Modular Neighbor-Distinguishing Edge Colorings of Graphs

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ABSTRACT

Let G be a connected graph of order 3 or more and $c: E(G) \to \mathbb{Z}_k$ $(k \geq 2)$ an edge coloring of G where adjacent edges may be colored the same. The color sum $s(v)$ of a vertex v of G is the sum in $\mathbb{Z}_k$ of the colors of the edges incident with v. An edge coloring c is a modular neighbor-distinguishing k-edge coloring of G if $s(u) \neq s(v)$ in $\mathbb{Z}_k$ for all pairs u, v of adjacent vertices of G. The modular chromatic index $\chi'_m(G)$ of G is the minimum k for which G has a modular neighbor-distinguishing k-edge coloring. For every graph G, it follows that $\chi'_m(G) \geq \chi(G)$. In particular, it is shown that if G is a graph with $\chi(G) \equiv 2 \pmod 4$ for which every proper $\chi(G)$-coloring of G results in color classes of odd size, then $\chi'_m(G) > \chi(G)$. The modular chromatic indices of several well-known classes of graphs are determined. It is shown that if G is a connected bipartite graph, then $2 \leq \chi'_m(G) \leq 3$ and it is determined when each of these two values occurs. There is a discussion on the relationship between $\chi'_m(G)$ and $\chi'_m(H)$ when H is a subgraph of G.

Key Words: modular neighbor-distinguishing edge coloring, modular chromatic index.

AMS Subject Classification: 05C15.

1 Introduction

Graph coloring is one of the most popular research areas in graph theory. Among the most studied colorings are proper vertex colorings and proper edge colorings. A *proper vertex coloring* of a graph G is an assignment of colors to the vertices of G such that adjacent vertices are assigned distinct colors and the minimum number of colors in a proper vertex coloring of G is the *chromatic number* $\chi(G)$ of G. A *proper edge coloring* of a graph G is an assignment of colors to the edges of G such that adjacent edges are assigned distinct colors and the minimum number of colors in a proper edge coloring of G is the *chromatic index* $\chi'(G)$ of G.

A coloring that provides a method of distinguishing every two adjacent vertices is said to be *neighbor-distinguishing*. Thus a proper vertex coloring of a graph is neighbor-distinguishing. A number of neighbor-distinguishing vertex colorings other than the standard proper colorings have been introduced (see [4, 5, 6], for example). Furthermore, edge colorings (proper or nonproper) have also been introduced to distinguish every pair of adjacent vertices in a graph (see [1, 2, 8, 11] or [7, p. 385], for example). Another neighbor-distinguishing vertex coloring was introduced in [9] for the purpose of finding solutions to the following checkerboard problem.

> *The squares of an $m \times n$ checkerboard (m rows and n columns) are alternately colored black and red. Two squares are said to be neighboring if they belong to the same row or the same column and there is no square between them. Is it possible to place coins on some of the squares of an $m \times n$ checkerboard (at most one coin per square) such that for every two squares of the same color the numbers of coins on neighboring squares are of the same parity, while for every two squares of different colors the numbers of coins on neighboring squares are of opposite parity?*

For a vertex v of a graph G, let $N(v)$ denote the neighborhood of v (the set of vertices adjacent to v). For a graph G without isolated vertices let $c : V(G) \to \mathbb{Z}_k$ ($k \geq 2$) be a vertex coloring of G where adjacent vertices may be colored the same. The *color sum* of a vertex v of G is defined as the sum in $\mathbb{Z}_k$ of the colors of the vertices in $N(v)$. The coloring c is called a *modular k-coloring* of G if every pair of adjacent vertices of G have different color sums in $\mathbb{Z}_k$. The *modular chromatic number* of G is the minimum k for which G has a modular k-coloring. This coloring has been studied further in [10], which led to a complete affirmative solution to the checkerboard problem under investigation.

We introduce here a neighbor-distinguishing edge coloring that is closely related to the modular vertex colorings mentioned above. For a graph G without isolated vertices, let $c : E(G) \to \mathbb{Z}_k$ ($k \geq 2$) be an edge coloring of

G where adjacent edges may be colored the same. The *color sum* $s(v)$ of a vertex v of G is defined as the sum in $\mathbb{Z}_k$ of the colors of the edges incident with v, that is, if E_v is the set of edges incident with v in G, then

$$s(v) = \sum\nolimits_{e \in E_v} c(e).$$

An edge coloring c is a *modular neighbor-distinguishing k-edge coloring* of G if $s(u) \neq s(v)$ in $\mathbb{Z}_k$ for all pairs u, v of adjacent vertices of G. We refer to such edge colorings more simply as *modular k-edge colorings.* An edge coloring c is a *modular edge coloring* if c is a modular k-edge coloring for some integer $k \geq 2$. The *modular chromatic index* $\chi'_m(G)$ of G is the minimum k for which G has a modular k-edge coloring. If G contains a component isomorphic to K_2, say $V(K_2) = \{u, v\}$, then $s(u) = s(v)$ for any edge coloring of G, which implies that G does not have a modular edge coloring. On the other hand, every graph containing neither isolated vertices nor components isomorphic to K_2 has a modular edge coloring.

Proposition 1.1 *If a graph contains neither isolated vertices nor components isomorphic to K_2, then its modular chromatic index exists.*

Proof. Let G be such a graph and $E(G) = \{e_1, e_2, \ldots, e_m\}$, where $m \geq 2$. Define an edge coloring c of G by $c(e_i) = 2^{i-1}$ for $1 \leq i \leq m$ and let $k = \sum_{i=1}^{m} 2^{i-1} = 2^m - 1$. Since $1 \leq s(v) \leq k$ for every $v \in V(G)$ and $s(u) \neq s(v)$ in $\mathbb{Z}_k$ for every two distinct vertices u and v in G, it follows that c is a modular k-edge coloring of G and so $\chi'_m(G)$ exists. ∎

The following observation will be useful to us.

