How Vertex Elimination Can Overachieve

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Abstract

Vertex elimination orderings play a central role in many portions of graph theory and are exemplified by the so-called 'perfect elimination orderings' of chordal graphs. But perfect elimination orderings and chordal graphs enjoy many special advantages that overlap in more general settings: the random way that simplicial vertices can be chosen, always having a choice of simplicial vertices, the hereditary nature of being simplicial, and the neutral effect of deleting a simplicial vertex on whether the graph is chordal. A graph metatheory of vertex elimination formalizes such distinctions for general vertex elimination and examines them with simple theorems and delineating counterexamples.

1 When ϕ-elimination "succeeds"

It is convenient to think in terms of a graph-theoretic language based on atomic formulas that are interpreted as equality and adjacency of vertices, with the usual connectives of negation ($\neg$), conjunction ($\wedge$), disjunction ($\vee$), and implication ($\rightarrow$), and with universal and existential quantifiers ($\forall$ and $\exists$); quantification will always be over vertices. Other atomic formulas could be allowed, and conjunction and disjunction could be allowed over infinite sets of formulas so as to extend the language significantly beyond first-order logic—the only essential restriction is that quantification is always over vertices. For formulas $\psi(v)$ and $\phi(v)$, let, as usual, $(\exists v : \phi(v))\psi(v)$ abbreviate $(\exists v)[\phi(v) \wedge \psi(v)]$ and $(\forall v : \phi(v))\psi(v)$ abbreviate $(\forall v)[\phi(v) \rightarrow \psi(v)]$. The standard model-theoretic notation $G \models \phi[v_0]$ means that the graph G satisfies the formula $\phi(v)$ when all occurrences of the free variable v are assigned to be a specific vertex $v_0 \in V(G)$. When it aids readability, $\Rightarrow$ and $\Leftrightarrow$ will abbreviate 'implies' and 'if and only if'; also, $\phi \Rightarrow \psi \Rightarrow \pi$ will abbreviate that $\phi \Rightarrow \psi$ and $\psi \Rightarrow \pi$.

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Some specialized notation is useful for discussing elimination. Suppose a graph G has $V(G)$ ordered as $(v_1, \ldots, v_n)$. Let G_i denote the subgraph of G induced by $\{v_i, \ldots, v_n\}$; in other words, $G_1 = G$ and $G_i = G - \{v_1, \ldots, v_{i-1}\}$ whenever $2 \le i \le n$ (so $G_n \cong K_1$). Define $(\exists^1 v)\psi(v) = (\exists v)\psi(v)$ and, for each integer $i \ge 2$, define

$$(\exists^i v)\psi(v) \text{ to abbreviate } (\exists v \neq v_1, \ldots, v_{i-1})\psi(v),$$

which itself abbreviates $(\exists v : \neg(v = v_1) \wedge \cdots \wedge \neg(v = v_{i-1}))\psi(v)$ and is equivalent to $(\exists v \in V(G_i))\psi(v)$. Similarly, define $(\forall^1 v)\psi(v) = (\forall v)\psi(v)$ and, for each $i \ge 2$, define

$$(\forall^i v)\psi(v) \text{ to abbreviate } (\forall v \neq v_1, \ldots, v_{i-1})\psi(v),$$

which abbreviates $(\forall v : \neg(v = v_1) \wedge \cdots \wedge \neg(v = v_{i-1}))\psi(v)$ and is equivalent to $(\forall v \in V(G_i))\psi(v)$. For each integer $i \ge 1$, define the relativization ϕ^i of the formula ϕ by replacing each $\exists$ and $\forall$ occurring in ϕ with $\exists^i$ and $\forall^i$ respectively. For instance, if ϕ is $(\exists v)(\forall w)\psi(v, w)$, then $\phi^1 \Leftrightarrow \phi$ and $\phi^3 \Leftrightarrow (\exists v \neq v_1, v_2)(\forall w \neq v_1, v_2)(\psi(v, w))^3$. If a formula σ has no free variables, then $G \models \sigma^i$ if and only if $G_i \models \sigma$ (if and only if $G_i \models \sigma^i$).

