A REMARK ON THE EDGE-BANDWIDTH OF TORI

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ABSTRACT. The edge-bandwidth of a graph G is the smallest number b for which there is an injective labeling of E(G) with integers such that the difference between the labels at any adjacent edges is at most b. The edge-bandwidth of a torus (a product of two cycles) has been computed within an additive error of 5. Here we improve the upper bound, reducing the error to 3.

1. INTRODUCTION

Let G be a graph with n vertices. Given a bijection $\eta : V(G) \to [n]$, let $B(\eta)$ be the maximal difference between $\eta(u)$ and $\eta(v)$ for adjacent vertices u and v of G. The *bandwidth* $B(G)$ of G is the minimal value of $B(\eta)$, taken over all such bijections η.

This classical problem was introduced by Harary (Problem 16 on p.167 in [6]) and Harper [8]. It has been extensively studied due to its connections to isoperimetric inequalities [4], VLSI design and other layout problems [5], multicasting [3], multi-channel transmission of data with noise [2], graph searching [7], and other.

The *edge-bandwidth* $B'(G)$ of G is the bandwidth of the line graph of G. In other words, it is the smallest integer k for which there is a bijection between $E(G)$ and $\{1, \dots, e(G)\}$ such that the difference between the labels at any two adjacent edges is at most k. This parameter was introduced by Hwang and Lagarias [9].

Let us consider $C_m \oplus C_n$, the $m \times n$-torus, where C_n denotes the cycle of order n and $\oplus$ denotes the Cartesian product of graphs. The bandwidth of tori was studied by Li, Tao, and Shen [10] who computed $B(C_m \oplus C_n)$ for all m, n. Balogh, Mubayi, and Pluhár [1] considered the edge bandwidth of the torus $C_n \oplus C_n$ and established the following bounds:

$$(1) \qquad 4n - 2\sqrt{2n} - 1 \leq B'(C_n \oplus C_n) \leq 4n, \quad n \geq 3.$$

In [11], the gap in (1) has been reduced by improving the lower bound so that for any $m \geq n \geq 3$, we have

$$(2) \qquad 4n - 5 \leq B'(C_m \oplus C_n) \leq 4n.$$

The lower bound follows from the following more precise theorem proved implicitly in [11].

Theorem 1. *Assume that m and n are integers with $m \geq n \geq 3$. Then*

$$B'(C_m \oplus C_n) \geq 4n - 4 + \frac{2m - 2n - 1}{\ell}$$

for some integer ℓ with $1 \leq \ell \leq m/2$.

Note that Theorem 1 implies that for m strictly larger than n

$$B'(C_m \oplus C_n) \geq 4n - \left\lfloor \frac{4n+2}{m} \right\rfloor.$$

In this note, we show that the upper bound in (2) can be improved as follows.

Theorem 2. *Assume that m and n are integers with $m \geq n \geq 3$.*

(a) *If $m \leq 2n + 2$, then $B'(C_m \oplus C_n) \leq 4n - 1$.*
(b) *If n is even, and $m \leq n + 2$, or n is odd and $m \leq n + 1$, then $B'(C_m \oplus C_n) \leq 4n - 2$.*
(c) *If $m = n \in \{3, 4, 6\}$, then $B'(C_m \oplus C_n) \leq 4n - 3$.*

Combining Theorems 1 and 2, we obtain the following bounds.