Observation 1.2 *If G is a disconnected graph consisting of components $G_1, G_2, \ldots, G_t$ each of which contains at least three vertices, then*

$$\chi'_m(G) = \max\{\chi'_m(G_i) : 1 \leq i \leq t\}.$$

In view of Proposition 1.1 and Observation 1.2, we consider connected graphs of order 3 or more in this work. If c is a modular k-edge coloring of a graph G, then $s(u) \neq s(v)$ in $\mathbb{Z}_k$ for every pair u, v of adjacent vertices of G. Thus the coloring c^* of G defined by $c^*(v) = s(v)$, $v \in V(G)$, is a proper vertex coloring of G with at most k colors. This observation shows that $\chi(G)$ is a lower bound for $\chi'_m(G)$.

Proposition 1.3 *For every connected graph G of order at least* 3,

$$\chi'_m(G) \geq \chi(G).$$

To illustrate the concepts introduced above, consider the tree T of order 10 in Figure 1(a). An edge coloring of T is shown in Figure 1(b), where each edge is colored with an element in $\mathbb{Z}_3 = \{0, 1, 2\}$ and each vertex is labeled with its color sum. Observe that $s(u) \neq s(v)$ in $\mathbb{Z}_3$ for every pair u, v of adjacent vertices of T. Thus this edge coloring is a modular 3-edge coloring of T and so $\chi'_m(T) \leq 3$. Since $\chi'_m(T) \geq 2$ by Proposition 1.3, it follows that $\chi'_m(T)$ is either 2 or 3. To show that $\chi'_m(T) = 3$, assume, to the contrary, that there exists a modular 2-edge coloring c of T. Thus $s(v) = 0$ or $s(v) = 1$ for each $v \in V(T)$. By the symmetry of the tree, we may assume that $s(u_i) = 0$ and $s(w_i) = 1$ for $1 \leq i \leq 5$. Hence, $c(u_i w_5) = 0$ and $c(u_5 w_i) = 1$ for $1 \leq i \leq 4$. However, this implies that $s(u_5) = c(u_5 w_5) = s(w_5)$, which is not possible. Therefore, $\chi'_m(T) \geq 3$, that is, $\chi'_m(T) = 3 > \chi(T)$.

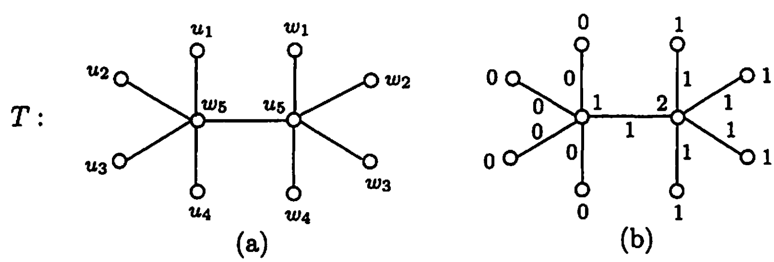


Figure 1: A modular 3-edge coloring of a graph

There are also graphs G for which $\chi'_m(G) = \chi(G)$. For example, consider the Petersen graph P in Figure 2. Since $\chi(P) = 3$ and there exists a modular 3-edge coloring of P (also shown in the figure), $\chi'_m(P) = 3 = \chi(P)$.

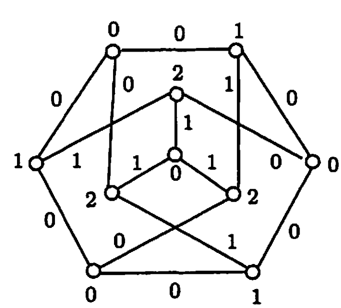


Figure 2: A modular 3-edge coloring of the Petersen graph

We refer to the book [3] for graph theory notation and terminology not described in this paper.

2 Modular Chromatic Indices of Complete Graphs and Cycles

In this section and the next we determine the modular chromatic indices of several classes of graphs. We begin with complete graphs. We first make a useful observation.

Observation 2.1 *If c is an edge coloring of a connected graph G, then*

$$\sum_{v \in V(G)} s(v) = 2 \sum_{e \in E(G)} c(e).$$

Thus if c is a modular k-edge coloring of G, then

$$\sum_{v \in V(G)} s(v) \equiv 2 \sum_{e \in E(G)} c(e) \pmod{k}.$$

We now determine $\chi'_m(K_n)$ for each integer $n \geq 3$.

Theorem 2.2 *For each integer $n \geq 3$,*

$$\chi'_m(K_n) = \begin{cases} n+1 & \text{if } n \equiv 2 \pmod 4 \\ n & \text{otherwise.} \end{cases}$$

Proof. Let $G = K_n$, where $V(G) = \{v_1, v_2, \ldots, v_n\}$. If n is odd, then let $c_1 : E(G) \to \mathbb{Z}_n$ be an edge coloring given by

$$c_1(e) = \begin{cases} i & \text{if } e = v_i v_n \ (1 \leq i \leq n-1) \\ 0 & \text{otherwise.} \end{cases}$$

Then $s(v_i) = i$ for $1 \leq i \leq n$, implying that c_1 is a modular n-edge coloring of G. It then follows by Proposition 1.3 that $\chi'_m(G) = n$ if n is odd. If n is even, then we consider two cases.

Case 1. $n \equiv 0 \pmod 4$. Let $n = 4p$ for some positive integer p. Define an edge coloring $c_2 : E(G) \to \mathbb{Z}_{4p}$ by

$$c_2(e) = \begin{cases} p & \text{if } e \in \{v_i v_{i+1} : 1 \leq i \leq 4p-2\} \cup \{v_1 v_{4p-1}\} \\ i & \text{if } e = v_i v_{4p} \text{ and } 1 \leq i \leq 4p-1 \text{ and } i \neq 2p \\ 0 & \text{otherwise.} \end{cases}$$

Thus for $1 \leq i \leq 4p$

$$s(v_i) = \begin{cases} 2p & \text{if } i = 2p \\ 0 & \text{if } i = 4p \\ 2p+i & \text{otherwise} \end{cases}$$

in $\mathbb{Z}_{4p}$. Hence c_2 is a modular $4p$-edge coloring of G. The result now follows by Proposition 1.3.

