On Difference Graphs

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Abstract

In this paper we give a survey of all graphs of order ≤ 5 which are difference graphs and we show that some families of graphs are difference graphs.

0.Introduction

Harary [5] calls a graph a difference graph if there is an bijection f from V to a set of non negative integers S such that $xy \in E$ if and only if $|f(x) - f(y)| \in S$. Bloom, Hell, and Taylor [1] have shown that the following graphs are difference graphs: trees, C_n, K_n, $K_{n,n}$, $K_{n-1,n}$, pyramids, and n-prisms. Gervacio [3] proved that wheels W_n are difference graphs if and only if $n = 3,4$, or 6. Sonntag [6] proved that cacti (that is, graphs in which every edge is contained in at most one cycle) with girth at least 6 are difference graphs and he conjectures that all cacti are difference graphs.

Here, we show that, the graph $K_n - me$, where the m deleted edges have a common vertex; the graph $K_n - 2e, n \geq 4$, where the two deleted edges have no vertex in common; the graph $K_n - 3e, n \geq 6$, where the three deleted edges have no vertex in common are difference graphs. Also, we show that the following graphs are difference graphs: the gear graph G_n for all $n \geq 3$; the grids $P_m \times P_n$ for all $n, m \geq 2$; the triangular snakes; the C_4-snakes; the dragon (that is, a graph formed by identifying the end vertex of the path of m edges ($m \geq 1$) and any vertex in the cycle C_n ($n \geq 3$)); the graph C_m^{**} consists of two cycles of the same order m joined by a bridge; and $C_{n,m}$ (that is, a graph obtained by identifying the center of a star S_m with a vertex of a cycle C_n).
Throughout this paper, we use the standard notations and conventions as in [2] and [4].

1.Simple Observations on Difference Graphs.

Observation 1.1. [1] The labeling of a difference graph is not unique. For example the path P_3 can be specified by any of the vertex label sets $\{1, 2, 4\}$, $\{2, 4, 8\}$, $\{3, 4, 7\}$, $\{0, 11, 27\}$.
Furthermore, if $S = \{s_1, s_2, s_3, \ldots, s_n\}$ is any vertex label set of the difference

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graph G then S and $kS = \{ks_1, ks_2, ks_3, \dots, ks_n\}$, where $k > 0$, always induce the same difference graph, since both vertex and edge values increase proportionally.

Observation 1.2. [1] Any vertex v with the label value $a_1 = 0$ is adjacent to every other vertex u to which any value $a_i > 0$ is assigned. This is clear so, since $|x - 0| = x$ for every $x \geq 0$.

Observation 1.3. [1] Not all graphs are difference graphs. The smallest, non difference graphs have five vertices. In Figure 19 we show all such non difference graphs.

2. Two Propositions on Difference Graphs.

Proposition 2.1. [1]
(1) Vertex label values s and $2s$ belong to adjacent vertices, (2) Vertex label values r and t, $r \neq t$ belong to vertices adjacent to a vertex labeled $r + t$, (3) No other adjacencies occur in difference graphs. (An edge between vertices with values s and $2s$ will be termed an edge of the first kind, an edge between vertices with values r and $r + t$ will be termed an edge of the second kind).
Observe, that in a difference graph at least $deg(v) - 2$ edges incident with vertex v must be of the second kind, because no more than two can be of the first kind.

Proposition 2.2. [1] If each of the components of a graph G is a difference graph, and does not contain the vertex label 0 then G is a difference graph.

Proof: Similar to proof in [1], let $G = G_1 \cup G_2 \cup G_3 \cup \dots \cup G_n$ and let $S_i = S(G_i)$ be the vertex label set of the i^{th} component. Take $S(G) = S_1 \cup k_2S_2 \cup k_3S_3 \cup \dots \cup k_nS_n$, where $k_2 = max\{s: s \in S_1\} + 1$, and $k_i = max\{s: s \in k_{i-1}S_{i-1}\} + 1$ for $i \geq 3$. Then it is clear that G is a difference graph.

Example 2.2.