Corollary 3. *Assume that m and n are integers with $m \geq n \geq 3$.*

(1) *If $m \geq 4n + 3$, then $B'(C_m \oplus C_n) = 4n$.*
(2) *If $2n + 3 \leq m \leq 4n + 2$, then $4n - 1 \leq B'(C_m \oplus C_n) \leq 4n$.*
(3) *If $m = 2n + 2$, then $B'(C_m \oplus C_n) = 4n - 1$.*
(4) *If $\frac{4n+2}{3} < m \leq 2n + 1$, then $4n - 2 \leq B'(C_m \oplus C_n) \leq 4n - 1$.*
(5) *If $\alpha_n - 1 \leq m \leq \frac{4n+2}{3}$, then $4n - 3 \leq B'(C_m \oplus C_n) \leq 4n - 1$.*
(6) *If $n + 1 \leq m \leq \alpha_n$, then $4n - 3 \leq B'(C_m \oplus C_n) \leq 4n - 2$.*
(7) *If $m = n \in \{3, 4, 6\}$, then $4n - 5 \leq B'(C_m \oplus C_n) \leq 4n - 3$.*
(8) *If $m = n \notin \{3, 4, 6\}$, then $4n - 5 \leq B'(C_m \oplus C_n) \leq 4n - 2$.*

Where α_n is equal to $n + 2$ if n is even and to $n + 1$ for n odd.

2. Notation

For a positive integer n, the set $\{1, 2, \ldots, n\}$ will be denoted by $[n]$. Given graphs F and H of orders m and n respectively, we will assume that $V(F) = [m]$, $V(H) = [n]$, and the *Cartesian product* $G = F \oplus H$ has the vertex set

$$V(G) = \{(i, j) : 1 \leq i \leq m, \ 1 \leq j \leq n\}$$

and the edge set

$$\{r_{i,D} : 1 \le i \le m,\ D \in E(H)\} \cup \{c_{D,j} : D \in E(F),\ 1 \le j \le n\}$$

with $r_{i,xy}$ incident to (i,x) and (i,y) and $c_{xy,j}$ incident to (x,j) and (y,j). (Here we abbreviate $\{x,y\}$ to xy.) The edges of the form $r_{i,D}$ are called *horizontal* and the edges $c_{D,j}$ are *vertical.*

For a cycle C_n, we assume that it traverses its vertex set in the natural order $1,2,\dots,n-1,n,1$. If F and H are cycles, then for any $i \in [m]$ and $j \in [n-1]$ the edge $r_{i,\{j,j+1\}}$ will be denoted shortly by $r_{i,j}$ and $r_{i,n} = r_{i,\{n,1\}}$. Likewise we define $c_{i,j}$ for $i \in [m]$ and $j \in [n]$. For $i = 1,\dots,m$, the *i-th row* is

$$R_i = \{r_{i,j} : j \in [n]\},$$

and the *i-th quasi-row* is

$$Q_i = \{c_{i,j} : j \in [n]\}.$$

For example, Figure 1 illustrates our notation for the torus $C_3 \oplus C_4$, where the edges on the right and the bottom are assumed to loop around and connect to the corresponding vertices on the left and the top.

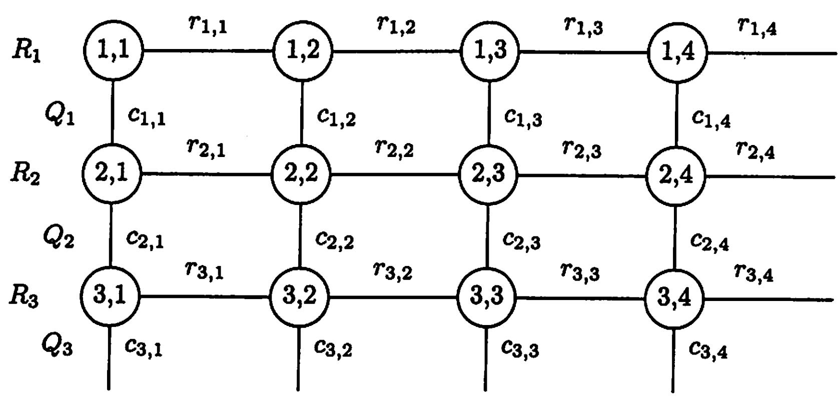

FIGURE 1. The torus $C_3 \oplus C_4$.