Case 2. $n \equiv 2 \pmod 4$. Let $n = 4p+2$ for some positive integer p. Define an edge coloring $c_3 : E(G) \to \mathbb{Z}_{4p+3}$ by

$$c_3(e) = \begin{cases} i-1 & \text{if } e = v_iv_{4p+2} \text{ and } 2 \le i \le 2p+1 \\ i+1 & \text{if } e = v_iv_{4p+2} \text{ and } 2p+2 \le i \le 4p+1 \\ 1 & \text{if } e \in \{v_iv_{i+1} : 1 \le i \le 2p\} \cup \{v_1v_{2p+1}\} \\ 0 & \text{otherwise.} \end{cases}$$

Thus for $1 \le i \le 4p+2$

$$s(v_i) = \begin{cases} 0 & \text{if } i = 4p+2 \\ i+1 & \text{otherwise} \end{cases}$$

in $\mathbb{Z}_{4p+3}$ and so c_3 is a modular $(4p+3)$-edge coloring of G. Thus, $\chi'_m(G) \le n+1$ if $n \equiv 2 \pmod 4$. On the other hand, assume, to the contrary, that there exists a modular $(4p+2)$-edge coloring c' of G. Then by Observation 2.1

$$2\sum_{e\in E(G)} c'(e) = \sum_{i=1}^{4p+2} s(v_i) = 0 + 1 + \cdots + (4p+1) = 2p+1$$

in $\mathbb{Z}_{4p+2}$, which is impossible. Therefore, $\chi'_m(G) \ge n+1$, which in turn implies that $\chi'_m(G) = n+1$ if $n \equiv 2 \pmod 4$. ∎

It is well known that if v is a vertex in a nontrivial graph G, then either $\chi(G-v) = \chi(G)$ or $\chi(G-v) = \chi(G)-1$. Also, if an edge e is deleted from an nonempty graph G, then $\chi(G-e) = \chi(G)$ or $\chi(G-e) = \chi(G)-1$. This, however, is not the case for the modular chromatic index of a graph. For example, let $G = K_n$ with $n \equiv 2 \pmod 4$. By Theorem 2.2, $\chi'_m(G) = n+1$, while $\chi'_m(G-v) = \chi'_m(K_{n-1}) = n-1$ as $n-1 \not\equiv 2 \pmod 4$, implying that $\chi'_m(G-v) = \chi'_m(G) - 2$ for each $v \in V(G)$. Furthermore, $\chi'_m(G-e) = \chi'_m(G) - 2$ for each $e \in E(G)$, as we show next. It is known that $\chi(K_n - e) = n-1$ for each integer $n \ge 3$.

Theorem 2.3 *For each integer $n \ge 3$, $\chi'_m(K_n - e) = n-1$.*

Proof. Let $G = K_n - e$ and $V(G) = \{v_1, v_2, \ldots, v_n\}$. We consider two cases.

Case 1. *n is odd.* Without loss of generality, assume that $v_{\lfloor n/2\rfloor}v_{\lceil n/2\rceil} \notin E(G)$. Let H be the connected spanning subgraph of G such that

$$\deg_H v_i = \begin{cases} i & \text{if } 1 \le i \le \lfloor \frac{n}{2} \rfloor \\ i-1 & \text{if } \lceil \frac{n}{2} \rceil \le i \le n. \end{cases}$$

Thus $\deg_H v_{\lfloor n/2 \rfloor} = \deg_H v_{\lceil n/2 \rceil}$ while $\deg_H x \neq \deg_H y$ for all pairs x, y of distinct vertices of H with $\{x, y\} \neq \{v_{\lfloor n/2 \rfloor}, v_{\lceil n/2 \rceil}\}$. Define an edge coloring c_1 of G by $c_1(e) = 1$ if $e \in E(H)$ and $c_1(e) = 0$ otherwise. Then $s(v) = \deg_H v$ for each $v \in V(G)$ and this is a modular $(n-1)$-edge coloring of G.

Case 2. n is even. Suppose that $v_{n/2-1}v_{n/2} \notin E(G)$. Construct the subgraph H' with the vertex set $\{v_1, v_2, \ldots, v_{n-1}\}$ as described in Case 1. Let H be the spanning subgraph of G obtained from H' by adding the isolated vertex v_n. Then

$$\deg_H v_i = \begin{cases} i & \text{if } 1 \leq i \leq \frac{n}{2} - 1 \\ i-1 & \text{if } \frac{n}{2} \leq i \leq n-1 \\ 0 & \text{if } i = n. \end{cases}$$

In this case, $\deg_H v_n = 0$, $\deg_H v_{n/2-1} = \deg_H v_{n/2}$ and $\deg_H x \neq \deg_H y$ for all pairs x, y of distinct vertices of H with $\{x, y\} \neq \{v_{n/2-1}, v_{n/2}\}$. Define an edge coloring c_2 of G by $c_2(e) = 1$ if $e \in E(H)$ and $c_2(e) = 0$ otherwise. Then $s(v) = \deg_H v$ for each $v \in V(G)$ and so c_2 is a modular $(n-1)$-edge coloring of G.

Since $\chi'_m(G) \geq n - 1$ by Proposition 1.3, the result now follows. ∎

A fundamental property of the chromatic number is that if H is a subgraph of a graph G, then $\chi(H) \leq \chi(G)$. For the modular chromatic index, the situation is different. If H is a subgraph of G for which $\chi'_m(H) = \chi(H)$, then $\chi'_m(H) = \chi(H) \leq \chi(G) \leq \chi'_m(G)$. On the other hand, if $H = K_n$ with $n \equiv 2 \pmod 4$ is a subgraph of a graph G, then it is possible that $\chi'_m(H) > \chi'_m(G)$. In fact, if $H \subset G \subseteq \mathrm{cor}(H)$, where $\mathrm{cor}(H)$ is the *corona* of H (the graph obtained from H by adding a pendant edge at each vertex of H), then $\chi'_m(G) = n < \chi'_m(H)$. Furthermore, if G is the Cartesian product $H \times K_2$, then $\chi'_m(G) = n < \chi'_m(H)$ as well. We now verify both statements.

