w) = white then
                    Add w to Q
                    if COLOR(x) = blue then
                        COLOR(w):= red; S_2 := S_2 ∪ {w}
                    else
                        COLOR(w):= blue; S_1 := S_1 ∪ {w}
            remove x from Q.
        for each v ∈ V(H̄) do           /Isolated vertices are coloured blue
            if COLOR(v) = white then
                COLOR(v):= blue
```

The correctness of algorithm SP follows from the bipartiteness of $\overline{H}_G$. This implies that $\overline{H}_G$ is two-colourable and it does not contain odd cycles. Therefore, every vertex of $\overline{H}_G$ receives a unique colour. Isolated vertices are coloured blue.

Theorem 4 *Algorithm SP requires $O(n^2)$ time to partition S into S_1 and S_2 satisfying Theorem 2.*

Proof: Step 1 needs $O(n^2)$ time to build the graph $\overline{H}_G$. Since Step 2 is a Breadth First Search algorithm, it requires $O(n^2)$ time to colour the vertices of $\overline{H}_G$.■

If the input to the algorithm is a split-comparability graph, instead of a split-permutation graph, in Step 2 we must verify if there is any conflict between the colours assigned to adjacent vertices. The existence of any conflict implies that $\overline{H}_G$ is not bipartite and this means that the input graph is not permutation. This verification does not increase the complexity of SP.

Recently, there has been interest in *certifying algorithms*, i.e., algorithms that provide a certificate with each answer they produce for a decision problem [6]. If we modify Step 2, we have a certifying algorithm for a given split-comparability graph being also permutation. It provides a certificate of membership (the partition of S) and a certificate of nonmembership: an induced C_3 in $\overline{H}_G$, that implies an induced G_3 in G, in $O(n^2)$ time. We only need to modify the *While* loop of Step 2, adding an array VCOLOR that contains for each vertex of $\overline{H}_G$, the vertex from which it was coloured. Variables $V1, V2$ and $V3$ contain the vertices of the cycle C_3, if it exists.

```
While Q ≠ ∅ do
    x:= the first vertex of Q
    for each w ∈ N(x) do
        if COLOR(w) = white then
            Add w to Q
            VCOLOR(w):= x
            if COLOR(x) = blue then
                COLOR(w):= red; S2 := S2 ∪ {w}
            else
                COLOR(w):= blue; S1 := S1 ∪ {w}
        else            /Verification of conflicts
            if COLOR(w) = COLOR(x) then
                V1 := x; V2 := w; V3 := VCOLOR(x)
                END     /The search finds an induced C3 : V1, V2, V3
    remove x from Q.
```

In case of conflict, let $N(V1) = [v_1, a_1] \cup [b_1, v_r]$, $N(V2) = [v_1, a_2] \cup [b_2, v_r]$ and $N(V3) = [v_1, a_3] \cup [b_3, v_r]$. Thus, the subgraph induced by $V1, V2, V3, a_1, a_2, b_2$ and b_3 is isomorphic to G_3.

Theorem 3 implies the following characterization of threshold graphs.

Corollary 4 *Let G be a split graph. G is a threshold graph if and only if for each vertex $s \in S$ the set $N(s)$ has the form $[v_1, v_i]$, $i \leq r$.*

Proof:
(If part) Let $G = (S \cup K, E)$ be a split graph. If G is such that $N(s)$ has the form $[v_1, v_i]$ with $i \leq r$ for each vertex $s \in S$, then V admits a vicinal preorder. This relation induces a chain in V such that vertices of S are placed before vertices of K.

(Only if part) If G is a threshold graph then it has threshold dimension one. Thus, by Theorem 3 we have that either $S_1 = \phi$ or $S_2 = \phi$. If $S_2 = \phi$ then $N(s) = [v_1, v_i]$ for some $i \leq r$ and G is the graph $G[T_1]$. On the other hand, if $S_1 = \phi$ then $N(s) = [v_j, v_r]$ for some $j \geq 1$ and G is the graph $G[T_2]$.■

We use the above characterization of threshold graphs in [10] to solve two difficult problems in this class: the chromatic index and the enumeration of all maximal independent sets.

4 Conclusions

We have characterized split-permutation graphs by decomposing a maximum clique into three cliques and its stable set into two subsets. We prove that if a split-permutation graph has no induced P_4 it has threshold dimension 1, i.e., it is a threshold graph, otherwise it is 2-threshold. Given a split-permutation graph it takes $O(n)$ time to determine its threshold dimension because it is enough to verify if a vertex of maximum degree is universal. We develop an algorithm that builds the threshold graphs whose edge union is the given graph in $O(n^2)$ time. We also show that every split-interval graph has threshold dimension two.

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