Define an ordering $(v_1, \ldots, v_n)$ of $V(G)$ to be a *ϕ-elimination ordering* of G if $G_i \models \phi[v_i]$ for each $i \in \{1, \ldots, n-1\}$. Say that *ϕ-elimination succeeds* in G if G has a ϕ-elimination ordering; more formally, ϕ-elimination succeeds if G satisfies

$$(\exists v_1 : \phi(v_1))^1 \cdots (\exists v_{n-1} : \phi(v_{n-1}))^{n-1}(v_1 = v_1) \qquad (1)$$

($v_1 = v_1$ could be replaced by anything that is constantly true). An equivalent, perhaps more suggestive way to say that ϕ-elimination succeeds in G would be for G to satisfy the following:

$$(\exists^1 v_1) \cdots (\exists^{n-1} v_{v-1}) \bigwedge_{i=1}^{n-1} \phi^i(v_i).$$

Let $\mathcal{S}(\phi)$ denote the class of all graphs in which ϕ-elimination succeeds.

Example 1 Interpret $\phi_1(v)$ as 'v is a *simplicial vertex*' (in other words, '$N(v)$ is complete'). Then ϕ_1-elimination succeeds in G if and only if G has a perfect elimination ordering, and $\mathcal{S}(\phi_1)$ is the class of *chordal graphs* (typically defined as those graphs in which every cycle of length four or more has a chord; see [3, 6, 8] for these and many other characterizations).

Chapter 5 of [3] contains a trove of graph classes that can be similarly characterized by some sort of ϕ-elimination succeeding; also see [6] and Chapter 12 of [11].

It must be emphasized that this formulation only applies to those forms of vertex elimination that involve repeatedly locating and then deleting a vertex—v_i—that has a particular property—$\phi(v)$—in the subgraph G_i. This is different from studying algorithms that recognize whether or not a given graph is in a particular class or studying efficient constructions for ϕ-elimination orderings.

This approach also excludes many well-studied elimination-type orderings. For example, a graph is *strongly chordal* if it is chordal and every cycle C of even length six or more has a chord whose endpoints divide $E(C)$ into two odd-length paths; see [3, 8]. Strongly chordal graphs can be characterized both by 'strong elimination orderings'—which are excluded—and by 'simple elimination orderings'—which are included (see [3] for both).

This approach is also different from algorithmic graph searching—see [4] for an overview—such as LexBFS. (There are, of course, connections between elimination orderings and graph searching that date back at least to [9]; again, see [4].)

2 When "random" ϕ-elimination succeeds

As with eliminating leaves from trees, the order in which simplicial vertices are eliminated from chordal graphs in Example 1 does not matter; backtracking is never required. Such ϕ-elimination orderings for G were informally referred to in [8] as being 'foolproof.' We will say here, instead, that *random ϕ-elimination succeeds* in G. (The adjectives 'indifferent' and 'oblivious' would be reasonable alternatives to 'random.') Define random ϕ-elimination succeeding in G formally by G satisfying all of the n expressions

$$(\forall v_1 : \phi(v_1))^1 \cdots (\forall v_i : \phi(v_i))^i (\exists v_{i+1} : \phi(v_{i+1}))^{i+1} \cdots (\exists v_{n-1} : \phi(v_{n-1}))^{n-1} (v_1 = v_1) \tag{2}$$