Let $n \geq 6$ be an integer where $n \equiv 2 \pmod 4$. Then $n = 4p + 2$ for some positive integer p. Let $H = K_n$ where $V(H) = \{u_0, u_1, \ldots, u_{4p+1}\}$ and let $C = (u_0, u_1, \ldots, u_{4p+1}, u_{4p+2} = u_0)$ be a Hamiltonian cycle of H. We define edge colorings $c_i : E(H) \to \mathbb{Z}_{4p+2}$ of H $(i = 1, 2)$ by

$$c_1(e) = \begin{cases} i & \text{if } e \in E(C) \text{ and } e \text{ is incident with } u_{2i} \text{ for } 0 \leq i \leq p \\ -i & \text{if } e \in E(C) \text{ and } e \text{ is incident with } v_{4p+2-2i} \text{ for } 1 \leq i \leq p \\ 0 & \text{otherwise} \end{cases}$$

and

$$c_2(e) = \begin{cases} 2p+1 & \text{if } e = u_0u_{2p+1} \\ c_1(e) & \text{otherwise.} \end{cases}$$

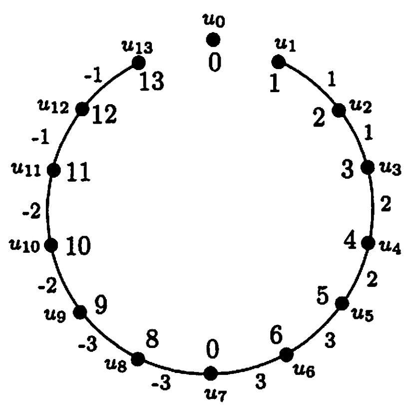
The color sums $s_1(u_i)$ in $\mathbb{Z}_{4p+2}$, $0 \le i \le 4p+1$, obtained from c_1 are

$$s_1(u_i) = \begin{cases} 0 & \text{if } i = 2p+1 \\ i & \text{otherwise;} \end{cases}$$

while the color sums $s_2(u_i)$ in $\mathbb{Z}_{4p+2}$, $0 \le i \le 4p+1$, obtained from c_2 are

$$s_2(u_i) = \begin{cases} 2p+1 & \text{if } i = 0, 2p+1 \\ s_1(u_i) & \text{otherwise.} \end{cases}$$

The colorings c_1, c_2, s_1 and s_2 are illustrated for $H = K_{14}$ in Figure 3 where only the edges not colored 0 are shown.


(a) The colorings c_1 and s_1

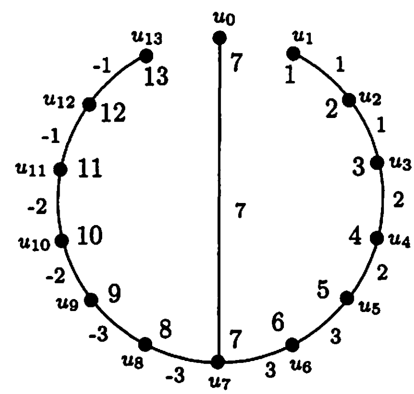

(b) The colorings c_2 and s_2

Figure 3: Illustrating the colorings c_1, c_2, s_1 and s_2

The edge colorings c_1 and c_2 do not result in proper vertex colorings s_1 and s_2, respectively. This, of course, is not surprising however since $\chi'_m(K_{4p+2}) = 4p+3$ and there is no modular edge coloring of K_{4p+2} using the elements of $\mathbb{Z}_{4p+2}$. In the colorings s_1 and s_2, only u_0 and u_{2p+1} are colored the same, namely $s_1(u_0) = s_1(u_{2p+1}) = 0$ and $s_2(u_0) = s_2(u_{2p+1}) = 2p+1$.

Now let G be a graph such that $H \subset G \subseteq \text{cor}(H)$ obtained from H by adding j new vertices $w_0, w_1, \ldots, w_{j-1}$ (for some integer j with $1 \le j \le 4p+2$) and joining w_i to u_i for $0 \le i \le j-1$. Then the edge coloring $c'_2 : E(G) \to \mathbb{Z}_{4p+2}$ defined by

$$c'_2(e) = \begin{cases} 2p+1 & \text{if } e = u_0w_0 \\ 0 & \text{if } e = u_iw_i \text{ and } 1 \le i \le j-1 \\ c_2(e) & \text{otherwise} \end{cases}$$

is a modular edge coloring of G and so $\chi'_m(G) = 4p+2 < \chi'_m(H)$.

If $G = H \times K_2$, then let H_1 be a copy of H whose edges are colored according to c_1 and H_2 a copy of H whose edges are colored according to c_2. Join the vertex u_0 in H_1 to the vertex u_0 in H_2 by an edge colored $2p+1$. For $1 \le i \le 4p+1$, join the vertex u_i in H_1 to the vertex u_{4p+2-i} in H_2 by an edge colored 0. This produces a modular edge coloring of G using the elements in $\mathbb{Z}_{4p+2}$ and so $\chi'_m(G) = 4p+2 < \chi'_m(H)$.