3. PROOF OF THEOREM 2.

Let $a_1, a_2, \dots, a_{m-1}$ be a sequence of nonnegative integers with $a_i \le n$ for every i, such that

$$\sum_{i=1}^{m-1} a_i = (m-2)\,n.$$

For each k with $0 \le k \le m-1$, set

$$s_k = \sum_{i=1}^{k} a_i.$$

In particular, $s_0 = 0$ and $s_{m-1} = (m-2)\,n$. Let $A_1, A_2, \ldots, A_{m-1}$ be the disjoint subsets of $\{2n+1, \ldots, 2mn-2n\}$, with A_i consisting of a_i consecutive integers, with blocks of n consecutive integers between them, that is we define

$$A_i = \{(i+1)\,n + s_{i-1} + 1, \ldots, (i+1)\,n + s_i\},$$

$i = 1, 2, \ldots, m-1$. In particular,

$$\begin{aligned} A_1 &= \{2n+1, 2n+2, \ldots, 2n+a_1\}, \\ A_2 &= \{3n+a_1+1, \ldots, 3n+s_2\}, \end{aligned}$$

and

$$A_{m-1} = \{(2m-2)\,n - a_{m-1} + 1, \ldots, (2m-2)\,n\}.$$

Denote by D_1 the interval $\{n+1, \ldots, 2n\}$ consisting of n integers that precede A_1, by D_m the interval $\{(m-2)\,n+1, (m-1)\,n\}$ with n integers that follow A_{m-1}, and by D_i $(i = 2, \ldots, m-1)$ the interval consisting of n integers that are between A_{i-1} and A_i.

We define a labeling $\eta : E\,(C_m \oplus C_n) \to [2mn]$ as a union $\eta = \eta_1 \cup \eta_2$ of two functions with disjoint domains, where

$$\eta_1 : R_1 \cup R_{\lceil m/2 \rceil + 1} \cup \bigcup_{i=1}^{m} Q_i \to [2mn] \setminus \bigcup_{i=1}^{m-1} A_i$$

and

$$\eta_2 : \bigcup_{i \in [m] \setminus \{1, \lceil m/2 \rceil + 1\}} R_i \to \bigcup_{i=1}^{m-1} A_i.$$

The function η_1 is specified as follows. The edges of the row R_1 are assigned the smallest n labels, that is we define

$$\eta_1\,(r_{1,j}) = j,$$

for every $j \in [n]$. The edges of $R_{\lceil m/2 \rceil + 1}$ get the largest n labels, however we shift the labeling to start with $r_{\lceil n/2 \rceil + 1, n}$, that is we define

$$\eta_1\left(r_{\lceil n/2 \rceil + 1, n}\right) = (2m-1)\,n + 1,$$

and

$$\eta_1\left(r_{\lceil n/2 \rceil + 1, j}\right) = (2m-1)\,n + j + 1,$$

for $j = 1, 2, \ldots, n-1$.

Let σ be the permutation of $[m]$ defined by

$$\sigma(i) = \begin{cases} \dfrac{i+1}{2} & \text{if } i \text{ is odd,} \\ m - \dfrac{i-2}{2} & \text{if } i \text{ is even.} \end{cases}$$

The edges of the quasi-row $Q_{\sigma(i)}$ are labeled with the elements of D_i. Namely, for each $i \in [m]$ and $j \in [n]$ we define

$$\eta_1\left(c_{\sigma(i),j}\right) = in + s_{i-1} + j.$$

In particular,

$$\begin{aligned} \eta_1\left(c_{\sigma(1),j}\right) &= n + j, \\ \eta_1\left(c_{\sigma(2),j}\right) &= n + a_1 + j, \end{aligned}$$

and

$$\eta_1\left(c_{\sigma(m),j}\right) = (2m-2)\,n + j,$$

for every $j \in [n]$. Figure 2 shows the labeling η_1 for m even, and Figure 3 for m odd.