(i universal quantifiers followed by $n-i-1$ existential quantifiers) running from $i = 0$—when (2) reduces to (1)—to $i = n-1$. (The $i = n-1$ instance, with all universal quantifiers, does not by itself define ϕ-elimination succeeding in G, because the ith instance is required to ensure that the $(i+1)$st instance does not hold vacuously. Nothing needs to be said about whether the one-vertex graph G_n satisfies $\phi[v_n]$; ϕ-elimination will always succeed in K_1.) An equivalent, perhaps more suggestive way to say that random ϕ-elimination succeeds in G would be for G to satisfy the following:

$$\bigwedge_{i=0}^{n-1} (\forall^1 v_1) \cdots (\forall^i v_i) \left[\bigwedge_{j=0}^{i} \phi^j(v_j) \rightarrow (\exists^{i+1} v_{i+1}) \cdots (\exists^{n-1} v_{n-1}) \bigwedge_{k=i+1}^{n-1} \phi^k(v_k) \right].$$

Example 2 Interpret $\phi_2(v)$ as '$N(v)$ is connected.' Then ϕ_2-elimination does not succeed in C_4 (there is no choice for v_1); ϕ_2-elimination succeeds in the wheel $C_4 + K_1$, but random ϕ_2-elimination does not succeed there (the degree-4 vertex satisfies $\phi_2(v)$ yet cannot be taken as v_1, since there would be no possible v_2); and random ϕ_2-elimination succeeds in K_4.

Define a vertex property $\phi(v)$ to be *hereditary* if, for every graph G, for every $v_0 \in V(G)$ such that $G \models \phi[v_0]$, and for every induced subgraph H of G such that $v_0 \in V(H)$, it follows that $H \models \phi[v_0]$. Property $\phi_1(v)$ from Example 1—being simplicial—is the prototypical example of a hereditary vertex property, while property $\phi_2(v)$ is not hereditary: take $G = K_{1,1,2}$ (C_4 with one chord) with v_0 and w the two degree-3 vertices and $H = G - w \cong K_{1,2}$. Call a graph G *nontrivial* if $|V(G)| \geq 2$.

Theorem 1 *For every vertex property $\phi(v)$ and graph G, the following conditions satisfy the implications* (1.1) $\Rightarrow$ (1.2) $\Rightarrow$ (1.3):

(1.1) *For every induced subgraph H of G, there exists a vertex v in H such that $H \models \phi[v]$.*

(1.2) *Random ϕ-elimination succeeds in G.*

(1.3) *ϕ-elimination succeeds in G.*

Moreover, if $\phi(v)$ is hereditary, then (1.1), (1.2), *and* (1.3) *are equivalent.*

Proof: The implications (1.1) $\Rightarrow$ (1.2) $\Rightarrow$ (1.3) are straightforward consequences of the definition of (random) ϕ-elimination succeeding.

Suppose $\phi(v)$ is hereditary and condition (1.3) holds [toward showing (1.1)]; specifically, suppose $(v_1, \ldots, v_n)$ is a ϕ-elimination ordering for G. Suppose H is an induced subgraph of G and h is the minimum subscript such that $v_h \in V(H)$. Then by (1.3), G_h contains a vertex v_h such that $G_h \models \phi[v_h]$. Since H is also an induced subgraph of G_h and since $\phi(v)$ is hereditary, $H \models \phi[v_h]$. Thus (1.1) holds. □

Example 3 Interpret $\phi_3(v)$ as the non-hereditary property 'v is neither a cut-vertex nor an isolated vertex of a nontrivial graph.' Then $\mathcal{S}(\phi_3)$ is the class of connected graphs. Random ϕ_3-elimination succeeds in all connected graphs, but the subgraph H induced by two nonadjacent vertices of $G = C_4$ contains no v_0 such that $H \models \phi_3[v_0]$. Hence, the converse of (1.1) $\Rightarrow$ (1.2) fails in general.

