# TOPOLOGICAL PROPERTIES OF THE SET OF ALL MINIMAL TOTAL DOMINATING FUNCTIONS OF A GRAPH

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**Abstract**

A total dominating function (TDF) of a graph $G = (V, E)$ is a function $f : V \to [0,1]$ such that for all $v \in V$, the sum of the function values over the open neighbourhood of $v$ is at least one. A minimal total dominating function (MTDF) $f$ is a TDF such that it is not a TDF if for any $v \in V$, the value of $f(v)$ is decreased. A convex combination of two MTDFs $f$ and $g$ of a graph $G$ is $h_\lambda = \lambda f + (1 - \lambda)g$, were $0 < \lambda < 1$. A basic minimal total dominating function (BMTDF) is an MTDF which cannot be expressed as a convex combination of two or more different MTDFs. In this paper we study the structure of the set of all minimal total dominating functions ($\mathfrak{F}_T$) of some classes of graphs and characterize the graphs having $\mathfrak{F}_T$ isomorphic to one simplex.

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## 1 Introduction

Throughout this paper the notation $G = (V, E)$ represents a finite undirected graph which does not contain loops or multiple edges. Unless specified otherwise, we follow the terminology of West [11]. A *dominating set* of $G = (V, E)$ is a subset $S$ of $V$ such that every vertex of $V - S$ is adjacent to a vertex in $S$. A dominating set $S$ is a *minimal dominating set* if no proper subset of $S$ is a dominating set.

The characteristic function of a dominating set is a function $f : V \to \{0, 1\}$ such that

$$\sum_{x \in N[v]} f(x) \geq 1$$

for all $v \in V$ , where $N[v]$ is the closed neighborhood of $v$. This function is generalized by allowing $f(v)$ to vary in the closed interval $[0, 1]$ instead of the two element set $\{0, 1\}$. A *dominating function* (DF) of a graph $G = (V, E)$ is a function $f : V \to [0, 1]$ such that

$$\sum_{x \in N[v]} f(x) \geq 1$$

for all $v \in V$. This fractional version of domination was first formally defined in 1987 by Hedetniemi and Wimer [7]. A *minimal dominating function* (MDF) is a DF such that $f$ is not a DF if for any $v \in V$, the value of $f(v)$ is decreased.

An analogous theory of total dominating functions was first developed by Cockayne et al. [3] in 1990's. A real valued function $f : V \to [0, 1]$ of a graph $G = (V, E)$ is called a *total dominating function* (TDF) if

$$\sum_{x \in N(v)} f(x) \geq 1$$

for all $v \in V$. The TDF $f$ is a minimal TDF (MTDF) if there does not exist another TDF $g$ such that $g(v) \leq f(v)$ for all $v \in V$ with strict inequality at some vertex. The theory of fractional total domination has been studied by many authors [2, 3, 4, 12].

The subset $S$ of $V$ is called a *total dominating set* if every vertex in $G$ is adjacent to at least one vertex in $S$. A *minimal total dominating set* is a total dominating set $S$, such that $S - \{v\}$ is not a total dominating set, for all $v \in S$. The $0 - 1$ valued TDFs (MTDFs) are the characteristic functions of total dominating sets (minimal total dominating sets) of a graph. The minimum and maximum of the cardinalities of all MDSs of a graph are called the *total domination number* ($\gamma_t(G)$) and the *upper total domination number* ($\Gamma_t(G)$) respectively.