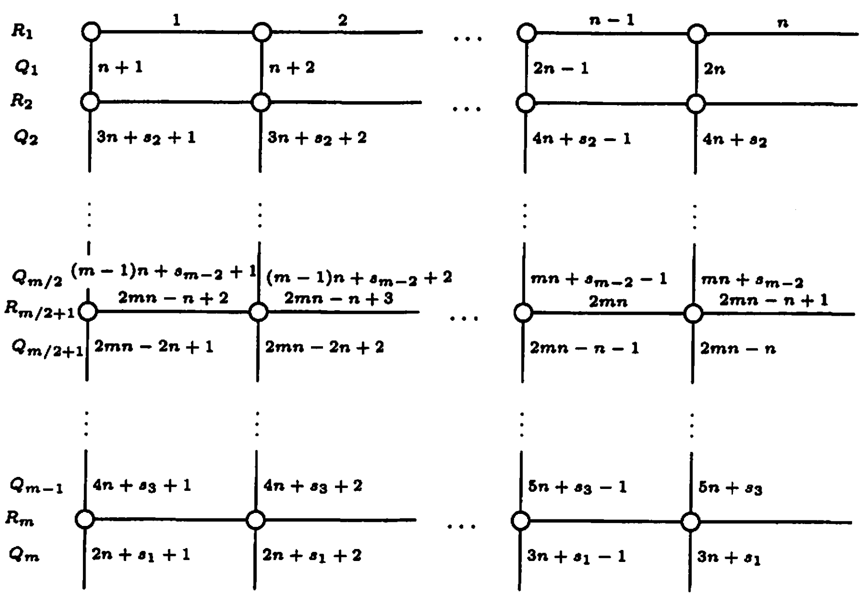

FIGURE 2. The labeling η_1 for m even.

Example 4. *Let $m = n = 6$, $a_1 = a_3 = a_5 = 6$, and $a_2 = a_4 = 3$. Figure 4 shows the resulting labeling η_1.*

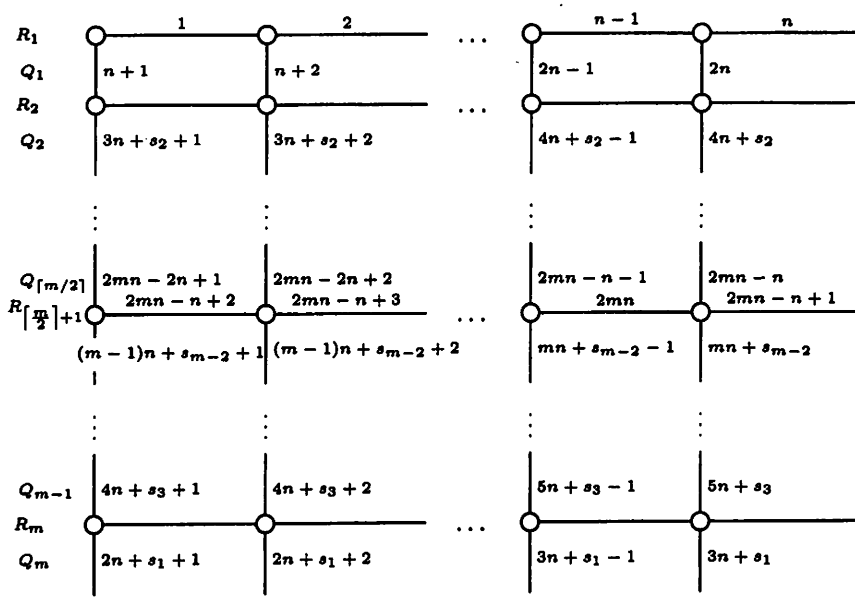

FIGURE 3. The labeling η_1 for m odd.

Lemma 5. *Let η_1^* be the maximal difference $|\eta_1(e) - \eta_1(e')|$ with the edges e, e' being adjacent. Then*

(3) $\eta_1^* = 2n + \max\{1 + a_1, a_1 + a_2, a_2 + a_3, \dots, a_{m-2} + a_{m-1}, a_{m-1} + 1\}$.