It is known that $\chi(C_n) = 2$ if n is even, while $\chi(C_n) = 3$ if n is odd. Next we determine $\chi'_m(C_n)$ for every integer $n \ge 3$. For an ordering $v_1, v_2, \ldots, v_n$ of the vertices of a connected graph G of order $n \ge 3$ and a modular edge coloring c of G, define the *color sum sequence* of G with respect to c by

$$s_c:\ s(v_1), s(v_1), \ldots, s(v_n).$$

Theorem 2.4 *For every integer $n \ge 3$,*

$$\chi'_m(C_n) = \begin{cases} 2 & \text{if } n \equiv 0 \pmod 4 \\ 3 & \text{if } n \equiv 1,2,3 \pmod 4. \end{cases}$$

Proof. Since $\chi'_m(C_3) = \chi'_m(K_3) = 3$ by Theorem 2.2, suppose that $n \ge 4$. Let $C_n = (v_1, v_2, \ldots, v_n, v_{n+1} = v_1)$. If $n \equiv 0 \pmod 4$, then let $c_1 : E(C_n) \to \mathbb{Z}_2$ be a 2-edge coloring of C_n such that $c_1(v_iv_{i+1}) = 1$ if and only if $i \equiv 0, 3 \pmod 4$. Then the color sum sequence of c_1 is $1, 0, 1, 0, \ldots, 1, 0$. Thus c_1 is a modular 2-edge coloring and so $\chi'_m(C_n) = 2$ by Proposition 1.3.

If $n \not\equiv 0 \pmod 4$, write $n = 4p + q$, where p is a positive integer and $q \in \{1, 2, 3\}$. Consider an edge coloring $c_2 : E(C_n) \to \mathbb{Z}_3$ such that

$$c_2(v_iv_{i+1}) = \begin{cases} 0 & \text{if } 1 \le i \le 4p \text{ and } i \equiv 1,2 \pmod 4 \\ 1 & \text{if (i) } 1 \le i \le 4p,\ i \equiv 0,3 \pmod 4,\ q \in \{1,2\} \text{ or} \\ & \text{(ii) } q = 3 \text{ and } i = n \\ 2 & \text{otherwise.} \end{cases}$$

The color sum sequence of c_2 is

$$s_{c_2}: \begin{cases} 2,0,1,2,1,0,1,2,1,\ldots,0,1,2,1,0,1,2,0 & \text{if } q = 1 \\ 2,0,1,2,1,0,1,2,1,\ldots,0,1,2,1,0,1,2,0,1 & \text{if } q = 2 \\ 1,0,1,2,1,0,1,2,1,\ldots,0,1,2,1,0,1,2,0,1,0 & \text{if } q = 3. \end{cases}$$

Thus c_2 is a modular 3-edge coloring of C_n. Therefore, $\chi'_m(C_n) \le 3$ if $n \not\equiv 0 \pmod 4$. Furthermore, $\chi'_m(C_n) \ge \chi(C_n) = 3$ if n is odd by Proposition 1.3.

It remains to show that $\chi'_m(C_n) \ge 3$ for $n \equiv 2 \pmod 4$. Let $n = 4p+2$ where p is a positive integer and assume, to the contrary, that there exists a modular 2-edge coloring c' of C_n. Then there are $2p+1$ vertices having color sum 0 and there are $2p+1$ vertices having color sum 1. However, by Observation 2.1

$$0 = 2\sum_{i=1}^{n} c'(v_i v_{i+1}) = \sum_{i=1}^{n} s(v_i) = 2p+1 = 1$$

in $\mathbb{Z}_2$, which is impossible. Hence, $\chi'_m(C_n) \geq 3$ if $n \equiv 2 \pmod 4$, completing the proof. ■

We have now presented two classes of graphs G for which $\chi'_m(G) > \chi(G)$, namely the complete graphs K_n and the cycles C_n where $n \equiv 2 \pmod 4$. That $\chi'_m(G) > \chi(G)$ in both instances is a special case of the following more general result.

Theorem 2.5 *Let G be a graph such that $\chi(G) \equiv 2 \pmod 4$. If each color class in every proper $\chi(G)$-coloring of G consists of an odd number of vertices, then $\chi'_m(G) > \chi(G)$.*

Proof. Suppose that $\chi(G) = 4p+2$ for some nonnegative integer p. If $\chi'_m(G) = \chi(G)$, then there exists a modular $(4p+2)$-edge coloring $c : E(G) \to \mathbb{Z}_{4p+2}$. Let $V_0, V_1, \ldots, V_{4p+1}$ be the resulting color sum classes from the coloring c, where $s(v) = i$ if $v \in V_i$ $(0 \leq i \leq 4p+1)$. By Observation 2.1,

$$\sum_{i=0}^{4p+1} i \cdot |V_i| \equiv 2r \pmod{4p+2}$$

for some integer r with $0 \leq r \leq 2p$. However, this is impossible since each $|V_i|$ is odd. ■

By Theorem 2.5, if $G = K_{n_1,n_2,\ldots,n_k}$ is a complete k-partite graph where $k \equiv 2 \pmod 4$ and each n_i, $1 \leq i \leq k$, is odd, then $\chi'_m(G) > \chi(G)$. In particular, the complete regular k-partite graph $G = K_{r,r,\ldots,r}$ where $k \equiv 2 \pmod 4$ and r is odd has the property that $\chi'_m(G) > \chi(G)$. In fact, $\chi'_m(G) = \chi(G)+1$. To see this, it suffices to show that G has a modular $(k+1)$-edge coloring. Let $V_0, V_2, \ldots, V_{k-1}$ be the partite sets of G. There are r pairwise vertex-disjoint copies $G_1, G_2, \ldots, G_r$ of K_k in G, where $V(G_i) = \{v_0, v_1, \ldots, v_{k-1}\}$ with $v_j \in V_j$ for $1 \leq i \leq r$ and $0 \leq j \leq k-1$. Coloring the edges of each G_i the same as the edges of K_k described earlier and assigning 0 to all other edges of G produces a modular $(k+1)$-edge coloring of G in which $s(v) = j$ if $v \in V_j$ for $0 \leq j \leq k-1$. Thus $\chi'_m(G) = \chi(G)+1$.

We make another observation here. Note that $H = K_k$ is a subgraph of the complete regular k-partite graph $G = K_{r,r,\ldots,r}$. If $k \equiv 2 \pmod 4$ and r is odd, then $\chi'_m(H) > \chi(H)$ and $\chi'_m(G) > \chi(G)$ by Theorems 2.2 and 2.5, while $\chi'_m(H) = \chi'_m(G)$.