For an MTDF $f$ of $G$, $f(N(v)) = \sum_{x \in N(v)} f(x)$ and $|f| = \sum_{x \in V} f(x)$. The *boundary* of $f$ or $B_f$ is $\{v \in V : \sum_{x \in N(v)} f(x) = 1\}$, the *positive set* of $f$ or $P_f$ is $\{v \in V : f(x) > 0\}$ and the *mid set* of $f$ or $P'_f$ is $\{v \in V : 1 > f(x) > 0\}$. For any two subsets $A$ and $B$ of $V$, we write $A \to_t B$ if every vertex in $B$ is adjacent to some vertex in $A$. The *fractional* *total domination number* ($\gamma_{ft}(G)$) and the *fractional upper total domination number* ($\Gamma_{ft}(G)$) are defined as follows:

$$\gamma_{ft}(G) = min\{|f| : f \text{ is an } MTDF \text{ of } G\},$$

$$\Gamma_{ft}(G) = max\{|f| : f \text{ is an } MTDF \text{ of } G\}.$$

A graph has either only one MTDF or infinitely may MTDFs. The following theorem helps us to identify MTDFs from a collection of TDFs.

**Theorem 1.1.** [1] *A total dominating function $f$ of the graph $G$ is a minimal total dominating function if and only if $B_f \rightarrow_t P_f$.*

Let $f$ and $g$ be two TDFs of $G$. A convex combination of $f$ and $g$ is $h_\lambda = \lambda f + (1-\lambda)g$ where $0 < \lambda < 1$. This function is clearly a TDF. Therefore the set of all TDFs forms a convex set. However it is evident from the following theorem that a convex combination of two MTDFs need not always be an MTDF.

**Theorem 1.2.** [5, 8] *A convex combination of $n$ MTDFs $f_1, f_2, \dots f_n$ is minimal if and only if $B_{f_1} \bigcap B_{f_2} \bigcap \cdots \bigcap B_{f_n} \rightarrow_t P_{f_1} \bigcup P_{f_2} \bigcup \cdots \bigcup P_{f_n}$.*

An MTDF of $G$ is called a *universal minimal total dominating function* if its convex combination with any other MTDF is minimal.

Since the set of TDFs is convex, some TDFs cannot be expressed as a convex combination two or more TDFs. Motivated by this fact Reji Kumar introduced the basic total dominating functions (BTDFs) and the basic minimal total dominating functions (BMTDFs) [8]. An MTDF is called a *basic minimal total dominating function* or BMTDF, if it cannot be expressed as a proper convex combination of two or more distinct MTDFs. A necessary and sufficient condition for an MTDF to be a basic MTDF is known and based on this an algorithm is developed to find whether a given MTDF is basic or not. The following results discuss it.

**Theorem 1.3.** [8] *An MTDF $f$ is a BMTDF if and only if there does not exist another MTDF $g$ such that $B_f = B_g$ and $P_f = P_g$.*

**Theorem 1.4.** [8] *Let $f$ be an MTDF of the graph $G$ with $B_f = \{v_1, v_2, \dots, v_m\}$ and $P_f' = \{u_1, u_2, \dots, u_n\}$. Let $A = (a_{ij})$ be an $m \times n$ matrix defined by*

$$a_{ij} = \begin{cases} 1 & \text{if } v_i \text{ is adjacent to } u_j \\ 0 & \text{otherwise.} \end{cases}$$

*Consider the system of linear equations given by*

$$\sum_j a_{ij} x_j = 0, \quad \text{where } 1 \le i \le m. \tag{1.1}$$

*The function f is a BMTDF if and only if (1.1) does not have a non-trivial solution.*

**Corollary 1.5.** [8] *Every* $0-1$ *MTDF of G is a BMTDF.*

It was proved in [8] that any finite graph $G$ has only finitely many BMTDFs. Also, $G$ has four BMTDFs, say $f_1, f_2, f_3$ and $f_4$ such that $\gamma_t(G) = |f_1|$, $\Gamma_t(G) = |f_2|$, $\gamma_{ft}(G) = |f_3|$ and $\Gamma_{ft}(G) = |f_4|$. So we can restrict our search for the four domination parameters to a relatively small set of BMTDFs and Theorem 1.3 and Theorem 1.4 provide an easy algorithm to find all BMTDFs of a graph. This motivated us to study further about the set of all BMTDFs of a graph.