Proof. Note that if e is an edge of the quasi-row Q_1 and e' is the adjacent edge of the quasi-row Q_m, then

$$|\eta_1(e) - \eta_1(e')| = n + a_1.$$

If e is an edge of $Q_{\lceil m/2\rceil}$ and e' is the adjacent edge of $Q_{\lceil m/2\rceil+1}$, then

$$|\eta_1(e) - \eta_1(e')| = n + a_{m-1}.$$

Otherwise, if the edges e, e' are in quasi-rows and are adjacent, then one of them is in $Q_{\sigma(i)}$ and the other is in $Q_{\sigma(i+2)}$ for some $i = 1, 2, \dots, m-2$. Then

$$|\eta_1(e) - \eta_1(e')| = 2n + a_i + a_{i+1}.$$

If e and e' are both in R_1 or both in $R_{\lceil m/2\rceil+1}$, then

$$|\eta_1(e) - \eta_1(e')| \le n - 1.$$

If e is in R_1 and e' is in Q_1 or Q_m, then

$$|\eta_1(e) - \eta_1(e')| \le 2n + a_1 + 1.$$

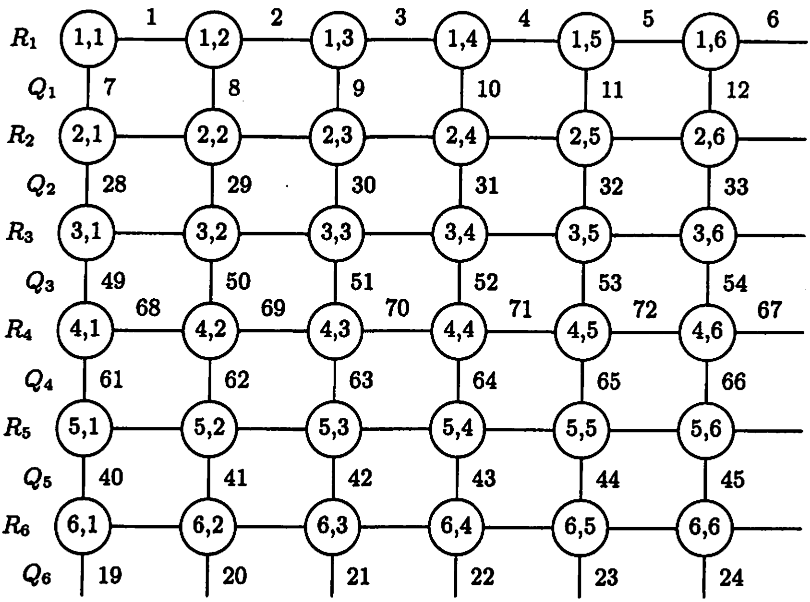

FIGURE 4. The labeling η_1 of $C_6 \oplus C_6$.

If e is in $R_{\lceil m/2 \rceil+1}$ and e' is in $Q_{\lceil m/2 \rceil}$ or $Q_{\lceil m/2 \rceil+1}$, then

$$|\eta_1(e) - \eta_1(e')| \leq 2n + a_{m-1} + 1.$$

It follows that the equality (3) holds. □

Now we define η_2. For every $i = 1, 2, \ldots, m-2$, let $\tau(i) = \max\{\sigma(i), \sigma(i+2)\}$, that is, let

$$\tau(i) = \begin{cases} \dfrac{i+3}{2} & \text{if } i \text{ is odd,} \\ m - \dfrac{i-2}{2} & \text{if } i \text{ is even.} \end{cases}$$

Note that, for every $i = 1, 2, \ldots, m-2$, the quasi-rows $Q_{\sigma(i)}$ and $Q_{\sigma(i+2)}$ are adjacent in the torus with the row $R_{\tau(i)}$ between them. Note also that

$$\{\tau(1), \ldots, \tau(m-2)\} = [m] \setminus \{1, \lceil m/2 \rceil + 1\},$$

so the domain of η_2 is the union $R_{\tau(1)} \cup \cdots \cup R_{\tau(m-2)}$. Let $B_1, \ldots, B_{m-2}$ be disjoint sets of cardinality n such that

$$\bigcup_{k=1}^{m-2} B_k = \bigcup_{i=1}^{m-1} A_i,$$

and $b_i < b_j$ whenever $0 \leq i < j \leq m-2$, $b_i \in B_i$, and $b_j \in B_j$. Let η_2 be any bijective function

$$\eta_2 : \bigcup_{k=1}^{m-2} R_{\tau(k)} \to \bigcup_{i=1}^{m-1} A_i$$

so that $\eta_2(e) \in B_k$ for every $e \in R_{\tau(k)}$. Figure 5 shows the labeling η_2 in Example 4.