3 Modular Chromatic Indices of Bipartite Graphs

For an arbitrary bipartite graph G, the possible values of the modular chromatic number of G is not known. In fact, it is not even known whether there is a constant C such that the modular chromatic number of every connected bipartite graph is bounded above by C. This, however, is not the case for the modular chromatic index of a bipartite graph, as we show in this section. We first determine the modular chromatic index of a path.

Theorem 3.1 *For each integer $n \geq 3$,*

$$\chi'_m(P_n) = \begin{cases} 2 & \text{if } n \equiv 0,1,3 \pmod 4 \\ 3 & \text{if } n \equiv 2 \pmod 4. \end{cases}$$

Proof. Let $P_n = (v_1, v_2, \ldots, v_n)$. For $n \equiv 0 \pmod 4$ or $n \equiv 3 \pmod 4$, define the 2-edge coloring $c_1 : E(P_n) \to \mathbb{Z}_2$ such that $c_1(v_iv_{i+1}) = 1$ if and only if $i \equiv 1, 2 \pmod 4$. Then the color sum sequence of c_1 is

$$s_{c_1} : \begin{cases} 1,0,1,0,\ldots,1,0 & \text{if } n \equiv 0 \pmod 4 \\ 1,0,1,0,\ldots,1,0,1 & \text{if } n \equiv 3 \pmod 4. \end{cases}$$

For $n \equiv 1 \pmod 4$, define the 2-edge coloring $c_2 : E(P_n) \to \mathbb{Z}_2$ such that $c_2(v_iv_{i+1}) = 1$ if and only if $i \equiv 2, 3 \pmod 4$. Then the color sum sequence of c_2 is $0, 1, 0, 1, 0, \ldots, 1, 0$. Hence c_1 and c_2 are modular 2-edge colorings and so $\chi'_m(P_n) = 2$ if $n \equiv 0, 1, 3 \pmod 4$.

For $n \equiv 2 \pmod 4$, define the 3-edge coloring $c_3 : E(P_n) \to \mathbb{Z}_3$ by

$$c_3(v_iv_{i+1}) = \begin{cases} 0 & \text{if } i = n-1 \\ 1 & \text{if } 1 \leq i \leq n-2 \text{ and } i \equiv 1,2 \pmod 4 \\ 2 & \text{if } 1 \leq i \leq n-2 \text{ and } i \equiv 0,3 \pmod 4. \end{cases}$$

Then the color sum sequence of c_3 is

$$1,2,0,1,0,2,0,1,0,\ldots,2,0,1,0,2,0,1,2,0.$$

Thus c_3 is a modular 3-edge coloring and so $\chi'_m(P_n) \leq 3$. By Theorem 2.5, $\chi'_m(P_n) = 3$. ■

Suppose that G is a connected bipartite graph of order $n \geq 3$ and let U and W be the partite sets of G with $|U| = r$ and $|W| = s$. If $\chi'_m(G) = 2$, then at least one of r and s must be even by Theorem 2.5. Let us next determine the modular chromatic indices of compete bipartite graphs.

Proposition 3.2 *For positive integers r and s where $r + s \geq 3$,*

$$\chi'_m(K_{r,s}) = \begin{cases} 3 & \text{if } r \text{ and } s \text{ are odd} \\ 2 & \text{otherwise.} \end{cases}$$

Proof. We may assume that $1 \leq r \leq s$. First suppose that $r = 1$ and $s \geq 2$. If s is even, then the coloring assigning the color 1 to every edge is a modular 2-edge coloring of $K_{1,s}$. Hence, $\chi'_m(K_{1,s}) = 2$ in this case. Suppose next that s is odd. By Theorem 2.5, $\chi'_m(K_{1,s}) \geq 3$. On the other hand, the coloring assigning the color 1 to two edges and the color 0 to the remaining $n-3$ edges is a modular 3-edge coloring of $K_{1,s}$. Thus the result holds for $r = 1$.

Next suppose that $r, s \geq 2$. By Proposition 1.3, $\chi'_m(K_{r,s}) \geq \chi(K_{r,s}) = 2$. Let U and W be the partite sets of $K_{r,s}$ with $|U| = r$ and $|W| = s$. If at least one of r and s, say r, is even, then let $w \in W$ and consider a 2-edge coloring assigning the color 1 to an edge e if and only if e is incident with w. Then this is a modular 2-edge coloring of $K_{r,s}$ and so $\chi'_m(K_{r,s}) = 2$.

If both r and s are odd, then $\chi'_m(K_{r,s}) \geq 3$. Write $r = 6p+q \geq 3$, where p is a nonnegative integer and $q \in \{1, 3, 5\}$, and let $w \in W$. If $q \neq 1$, then the edge coloring c_1 given by $c_1(e) = 1$ if e is incident with w and $c_1(e) = 0$ otherwise is a modular 3-edge coloring of $K_{r,s}$. If $q = 1$, then $r \geq 7$. Let $U = \{u_1, u_2, \ldots, u_r\}$ and observe that the edge coloring c_2 given by

$$c_2(e) = \begin{cases} 2 & \text{if } e \in \{u_1w, u_2w\} \\ 1 & \text{if } e = u_iw \ (3 \leq i \leq r) \\ 0 & \text{otherwise} \end{cases}$$

is a modular 3-edge coloring of $K_{r,s}$. ∎

We now turn our attention to trees and show that the modular chromatic index of every tree of order 3 or more is either 2 or 3. Moreover, we characterize all trees whose modular chromatic index is 2 (or is 3).