We denote the set of all MTDFs of $G$ by $\mathfrak{F}_T(G)$ and the set of all BMTDFs by $\mathfrak{F}_{BT}(G)$. For a graph $G$, we define:

$$C_0(G) = \{v \in V : f(v) = 0 \text{ for any MTDF } f \text{ of } G\} \text{ and}$$

$$C_1(G) = \{v \in V : f(v) = 1 \text{ for any MTDF } f \text{ of } G\}.$$

Cockayne et al. [3] proved that for any $v \in V(G)$, $v \in C_0(G)$ if and only if $v$ is in no MTDS of $G$ and $v \in C_1(G)$ if and only if $v$ is in every MTDS of $G$.

The set of all *leaves* of $G$ is $L = \{v \in V : d(v) = 1\}$ and the set of all *remote vertices* is $R = \{v \in V : v \in N(u) \text{ for } u \in L\}$. The following relations, established by Cockayne et al. [3] are useful in our discussion. For any graph $G$, $C_1(G) = R$. For any vertex $v \in V(G)$, $v \in C_0(G)$ if and only if for any $u \in N(v)$ there exists a vertex $w$, satisfying $N(w) \subseteq N(u) - \{v\}$.

Let $K$ be a convex subset of $\mathbf{R}^n$. A point $x \in K$ is an *extreme point* of $K$ if $y, z \in K$, $0 < \lambda < 1$, and $x = \lambda y + (1-\lambda)z$ imply $x = y = z$. The set of all extreme points of $K$ is denoted by $ext(K)$. A set $F \subseteq K$ is a face of $K$ if either $F = \emptyset$ or $F = K$ or there exists a supporting hyperplane $H$ of $K$ such that $F = K \bigcap H$. An $n$ - *simplex* in the Euclidean space is the convex hull of $n+1$ affinely independent points. A *convex polytope* is the convex hull of a finite set. A finite family $\mathfrak{B}$ of convex polytopes in $\mathbf{R}^n$ is called a *simplicial complex* if it satisfies the following conditions.

1. Every face of a member of $\mathfrak{B}$ is itself a member of $\mathfrak{B}$;

2. The intersection of any two members of $\mathfrak{B}$ is a face of each of them.

For further study of simplices, polytopes and complexes, the reader is referred to [6].

If $x = uv$ is an edge and $w$ is not a vertex of $G$, then $x$ is said to be *subdivided* when it is replaced by the edges $uw$ and $wv$. If each edge of $G$ is subdivided, then the resulting graph is called the *subdivision graph* of $G$ and is denoted by $S(G)$.

## 2 Structure of the set of all MTDFs of some classes of graphs

There exists a bijection, say $\theta$, from the set of all functions $f : V \to [0,1]$ of a graph $G = (V, E)$ to the $n$ dimensional cube $I^n$. Let $\theta(f)$ denote the image of a function $f \in \mathfrak{F}_T(G)$ and $\theta(\mathfrak{F}_T(G)) = \{\theta(f) : f \in \mathfrak{F}_T(G)\}$. The line segment connecting any two points in $\theta(\mathfrak{F}_T(G))$ is a subset of $\theta(\mathfrak{F}_T(G))$ if and only if the corresponding functions make a minimal convex combination in $\mathfrak{F}_T(G)$. Let $A \subseteq \mathfrak{F}_{BT}(G)$ and $\mathfrak{C}_A = \{h : h = \sum_{f\in A} \lambda_f f$ where $1 > \lambda_f > 0$ and $\sum_{f\in A} \lambda_f = 1\}$. The set $\mathfrak{C}_A$ is the set of all proper convex combinations of the BMTDFs in $A$. If a convex combination is minimal, then by Theorem 1.2, all convex combinations are minimal and they have the same boundary and positive set. We denote the boundary of this convex combinations by $B_A$ and the positive set by $P_A$. For convenience we denote an element of $\mathfrak{C}_A$ by $f_A$. We have already seen that only a finite number of MTDFs are basic MTDFs. All other MTDFs can be expressed as a convex combination of the functions in a subset of the set of all BMTDFs. The next result is a consequence of all these observations, and Theorem 1.3.