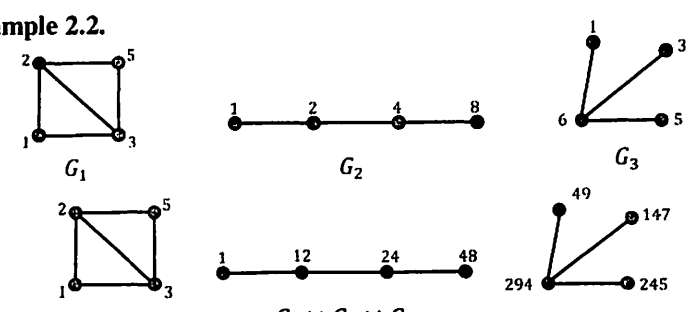

Figure 1

3. Families of Difference Graphs.

Theorem 3.1. The graph $K_n - (me)$, where the m deleted edges has a common vertex is a difference graph.

Proof: Label the common vertex by $n + m$, and the other vertices of the m deleted edges by $1,2,3 \ldots, m$. The remaining vertices will be labeled by $m + 1, m + 1, m + 1, \ldots, n - 1$.

Example 3.1.

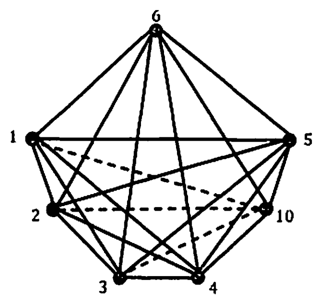

Figure 2. difference labeling of the graph $K_7 - (3e)$.

Theorem 3.2. The graph $K_n - 2e, n \geq 4$, where the two deleted edges have no vertex in common is a difference graph.

Proof: Label the end vertices of the two deleted edges by $1, n$ and $2, n + 1$.The remaining vertices will be labeled by $3, 4, 5, \ldots, n - 2$.

Example 3.2.

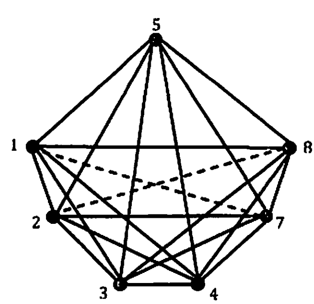

Figure 3. difference labeling of the graph $K_7 - 2e$.

Theorem 3.3. The graph $K_n - 3e, n \geq 6$, where the three deleted edges have no vertex in common is a difference graph.

Proof: Label the end vertices of the three deleted edges by $1, n-1$; $2, n$; and $3, n+1$. The remaining vertices will be labeled by $4, 5, 6, \dots, n-3$.

Example 3.3.

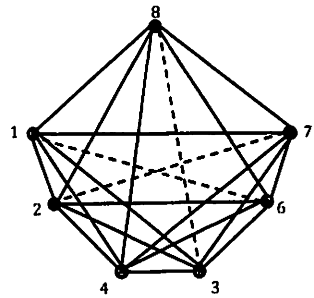

Figure 4. difference labeling of the graph $K_7 - 3e$.

The Cartesian product $G_1 \times G_2$ of G_1 and G_2 is the graph having vertex set $V(G_1) \times V(G_2)$ and edge set $\{(u_1, v_1)(u_2, v_2): (u_1 = u_2 \text{ and } v_1v_2 \in E(G_2)) \text{ or } (v_1 = v_2 \text{ and } u_1u_2 \in E(G_1))\}$ [4].

Theorem 3.4. The graph $P_m \times P_n$ is a difference graph for $n, m \geq 2$.
Proof: Let $P_m \times P_n$ be described as indicated in Figure 5.

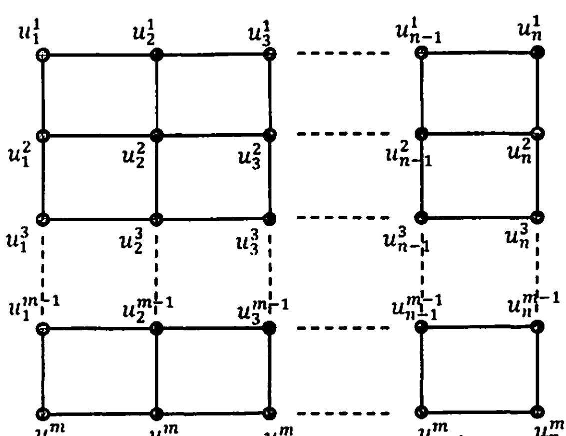

Figure 5.