Theorem 3.3 *Let T be a tree of order $r + s \geq 3$ whose partite sets have orders r and s. Then*

$$\chi'_m(T) = \begin{cases} 3 & \text{if } r \text{ and } s \text{ are odd} \\ 2 & \text{otherwise.} \end{cases}$$

Proof. We first show that every nontrivial tree of odd order is modular 2-edge colorable. Assume, to the contrary, that there exists a tree of odd order whose modular chromatic index is greater than 2. Let T be such a tree of the minimum order $r + s$ and suppose that U and W are the partite sets of T with $|U| = r$ and $|W| = s$. It follows by Proposition 3.2 that T is not a star and so we may assume that $r + s \geq 5$ where $r \geq 2$ is even and $s \geq 3$ is odd. Also, since T is not a path by Theorem 3.1, there are at least three end-vertices, implying that there are two end-vertices x and y belonging to the same partite set. Let T' be the tree obtained from T by deleting x and y. Therefore, $\chi'_m(T') = 2$ by assumption and so let c' be a modular 2-edge coloring of T'. Furthermore, let $U' \subseteq U$ and $W' \subseteq W$

be the partite sets of T' and observe that $|U'|$ is even while $|W'|$ is odd. Hence, c' assigns colors to the edges of T' so that $s_{c'}(v) = 1$ if and only if $v \in U'$ by Observation 2.1. If $x, y \in W$, then the edge coloring c of T given by $c(e) = c'(e)$ if $e \in E(T')$ and $c(e) = 0$ otherwise is a modular 2-edge coloring of T, which contradicts our assumption. Thus, we may assume that $x, y \in U$. Let $w_1 \in N(x)$ and $w_2 \in N(y)$ and consider the $w_1 - w_2$ path P in T'. (If $d(x, y) = 2$, then $w_1 = w_2$ and so $E(P) = \emptyset$.) We define an edge coloring c of T as follows:

$$c(e) = \begin{cases} c'(e) + 1 & \text{if } e \in E(P) \\ 1 & \text{if } e \in \{xw_1, yw_2\} \\ c'(e) & \text{otherwise.} \end{cases}$$

We verify that c is a modular 2-edge coloring of T. If $v \in V(T') - V(P)$, then $s_c(v) = s_{c'}(v)$; while if $v \in V(P)$, then $s_c(v) = s_{c'}(v) + 2 = s_{c'}(v)$. Hence, $s_c(v) = s_{c'}(v)$ for every $v \in V(T')$, that is, $s_c(v) = 1$ if $v \in U'$ and $s_c(v) = 0$ if $v \in W'$. Since $s_c(x) = s_c(y) = 1$, this is indeed a modular 2-edge coloring of T, which is again impossible. Hence, such a tree T does not exist and so $\chi'_m(T) = 2$ if $r + s$ is odd.

Next assume that $r + s \geq 4$ is even. If both r and s are even, then it can be verified that T is modular 2-edge colorable by an argument similar to the one used in the case when $r + s$ is odd. Thus we may assume that both r and s are odd. Let $r + s = 2k$ where $k \geq 2$. We need only verify that $\chi'_m(T) \leq 3$ by Theorem 2.5. We proceed by induction on k. For $k = 2$, $T = K_{1,3}$ and the result immediately follows by Proposition 3.2. Suppose that for some $k \geq 2$ every tree of order $2k$ that is a spanning subgraph of $K_{r,2k-r}$ for some odd integer r ($1 \leq r \leq 2k - 1$) is modular 3-edge colorable. Let T be a tree of order $2(k+1)$ with $T \subseteq K_{r,2(k+1)-r}$ for some odd integer r with $1 \leq r \leq 2(k+1) - 1$. Since T is not a star, let U and W be the partite sets of T such that $|U| = r \geq 3$ and $|W| = 2(k+1) - r \geq 3$. Also, since T is not a path, there exist at least three end-vertices in T, two of which belong to the same partite set. We may assume that x and y are end-vertices both belonging to U. Also, let w_1 and w_2 be the vertices in W such that $xw_1, yw_2 \in E(T)$. Consider the tree T' of order $2k$ obtained from T by deleting x and y. Then the sets $U' = U - \{x, y\}$ and $W' = W$ are the partite sets of T' and, furthermore, both $|U'|$ and $|W'|$ are odd. Hence, $\chi'_m(T') = 3$ and so let $c' : E(T') \to \mathbb{Z}_3$ be a modular 3-edge coloring of T'. We consider the following three cases.

Case 1. $0 \in \{s_{c'}(v) : v \in U'\}$. Then the edge coloring c given by $c(xw_1) = c(yw_2) = 0$ and $c(e) = c'(e)$ for every $e \in E(T')$ is a modular 3-edge coloring of T.

Case 2. $\{s_{c'}(v) : v \in W'\} = \{0\}$. Note that $d(x, y) = d$ is a positive even integer. Let $P = (w_1 = v_1, v_2, \ldots, v_{d-1} = w_2)$ be the $w_1 - w_2$ path in T'. (If $d = 2$, then $w_1 = w_2$ and $E(P) = \emptyset$.) Therefore, $v_i \in W'$ if i is odd and $v_i \in U'$ if i is even. Define an edge coloring c of T by

$$c(e) = \begin{cases} c'(e) + 1 & \text{if } e = v_i v_{i+1} \in E(P) \text{ and } i \text{ is odd} \\ c'(e) + 2 & \text{if } e = v_i v_{i+1} \in E(P) \text{ and } i \text{ is even} \\ 2 & \text{if } e = xw_1 \\ 1 & \text{if } e = yw_2 \\ c'(e) & \text{otherwise.} \end{cases}$$

To verify that c is a modular 3-edge coloring of T, first observe that $s_c(v) = s_{c'}(v)$ for every $v \in V(T') - V(P)$. Also, $s_c(v) = s_{c'}(v) + 3 = s_{c'}(v)$ for every $v \in V(P)$. In particular, $s_c(w_1) = s_c(w_2) = 0$. Thus, $s_c(x) = 1 \neq s_c(w_1)$ and $s_c(y) = 2 \neq s_c(w_2)$, implying that c is indeed a modular 3-edge coloring of T.

Case 3. $\{s_{c'}(v) : v \in U\} = \{A\}$ *and* $B \in \{s_{c'}(v) : v \in W\}$ *where* $\{A, B\} = \{1, 2\}$. We consider three subcases.

Subcase 3.1. $d(x, y) = d \geq 4$. If $s_{c'}(w_1) = s_{c'}(w_2) = B$, then let c be an edge coloring of T such that $c(xw_1) = c(yw_2) = A$ and $c(e) = c'(e)$ for every $e \in E(T')$ and observe that c is a modular 3-edge coloring of T'.