Example 4 Interpret $\phi_4(v)$ as the non-hereditary property 'v has a maximum neighbor' (where $w \in N[v]$ is a *maximum neighbor* of v if $N[u] \subseteq N[w]$ for every $u \in N[v]$; note that $w = v$ is allowed). Reference [2] (or Thm. 8.3.1 in [3]) shows that $\mathcal{S}(\phi_4)$ is the class of *dually chordal graphs* (often defined as those graphs that have maximum neighbor elimination orderings; see [3, 8] for other characterizations and that dually chordal graphs are always strongly chordal). Random ϕ_4-elimination does not succeed in all dually chordal graphs (in the wheel $C_4 + K_1$, the degree-4 vertex w_0 is its own maximum neighbor and so w_0 satisfies $\phi_2(v)$; yet w_0 cannot be used as v_1, since there would be no possible v_2). Hence, the converse of (1.2) ⇒ (1.3) fails in general.

Corollary 2 *For every vertex property $\phi(v)$, the following conditions satisfy the implications* (2.1) ⇒ (2.2) ⇒ (2.3):

(2.1) *$\phi(v)$ is hereditary.*

(2.2) *$\mathcal{S}(\phi)$ is hereditary* (meaning $\mathcal{S}(\phi)$ is closed under induced subgraphs).

(2.3) *Random ϕ-elimination succeeds in each $G \in \mathcal{S}(\phi)$.*

Proof: The implication (2.1) ⇒ (2.2) follows from (2.1) ⇒ [(1.3) ⇒ (1.1)], while (2.2) ⇒ (2.3) follows from (1.1) ⇒ (1.2) and the definition of random ϕ-elimination succeeding. □

Example 5 Interpret $\phi_5(v)$ as the non-hereditary property 'deg(v) = 1 or v has a twin' (where w is a *twin* of v if v and w have exactly the same neighbors except possibly for v and w themselves). Reference [1] (or Thm. 11.6.7 in [3]) shows that $\mathcal{S}(\phi_5)$ is the class of *distance-hereditary graphs*—those graphs G in which the distance between vertices in a connected induced subgraph of G always equals their distance in G (see [3] for other characterizations). Random ϕ_5-elimination succeeds in every distance-hereditary graph (since (2.2) holds).

Observe that ϕ_5-elimination is a counterexample to (2.2) ⇒ (2.1), and that ϕ_3-elimination is a counterexample to (2.3) ⇒ (2.2).

Corollary 2 can be rephrased as saying that the vertex property $\phi(v)$ being hereditary (or the weaker condition that the class $\mathcal{S}(\phi)$ is hereditary) implies that ϕ-elimination 'overachieves' in the sense of condition (2.3): ϕ-elimination succeeds implies random ϕ-elimination succeeds. But Example 3 shows that ϕ-elimination can also overachieve in non-hereditary cases.

Suppose $\mathcal{C}$ is any graph class (for instance, the class of chordal graphs or the class of connected graphs). Define a vertex property $\phi(v)$ to be *neutral* for $\mathcal{C}$ to mean that, for every graph G and $v_0 \in V(G)$, if $G \models \phi[v_0]$ then $G \in \mathcal{C} \Leftrightarrow G - v_0 \in \mathcal{C}$. For example, property $\phi_1(v)$ ('v is a simplicial vertex') is neutral for the class of chordal graphs; property $\phi_3(v)$ ('v is neither a cut-vertex nor an isolated vertex') is neutral for the class of connected graphs; property $\phi_4(v)$ ('v has a maximum neighbor') is not neutral for the class of dually chordal graphs (take $G = C_4 + K_1$ and deg(v_0) = 4);

and property $\phi_5(v)$ ('v is a leaf or has a twin') is neutral for the class of distance-hereditary graphs.

It is of interest to note that this concept of being neutral for a graph class appears in a contemplative discussion of ϕ-elimination in [11] in almost the same words as above (on page 181, called 'local testing' in the context of $\phi_5(v)$ and distance-hereditary graphs). Theorem 3 will show that being neutral for a class characterizes when *random* ϕ-elimination is equivalent to (general) ϕ-elimination.