We define the labeling function $f: V(P_m \times P_n) \to S$ (positive integers) as follows:

$f(u_1^1) = 2$, $f(u_i^2) = 2^{i+1}, i = 1,2,3, \dots, n$

$f(u_n^j) = 2^{j+n-1}, j = 3,4,5,\dots,m$

$f(u_i^1) = f(u_{i-1}^1) + f(u_i^2) \quad ,i = 2,3,4,\dots,n$

$f(u_i^3) = f(u_i^2) + f(u_{i+1}^3) \quad ,i = n-1, n-2,\dots,1$

$f(u_i^4) = f(u_i^3) + f(u_{i+1}^4) \quad ,i = n-1, n-2,\dots,1$

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$f(u_i^m) = f(u_i^{m-1}) + f(u_{i+1}^m) \quad ,i = n-1, n-2,\dots,1$

Example 3.4.

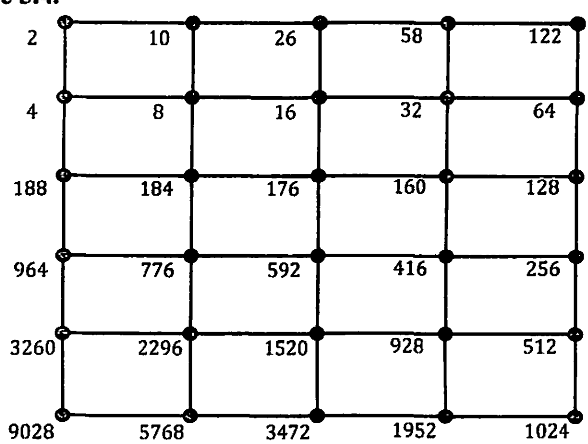

Figure 6. difference labeling of the graph $P_6 \times P_5$

A gear graph G_m is obtained from the wheel by adding a vertex between every pair of adjacent vertices of the cycle.

Theorem 3.5. The gear graph G_m is a difference graph for $m \geq 3$.
Proof: Let G_m be described as indicated in Figure 7.

Figure 7.

We define the labeling function $f: V(G_m) \to S$ (positive integers) as follows:
$f(v_0) = 1 \quad , \quad f(v_1) = 2$

$$f(v_i) = \begin{cases} 2f(v_{i-1}) & \text{if } i \text{ is even} \\ f(v_0) + f(v_{i-1}) & \text{if } i \text{ is odd} \end{cases}$$

$$f(v_{2n}) = f(v_1) + f(v_{2n-1})$$

Example 3.5.

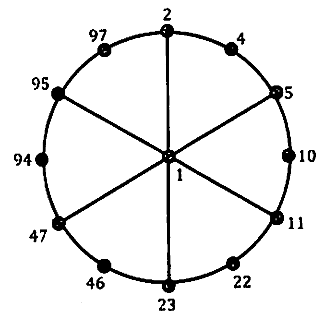

Figure 8. difference labeling of the gear G_6

Theorem 3.6. The dragon $D_{m,n}$ (that is, a graph formed by identifying the end vertex of the path of m vertices $(m \geq 2)$ and any vertex in the cycle $C_n, (n \geq 3)$) is a difference graph for $n \geq 3, m \geq 2$.

Proof: Let $D_{m,n}$ be described as indicated in Figure 9.

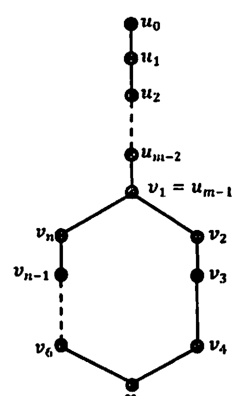

Figure 9.

We define the labeling function $f: V(D_{m,n}) \to S$ (positive integers) as follows:

$f(u_i) = 2^i \quad , \; i = 0,1,2, \dots, m-1$

$f(u_{m-1}) = f(v_1) = 2^{m-1}$

$f(v_i) = 2^{m+(i-2)}$, $i = 2,3,4,\dots,n-1$

$f(v_n) = f(v_{n-1}) + f(v_1)$

Example 3.6.