**Theorem 2.1.** *Let $A \subseteq \mathfrak{F}_{BT}(G)$ such that for any subset $A_i$ of $A$, the convex combination $f_{A_i}$ is an MTDF and $B_{A_1} \neq B_{A_2}$ or $P_{A_1} \neq P_{A_2}$ for any two nonempty subsets $A_1$ and $A_2$. Then $\mathfrak{C}_A$ is a simplex with dimension $|A| - 1$.*

*Proof.* We shall prove this by induction on the cardinality of $A$. The result is trivially true when $|A| = 1$. When $|A| = 2$, let $A = \{f_1, f_2\}$. Any $f \in \mathfrak{C}_A$ is a convex combination of $f_1$ and $f_2$. Therefore $\theta(\mathfrak{C}_A)$ is a 1 - simplex. Next assume that the result is true for all sets with cardinality at most $n$ that satisfy the given condition. Let $A' = \{f_1, f_2, \dots, f_{n+1}\}$ be a set containing $(n+1)$ BMTDFs. Form a set of $n$ BMTDFs - we shall denote it by $A$ - by removing $f_{n+1}$. By the induction assumption, $\mathfrak{C}_A$ is an $(n-1)$ - simplex. If $\mathfrak{C}_{A'}$ is an $n$ - simplex, then we are done. Suppose that $\mathfrak{C}_{A'}$ is not an $n$ - simplex. Then the image of $f_{(n+1)}$ lies in the same hyperplane, in which $\mathfrak{C}_A$ lies. Clearly $\theta(f_{n+1})$ is not an element of $\mathfrak{C}_A$. Otherwise $B_A = B_{A'}$ and $P_A = P_{A'}$, which is a contradiction. Now the line segment connecting any interior point of $\mathfrak{C}_A$ and $\theta(f_{n+1})$ intersects with a face of the simplex at some point. Let the pre-images of the interior point and the intersecting point be $g$ and $h$ respectively. Then $B_g = B_h$ and $P_g = P_h$. Since $h$ is a point on the face of the simplex, there exists a subset $\widehat{A}$ of $A$ such that $h = f_{\widehat{A}}$. This contradicts the given condition that for any non-empty subsets $A_1$ and $A_2$, $B_{A_1} \neq B_{A_2}$ or $P_{A_1} \neq P_{A_2}$. □

The converse of the above result is not true in general. The structure of $\mathfrak{F}_T(C_n)$ is studied in [10] for all values of $n$. It is interesting to note that the set $\mathfrak{F}_T(C_4)$ is convex, but it is not a simplex. Let us denote the four 0 - 1 MTDFs of $C_4$ by $f_1 = (1,1,0,0)$, $f_2 = (0,1,1,0)$, $f_3 = (0,0,1,1)$ and $f_4 = (1,0,0,1)$. We shall show that any MTDF $g$ of $C_4$ can be expressed as a convex combination of these four MTDFs. It is clear that all vertices of the four cycle are in $B_g$. If $g(v_1) = \delta$, then $g(v_3) = (1-\delta)$ and if $g(v_2) = \Delta$, then $g(v_4) = (1-\Delta)$. By equating the function values at each vertex, we get the system of equations, $\lambda_1 + \lambda_4 = \Delta$, $\lambda_1 + \lambda_2 = \delta$, $\lambda_2 + \lambda_3 = (1-\Delta)$ and $\lambda_3 + \lambda_4 = (1-\delta)$. This system is consistent. Solutions are obtained by assigning arbitrary value to one of the $\lambda_i$'s. Let $f_{i,j}$ denote the convex combination of the functions $f_i$ and $f_j$. Then the convex combinations $f_{1,2}$, $f_{1,3}$ and $f_{2,3}$ have either different boundary sets or different positive sets. But $B_{f_{1,2,3}} = B_{f_{1,3}}$ and $P_{f_{1,2,3}} = P_{f_{1,3}}$. Still the convex combination of the three points $(1,1,0,0)$, $(0,1,1,0)$ and $(0,0,1,1)$ make a two simplex in $I^4$.