Theorem 3 *Suppose a vertex property $\phi(v)$ is neutral for a graph class $\mathcal{C}$ that contains K_1, and suppose every nontrivial $G \in \mathcal{C}$ contains a vertex v_G such that $G \models \phi[v_G]$. Then $\mathcal{S}(\phi) = \mathcal{C}$ and random ϕ-elimination succeeds in every $G \in \mathcal{S}(\phi)$.*

Conversely, if random ϕ-elimination succeeds in every $G \in \mathcal{S}(\phi)$, then $\phi(v)$ is neutral for $\mathcal{S}(\phi)$.

Proof. First suppose $\mathcal{C}$ and $\phi(v)$ are as described in the first sentence of the theorem, toward showing that **(i)** $\mathcal{S}(\phi) = \mathcal{C}$ and **(ii)** $G \in \mathcal{C}$ implies random ϕ-elimination succeeds in G. Argue by induction on $|V(G)|$, from the basis $G = K_1 \in \mathcal{S}(\phi) \cap \mathcal{C}$ (remembering that ϕ-elimination always succeeds in K_1).

For **(i)**: If $G \in \mathcal{S}(\phi)$, then there is a v_1 such that $G \models \phi[v_1]$ and $G - v_1 = G_2 \in \mathcal{S}(\phi)$; therefore $G - v_1 \in \mathcal{C}$ (by induction hypothesis), and so $G \in \mathcal{C}$ (because $G \models \phi[v_1]$ and $\phi(v)$ is neutral for $\mathcal{C}$). Similarly, $G \in \mathcal{C}$ implies $G - v_G \in \mathcal{C}$; therefore $G - v_G \in \mathcal{S}(\phi)$ (by induction hypothesis), and so $G \in \mathcal{S}(\phi)$ (with $v_1 = v_G$).

For **(ii)**: Suppose $G \in \mathcal{C} = \mathcal{S}(\phi)$ and let v_1 be an arbitrary vertex such that $G \models \phi[v_1]$ ($v_1 = v_G$ ensures that such vertices exist). Then $G - v_1 \in \mathcal{C}$ (because $\phi(v)$ is neutral for $\mathcal{C}$). Therefore random ϕ-elimination succeeds in $G - v_1$ (by induction hypothesis), which implies that random ϕ-elimination succeeds in G (by the arbitrary choice of v_1).

For the converse, suppose random ϕ-elimination succeeds in each $G \in \mathcal{S}(\phi)$, and that $v_1 \in V(G)$ and $G \models \phi[v_1]$ [toward showing that $G \in \mathcal{S}(\phi) \Leftrightarrow G - v_1 \in \mathcal{S}(\phi)$]. The $\Rightarrow$ direction is immediate. The $\Leftarrow$ direction follows from every ϕ-elimination ordering $(v_2, \ldots, v_n)$ of $G - v_1$ automatically extending to a ϕ-elimination ordering $(v_1, v_2, \ldots, v_n)$ of G (because $G \models \phi[v_1]$). □

Up to now, we have considered various ways that ϕ-elimination can overachieve in the sense of (2.3). In the following sections, we consider two related senses of overachievement.

3 When random ϕ-elimination "flourishes"

The specific choice of each vertex v_i never matters in random ϕ-elimination, but this freedom is hollow when there is only one possible v_i to choose from. This suggests a related notion in which the vertex property $\phi(v)$ itself overachieves—when, at every stage of ϕ-elimination before the last, there exist at least two choices for v_i (meaning at least two v_i such that each $G_i \models \phi[v_i]$) that are *viable choices* (meaning these v_i can go on to be in ϕ-elimination orderings). When this happens, we will say that *ϕ-elimination flourishes* in G. If ϕ-elimination flourishes in G, then G has at least 2^{n-1} ϕ-elimination orderings. Theorem 4 generalizes a well-known property of ϕ_1-elimination (perfect elimination) orderings; see [6, page 84].

Theorem 4 *For every vertex property $\phi(v)$ and graph G, ϕ-elimination flourishes in G if and only if each vertex v is the final vertex in some ϕ-elimination ordering.*

Proof. If ϕ-elimination flourishes in G and v is any pre-determined vertex of G, then each v_i with $1 \leq i < n$ can be chosen to be different from $v = v_n$.