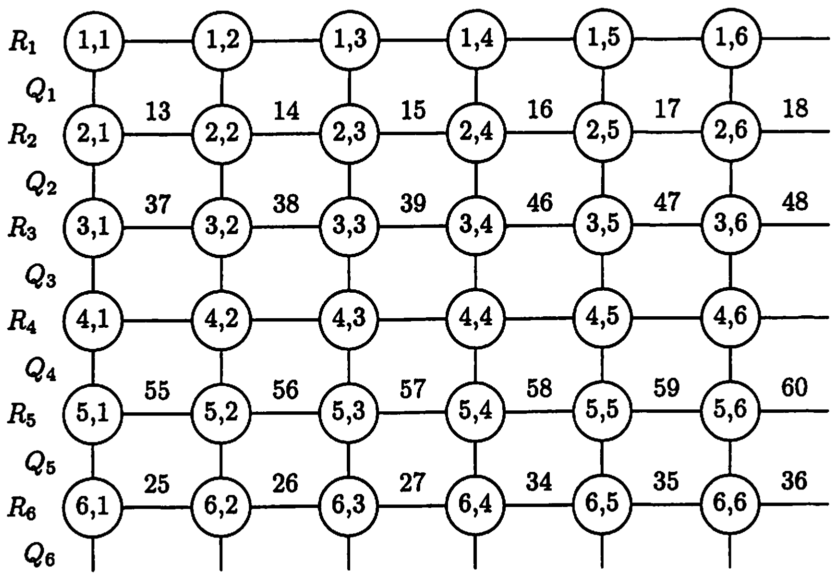

FIGURE 5. The labeling η_2 of $C_6 \oplus C_6$.

Lemma 6. *Let η^* be the maximal difference $|\eta(e) - \eta(e')|$ with the edges e, e' being adjacent. Then $\eta^* = \eta_1^*$.*

Proof. Since $0 \leq a_i \leq n$ for every $i \in [m-1]$, and

$$\sum_{i=1}^{m-1} a_i = n\,(m-2)\,,$$

it follows that

$$B_k \subseteq A_k \cup A_{k+1},$$

for every $k \in [m-2]$. Therefore any two elements of B_k differ by at most $2n-1$, implying that if e and e' are in the same row R_i with $i \in [m] \setminus \{1, \lceil m/2 \rceil + 1\}$, then

$$|\eta(e) - \eta(e')| \leq 2n - 1 \leq \eta_1^*.$$

Assume that e is in $R_{\tau(k)}$ for some $k \in [m-2]$, and e' is a vertical edge adjacent to e. Let e'' be the vertical edge adjacent to both e and e'. Without loss of generality, $e' \in Q_{\sigma(k)}$ and $e'' \in Q_{\sigma(k+2)}$. Then $\eta(e) \in B_k$, $\eta(e') \in D_k$, and $\eta(e'') \in D_{k+2}$, which implies that

$$\eta(e') < \eta(e) < \eta(e''),$$

and consequently

$$|\eta(e) - \eta(e')| < |\eta(e') - \eta(e'')| \leq \eta_1^*.$$

□

Now we are ready to complete the proof of Theorem 2. Let m and n be integers with $m \geq n \geq 3$. Assume that $m \leq 2n+2$. Then there exist nonnegative integers a_i, $i = 1, 2, \ldots, m-1$, such that $a_i \leq n$ for i odd and $a_i \leq n-1$ for i even and