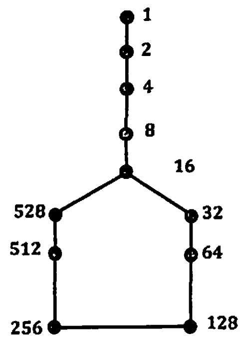

Figure 10. difference labeling of the graph $D_{5,7}$.

Theorem 3.7. The graph $C_{n,m}$ (that is, a graph obtained by identifying the center of a star S_m with a vertex of a cycle C_n) is a difference graph for $n \geq 3$, $m \geq 2$.

Proof: Let $C_{n,m}$ be described as indicated in Figure 11.

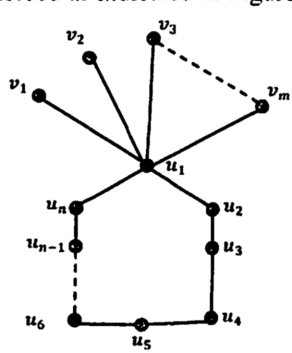

Figure 11.

We define the labeling function $f: V(C_{n,m}) \rightarrow S$ (positive integers) as follows:

$f(v_i) = 2i - 1$, $i = 1,2,3,\dots,m$

$f(u_i) = m2^i \qquad , i = 1,2,3,\dots,n-1$

$f(u_n) = 2m(1 + 2^{n-2})$

Example 3.7.

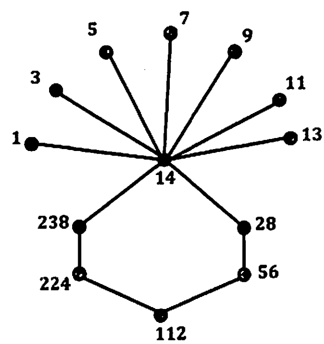

Figure 12. difference labeling of the graph $C_{6,7}$.

A triangular snake T_n is obtained from from a path $u_1, u_2, \dots, u_n$ by joining u_i and $u_i + 1$ to a new vertex v_i for $i = 1,2,\dots,n-1$.

Theorem 3.8. The triangular snake is a difference graph.
Proof: Let the triangular snake T_n be described as indicated in Figure 13.

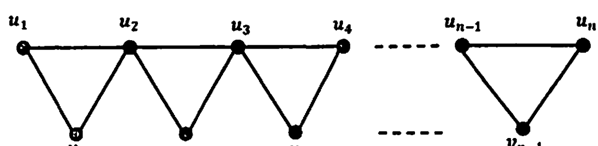

Figure 13.

We define the labeling function $f: V(T_n) \to S$ (positive integers) as follows:

$f(u_1) = 2, \quad f(v_1) = 1$

$f(u_i) = 3i \qquad , i = 1,2,3,\dots,n$

$f(v_i) = f(u_i + 1) - f(u_i) \qquad , i = 2,3,4,\dots,n-1$

Example 3.8.

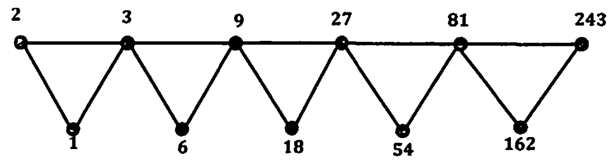

Figure 14. difference labeling of the graph T_6 .

Theorem 3.9. The C_4-snake (some extension to the triangular snake) is a difference graph.

Proof: Let the C_4-snake be described as indicated in Figure 15.

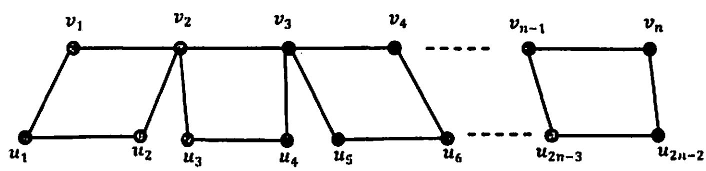

Figure 15.

We define the labeling function $f: V(C_4\ snake) \to S$ (positive integers) as follows:

$f(v_i) = 2.5^{i-1}$, $i = 1,2,3,\ldots,n$

$f(u_j) = 4.5^{(j-1)/2}$, $j = 1,3,5,\ldots,2n-3$

$f(u_j) = 8.5^{(j-2)/2}$, $j = 2,4,6,\ldots,2n-2$

Example 3.9.