Conversely, suppose G is a minimum-order graph such that each vertex is the final vertex in some ϕ-elimination ordering of G, yet ϕ-elimination does not flourish in G [arguing by contradiction]. Then the minimality of G would require that v_1 is uniquely determined, contradicting that $v = v_1$ could be saved until last. □

If, in addition to ϕ-elimination flourishing in G, *every* v_i such that $G_i \models \phi[v_i]$ is a viable choice for every i, we will say that *random ϕ-elimination flourishes* in G. It is well-known that random ϕ_1-elimination flourishes in chordal graphs, but random ϕ_2-elimination does not flourish in $G = C_4 + K_1$: although each of the five vertices of G satisfies $\phi_2(v)$, choosing v_1 to be the degree-4 vertex is not viable since no possibility would exist for v_2. (Yet ϕ_2-elimination does flourish in G.)

A formal definition of random ϕ-elimination flourishing in G results from random ϕ-elimination succeeding in G—in other words, from G satisfying all the expressions at (2)—in addition to G satisfying the modified versions

$$(\forall v_1 : \phi(v_1))^1 \cdots (\forall v_i : \phi(v_i))^i (\forall w : \phi(w))^{i+1} (\exists v_{i+1} \neq w : \phi(v_{i+1}))^{i+1} \quad (3)$$
$$\cdots (\exists v_{n-1} : \phi(v_{n-1}))^{n-1} (v_1 = v_1)$$

of those expressions for $i = 0, \ldots, n-3$—where $(\exists v_{i+1} : \phi(v_{i+1}))^{i+1}$ in (2) is replaced with $(\forall w : \phi(w))^{i+1} (\exists v_{i+1} \neq w : \phi(v_{i+1}))^{i+1}$ in (3)—ensuring that there are always at least two viable choices for those v_{i+1}. An equivalent, perhaps more suggestive way to write (3) would be the following:

$$\bigwedge_{i=0}^{n-1} (\forall^1 v_1) \cdots (\forall^i v_i)(\forall^i w \neq v_i) \left[\bigwedge_{j=0}^{i} \phi^j(v_j) \wedge \phi^i(w) \to (\exists^{i+1} v_{i+1}) \cdots (\exists^{n-1} v_{n-1}) \bigwedge_{k=i+1}^{n-1} \phi^k(v_k) \right].$$

Theorem 5 *For every vertex property $\phi(v)$ and graph G, the following conditions satisfy the implications* (5.1) $\Rightarrow$ (5.2) $\Rightarrow$ (5.3):

(5.1) *For every nontrivial induced subgraph H of G, there exist at least two vertices v in H such that $H \models \phi[v]$.*

(5.2) *Random ϕ-elimination flourishes in G.*

(5.3) *ϕ-elimination flourishes in G.*

Moreover, if $\phi(v)$ is hereditary, then (5.1), (5.2), *and* (5.3) *are equivalent.*

Proof: The proof of Theorem 5 strictly parallels the proof of Theorem 1. □

Let $\mathcal{F}(\phi)$ denote the class of all graphs in which ϕ-elimination flourishes.

Corollary 6 *For every vertex property $\phi(v)$, the following conditions satisfy the implications* (6.1) $\Rightarrow$ (6.2) $\Rightarrow$ (6.3):

(6.1) *$\phi(v)$ is hereditary.*

(6.2) *$\mathcal{F}(\phi)$ is hereditary* (meaning it is closed under induced subgraphs).