If $s_{c'}(w_1) = 0$ or $s_{c'}(w_2) = 0$, say the former, then let $P = (w_1 = v_1, v_2, \ldots, v_{d-1} = w_2)$ be the $w_1 - w_2$ path in T' and define an edge coloring c of T by

$$c(e) = \begin{cases} c'(e) + A & \text{if } e = v_i v_{i+1} \in E(P) \text{ and } i \text{ is odd} \\ c'(e) + B & \text{if } e = v_i v_{i+1} \in E(P) \text{ and } i \text{ is even} \\ A & \text{if } e \in \{xw_1, yw_2\} \\ c'(e) & \text{otherwise.} \end{cases}$$

Then

$$s_c(v) = \begin{cases} A & \text{if } v \in \{x, y\} \\ 2A = B & \text{if } v = w_1 \\ s_{c'}(v) & \text{otherwise} \end{cases}$$

and it is straightforward to verify that this is a modular 3-edge coloring of T.

Subcase 3.2. $d(x, y) = 2$. Let $w_1 = w_2 = w$. If $s_{c'}(w) = 0$, then let c be an edge coloring such that $c(xw) = c(yw) = A$ and $c(e) = c'(e)$ for every $e \in E(T')$ and observe that this is a modular 3-edge coloring of T.

Hence, suppose finally that $s_{c'}(w) = B$. Since T is not a star, there exists an end-vertex z in T such that $d(x, z) \geq 3$. Let $P = (w = v_1, v_2, \ldots, v_d = z)$ be the $w - z$ path in T', where $d = d(x, z)$.

Subcase 3.2.1. *d is odd.* Then $z \in W$ and so $s_{c'}(z) \in \{0, B\}$. Then the edge coloring c defined by

$$c(e) = \begin{cases} c'(e) - s_{c'}(z) + A & \text{if } e = v_i v_{i+1} \in E(P) \text{ and } i \text{ is odd} \\ c'(e) + s_{c'}(z) + B & \text{if } e = v_i v_{i+1} \in E(P) \text{ and } i \text{ is even} \\ A & \text{if } e \in \{xw, yw\} \\ c'(e) & \text{otherwise} \end{cases}$$

is a modular 3-edge coloring of T, since $s_{c'}(z) \in \{0, B\}$ and

$$s_c(v) = \begin{cases} A & \text{if } v \in \{x, y\} \\ 2s_{c'}(z) + B \in \{0, B\} & \text{if } v = z \\ B - s_{c'}(z) \in \{0, B\} & \text{if } v = w \\ s_{c'}(v) & \text{otherwise.} \end{cases}$$

Subcase 3.2.2. *d is even.* Then $z \in U$ and so $s_{c'}(z) = A$. Let w_3 be the neighbor of z in T, that is, $w_3 = v_{d-1}$. Then consider the edge coloring c defined by

$$c(e) = \begin{cases} c'(e) - s_{c'}(w_3) + A & \text{if } e = v_i v_{i+1} \in E(P) \text{ and } i \text{ is odd and } i \neq d-1 \\ c'(e) + s_{c'}(w_3) + B & \text{if } e = v_i v_{i+1} \in E(P) \text{ and } i \text{ is even} \\ A & \text{if } e \in \{xw, yw, zw_3\} \\ c'(e) & \text{otherwise} \end{cases}$$

and one can verify that c is a modular 3-edge coloring of T. ■

With the aid of Theorem 3.3, we are now able to classify all connected bipartite graphs according to their modular chromatic indices.

Theorem 3.4 *If G is a connected bipartite graph of order $r + s \geq 3$ such that $G \subseteq K_{r,s}$, then*

$$\chi'_m(G) = \begin{cases} 3 & \textit{if } r \textit{ and } s \textit{ are odd} \\ 2 & \textit{otherwise.} \end{cases}$$

Proof. If G is a tree, then the result clearly holds by Theorem 3.3. If G is not a tree, then let T be a spanning tree of G and observe that $T \subseteq K_{r,s}$. Let c_T be a modular edge coloring of T and define an edge coloring c of G by $c(e) = c_T(e)$ if $e \in E(T)$ and $c(e) = 0$ otherwise. Then $s_c(v) = s_{c_T}(v)$ for every vertex v in G. Therefore, every modular edge coloring of T induces a modular edge coloring of G using the same number of colors, which implies that $\chi'_m(G) \leq \chi'_m(T)$. The result now follows by Theorems 2.5 and 3.3. ■

If H is any connected bipartite graph each of whose partite sets contains an odd number of vertices, then $\chi'_m(H) = 3 > \chi(H)$ by Theorem 3.4. If G

is a graph such that $H \subset G \subseteq \text{cor}(H)$ with an odd number of pendant edges or $G = H \times K_2$, then G is also bipartite and $\chi'_m(G) = 2$ by Theorem 3.4. This provides us with another well-known class of graphs G containing a subgraph H such that $\chi'_m(H) > \chi'_m(G)$.

4 Open Questions

For every graph G encountered in this paper, we have seen that either $\chi'_m(G) = \chi(G)$ or $\chi'_m(G) = \chi(G) + 1$.

Problem 4.1 *Is it true that* $\chi(G) \le \chi'_m(G) \le \chi(G)+1$ *for every connected graph* G *of order* 3 *or more?*

For every graph G encountered in this paper for which $\chi'_m(G) = \chi(G) + 1$, the order of G is even and $\chi(G) \equiv 2 \pmod 4$.

Problem 4.2 *If* G *is a connected graph of order* 3 *or more with* $\chi'_m(G) = \chi(G) + 1$*, then does* G *have even order and* $\chi(G) \equiv 2 \pmod 4$*?*

Problem 4.3 *If* G *is a connected graph of order* 3 *or more with* $\chi'_m(G) > \chi(G)$*, then must every proper* $\chi(G)$*-coloring of* G *result in color classes of odd size?*

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