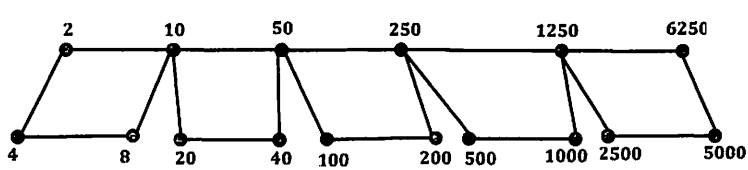

Figure 16.

Theorem 3.10. The graph C_m^{**} (that is, a graph consists of two cycles of the same order m joined by a bridge as shown in Figure 4.17) is a difference graph for $m \geq 3$.

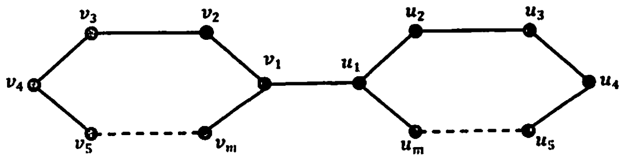

Figure 17.

Proof: We define the labeling function $f: V(C_m^{**}) \to S$ (positive integers) as follows:

$f(u_i) = 2^{i-1}$, $i = 2, 3, 4, \dots, m$

$f(u_1) = f(u_2) + f(u_m) = 2(2^{m-2} + 1)$

$f(v_i) = 2^{i+1}(2^{m-2} + 1)$, $i = 1, 2, 3, 4, \dots, m-1$

$f(v_m) = f(v_{m-1}) + f(v_1)$

Example 3.10.

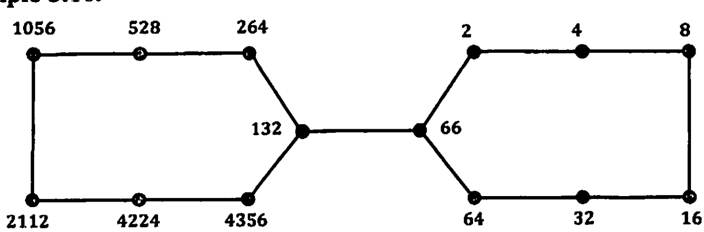

Figure 18.

4. Survey of Difference Graphs of Order ≤ 5.

Theorem 4.1. All graphs of order ≤ 5 are difference graphs, except the following graphs.

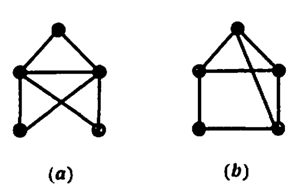

Figure 19.

Proof: (1) All disconnected graphs of order ≤ 5 are difference graphs by *proposition 2.2.* and *observation 1.3.*(2) All non isomorphic trees of order ≤ 5, $K_3, K_4, K_5, K_{1,3}, K_{1,4}, C_4$, and C_5 are difference graphs by Theorems due to Bloom, Hell and Taylor [1]. (3) The wheels W_3 and W_4 are difference graphs by Theorems due to Gervacio [3]. (4) The dragons and $C_{2,3}$ graphs are difference graphs by *Theorem 3.6*, *Theorem 3.7* in this paper. (5) The graphs $K_4 - e, K_5 - e, K_5 - 2e$ and $K_5 - 3e$ are difference graphs by *Theorem 3.1* in this paper. (6) The proof that these two graphs in Figure19 are non difference graph is too lengthy and tedious, but straightforward.
(7) The following graphs are difference graphs.

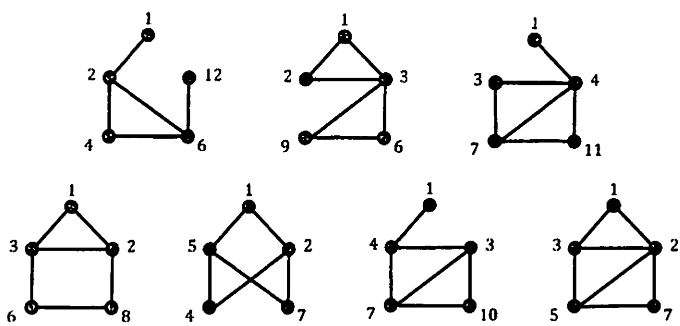

Figure 20.

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