$$\sum_{i=1}^{m-1} a_i = n\,(m-2).$$

Then

$$\max\{1+a_1, a_1+a_2, a_2+a_3, \ldots, a_{m-2}+a_{m-1}, a_{m-1}+1\} \leq 2n-1,$$

and it follows from Lemmas 5 and 6 that

$$B'(C_m \oplus C_n) \leq 4n-1.$$

If $m \leq n+2$ with n even or $m \leq n+1$ with n odd, then there exist nonnegative integers a_i, $i = 1, 2, \ldots, m-1$, such that $a_i \leq n$ for i odd and $a_i \leq n-2$ for i even and

$$\sum_{i=1}^{m-1} a_i = n\,(m-2),$$

which gives

$$\max\{1+a_1, a_1+a_2, a_2+a_3, \ldots, a_{m-2}+a_{m-1}, a_{m-1}+1\} \leq 2n-2,$$

and implies that

$$B'(C_m \oplus C_n) \leq 4n-2.$$

If $m = n = 3$, then let $a_1 = 2$ and $a_2 = 1$, see Figure 6. If $m = n = 4$, then let $a_1 = a_3 = 4$ and $a_2 = 0$, see Figure 7. If $m = n = 6$, then let $a_1 = a_3 = a_5 = 6$ and $a_2 = a_4 = 3$, see Example 4 (Figures 4 and 5).
In all these cases

$$\max\{1+a_1, a_1+a_2, a_2+a_3, \ldots, a_{m-2}+a_{m-1}, a_{m-1}+1\} = 2n-3,$$

implying that

$$B'(C_m \oplus C_n) \leq 4n-3.$$

Thus the proof is complete.

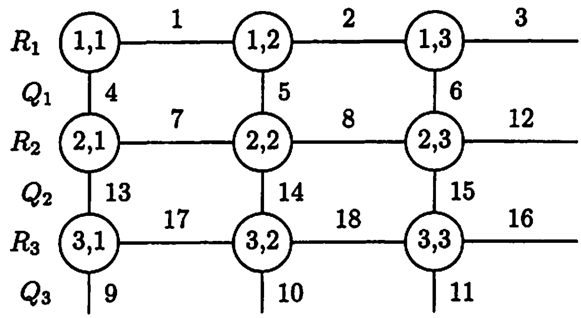

FIGURE 6. The labeling η of $C_3 \oplus C_3$.

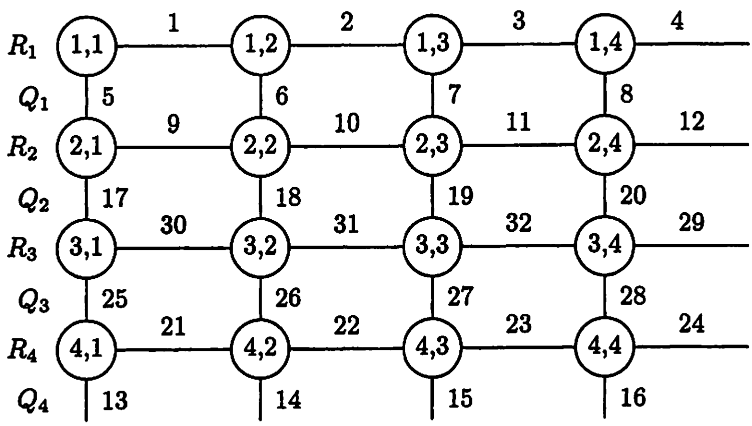

FIGURE 7. The labeling η of $C_4 \oplus C_4$.

4. Concluding Remark

The upper bound on the edge-bandwidth of a torus given in Theorem 2 seems to be tight and we would like to conjecture that it is actually the exact value. To prove the equality, however some refined techniques might be necessary. The situation seems to resemble the proof of the exact value of the edge-bandwidth of grids in [11]. There one technique was used to calculate the lower bound within an additive error of 1, and a different approach was required to remove this error. Perhaps a modification of that method would work for tori.

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