(6.3) *Random ϕ-elimination flourishes in each $G \in \mathcal{F}(\phi)$.*

Proof: The proof of Corollary 6 strictly parallels the proof of Corollary 2. □

Random ϕ_3-elimination flourishes in all connected graphs and is a counterexample to both (6.3) $\Rightarrow$ (6.2) and (5.2) $\Rightarrow$ (5.1) holding in general. Random ϕ_4-elimination does not flourish in all dually chordal graphs, although ϕ_4-elimination does flourish in all dually chordal graphs by Theorem 4 and [5, Thm. 5]; thus ϕ_4-elimination is a counterexample to (5.3) $\Rightarrow$ (5.2) holding in general. The proof of [1, Thm. 1] shows that random ϕ_5-elimination flourishes in all distance-hereditary graphs and is a counterexample to (6.2) $\Rightarrow$ (6.1) holding in general.

In random ϕ_1-elimination, two viable vertices always exist that are maximally distant from each other in G_i when $i < n$ (this is the notion of 'diametral elimination ordering' in [3]). But such maximally distant viable vertices do not necessarily exist at every stage of random ϕ_5-elimination (take G to be the distance-hereditary graph formed from $C_4 + K_1$ by creating one new degree-1 vertex that is incident to one of the existing degree-3 vertices). Yet [3, Thm. 5.1.5] presents another vertex property $\phi(v)$ that has $\mathcal{S}(\phi) = \mathcal{S}(\phi_5)$ for which maximally distant viable pairs of vertices do always exist (namely, interpret $\phi(v)$ as 'v is 2-*simplicial*'—meaning that there is no induced P_4 in the subgraph induced by $\{w : \text{dist}(v, w) \leq 2\}$).

Theorem 7 *Suppose a vertex property $\phi(v)$ is neutral for a graph class $\mathcal{C}$ that contains K_1, and suppose every nontrivial $G \in \mathcal{C}$ contains two vertices v_G such that $G \models \phi[v_G]$. Then $\mathcal{F}(\phi) = \mathcal{C}$ and random ϕ-elimination flourishes in every $G \in \mathcal{F}(\phi)$.*

Conversely, if random ϕ-elimination flourishes in every $G \in \mathcal{F}(\phi)$, then $\phi(v)$ is neutral for $\mathcal{F}(\phi)$.

Proof. The proof of Theorem 7 strictly parallels the proof of Theorem 3. □

Example 6 Interpret $\phi_6(v)$ as the non-hereditary property 'v is a maximum-degree simplicial vertex.' Then $\mathcal{S}(\phi_6)$ is the class of chordal graphs and random ϕ_6-elimination succeeds in every chordal graph (since (1.1) holds). But ϕ_6-elimination does not flourish in either of the two chordal graphs shown on the left in Fig. 1 ($v_1 = a$ is forced in each).

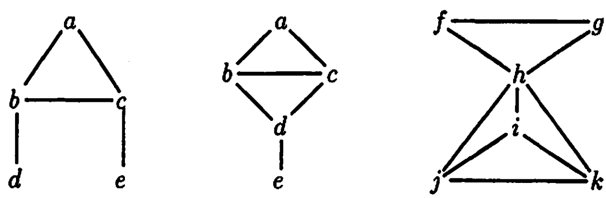

Figure 1: Three examples related to ϕ_6-elimination.

(Example 6 might seem related to 'maximum cardinality search' as defined in [12], but it is easy to see that, in either of the two chordal graphs on the left in Fig. 1, (a, e, d, b, c) is a ϕ_6-elimination ordering that does not correspond to a maximum cardinality search. See [10] for a general discussion of perfect elimination orderings, including their relationship to maximum cardinality search.)

In the ϕ_1-elimination, simplicial vertex example that motivated Theorem 4, any arbitrary complete subgraph of order k can be held back to become the final k vertices eliminated (see the ruminative discussion on page 84 of [6]). In contrast, random ϕ_6-elimination can flourish in G without it being possible to hold back even two adjacent vertices to become the final two vertices: If G is the chordal graph shown on the right in Fig. 1, then one cannot ensure that $\{v_5, v_6\} = \{j, k\}$ (doing so would force $v_1 = i$ and $v_2 \in \{f, g\}$, leaving no choice for $v_3 \notin \{j, k\}$ in G_3).

4 When "setwise" ϕ-elimination succeeds

Instead of considering the set of viable choices for each v_i and then choosing just one to eliminate, one can consider simultaneously removing all the eligible vertices v_i that satisfy $\phi(v)$ in a batch, and then repeating. This replaces vertex elimination in a graph G with 'setwise elimination' (partially motivated for chordal graphs in [7]). Neither of the adjectives 'random' nor 'flourishing' is appropriate for setwise ϕ-elimination.

For $i \geq 1$, define

$$\Phi_{[i]} = \{w_0 \in V(G_{[i]}) : G_{[i]} \models \phi[w_0]\},$$

where $G_{[1]} = G$ and, when $i \geq 2$, $G_{[i]} = G - (\Phi_{[1]} \cup \cdots \cup \Phi_{[i-1]})$. Say that *setwise ϕ-elimination succeeds* in G if some $V(G_{[i]}) = \emptyset$. Random ϕ-elimination succeeding in G clearly implies that setwise ϕ-elimination succeeds in G.

Perhaps surprisingly, ϕ-elimination succeeding and setwise ϕ-elimination succeeding are logically independent: Random ϕ_3-elimination succeeds in the graph G shown on the left in Fig. 2, but setwise ϕ_3-elimination does not succeed there (setwise ϕ_3-elimination stops with $\Phi_{[2]} = \emptyset$ and $G_{[2]} \cong 2K_1$). Setwise ϕ_5-elimination succeeds in the graph G shown on the right in Fig. 2 (with $G_{[2]} \cong P_4$, $G_{[3]} \cong P_2$, and $V(G_{[4]}) = \emptyset$), but ϕ_5-elimination does not succeed there.

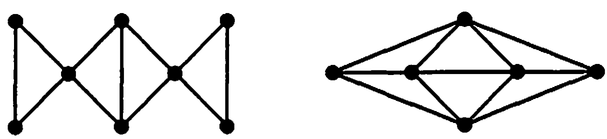

Figure 2: Two examples related to setwise ϕ-elimination.

Theorem 8 *If vertex property $\phi(v)$ is hereditary and G is any graph, then ϕ-elimination succeeds in G if and only if setwise ϕ-elimination succeeds in G.*

Proof. Suppose $\phi(v)$ is hereditary and G is given.

First suppose $(v_1, \ldots, v_n)$ is a ϕ-elimination ordering of G. Suppose setwise ϕ-elimination stops with $\Phi_{[k+1]} = \emptyset$ and $V(G_{[k+1]}) \neq \emptyset$ [arguing by contradiction]. Let $i \leq n$ be minimum such that $v_i \in V(G_{[k+1]})$, making $G_{[k+1]}$ an induced subgraph of G_i. Then $\phi(v_i)$ will not hold in $G_{[k+1]}$ (since $v_i \notin \Phi_{[k+1]} = \emptyset$). Thus $\phi(v_i)$ will not hold in G_i (because $\phi(v)$ is hereditary), contradicting that $(v_1, \ldots, v_n)$ is a ϕ-elimination ordering of G.

Conversely, suppose setwise ϕ-elimination succeeds in G. Let $\Phi_{[1]} = \{v_1, \ldots, v_{k_1}\}$, $\Phi_{[2]} = \{v_{k_1+1}, \ldots, v_{k_2}\}$, and so on, and let $\sigma = (v_1, \ldots, v_n)$. If

$v_i \in \Phi_{[j+1]} = \{v_{k_j+1}, \ldots, v_{k_{j+1}}\}$, then $\phi(v_i)$ will hold in $G_{[j+1]}$. Thus $\phi(v_i)$ will hold in the induced subgraph G_i of $G_{[j+1]}$ (because $\phi(v)$ is hereditary). This shows that σ is a ϕ-elimination ordering for G, and so shows that ϕ-elimination succeeds in G. □

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