If $\theta(\mathfrak{C}_A)$ where $A \subseteq \mathfrak{F}_{BT}(G)$ is isomorphic to a $(|A|-1)$ - simplex, then we denote this simplex by $I(A)$. Let $G$ be a graph with $|V| = n$. The Euclidean dimension of $\mathfrak{F}_T(G)$ is at most $n$ because $I(\mathfrak{F}_T(G)) \subset R^n$. Applying Theorem 2.1, we can prove the following result.

**Lemma 2.2.** *Let $G$ be a graph having order $n$ such that $|\mathfrak{F}_{BT}(G)| = r$, and $\mathfrak{F}_T(G)$ is convex.*

1. *If $r \leq (n+1)$ and for all different subsets $A_1$ and $A_2$ of $\mathfrak{F}_{BT}(G)$, $B_{f_{A_1}} \neq B_{f_{A_2}}$ or $P_{f_{A_1}} \neq P_{f_{A_2}}$, then $\mathfrak{F}_T(G)$ is an $(r-1)$ - simplex. Otherwise $\mathfrak{F}_T(G)$ is a convex polytope having dimension at most $n-1$.*

2. *If $r > (n+1)$, then $\mathfrak{F}_T(G)$ is a convex polytope having dimension at most $n$ and there exist two subsets $A_1$ and $A_2$ of $\mathfrak{F}_{BT}(G)$, such that $B_{f_{A_1}} = B_{f_{A_2}}$ and $P_{f_{A_1}} = P_{f_{A_2}}$.*

**Lemma 2.3.** *If $\mathfrak{F}_T(G)$ is not convex, then it is a simplicial complex.*

*Proof.* There exist maximal subsets $A_1, A_2, \ldots, A_s$ of $\mathfrak{F}_{BT}(G)$ such that $I(A_1), I(A_2), \ldots, I(A_s)$ are simplices in $R^n$. The union of all these simplices is a simplicial complex. □

**Theorem 2.4.** *For a complete bipartite graph $G = K_{m,n}$, the set $\mathfrak{F}_T(K_{m,n})$ is isomorphic to*

1. *an $(n-1)$ - simplex if $m = 1$ and $n \geq 2$;*

2. *a convex polytope, otherwise.*

*Proof.* Let $V_1 = \{u_1, u_2, \ldots, u_m\}$ and $V_2 = \{v_1, v_2, \ldots, v_n\}$ be the partition of $G = K_{m,n}$. We define the functions $f_{(u_i,v_j)}$ for $i = 1,2,\ldots,m$ and $j = 1,2,\ldots,n$ as follows.

$$f_{(u_i,v_j)}(x) = \begin{cases} 1 & \text{if } x = u_i \text{ or } v_j; \\ 0 & \text{otherwise.} \end{cases}$$

These functions are BMTDFs of the graph. Now we claim that the graph has no BMTDF other than these. If the graph has another 0 - 1 MTDF, then the function values of at least two vertices in either $V_1$ or $V_2$ must be equal to 1. But such a function cannot be an MTDF. Next suppose that the graph has a BMTDF $g$ which is not a 0 - 1 function. Then $P'_g \neq \emptyset$ and by Theorem 1.3, the system of equations contains only two equations. The rank of this system is two and since $g$ is basic, $|P'_g|$ is 2. This is possible only if each equation contains only one variable. Let us take $P'_g = \{u_1, v_1\}$. Then the system is $x_1 = 0, y_1 = 0$, where $x_1$ and $y_1$ are the variables representing the vertices $u_1$ and $v_1$ respectively. This is impossible because the boundary of every MTDF of $K_{m,n}$ is $V(K_{m,n})$ and if $|P'_g \bigcap V_i| \neq \emptyset$, then $|P'_g \bigcap V_i| > 1$ when $i = 1$ or 2. Thus it is clear that $K_{m,n}$ has only $0-1$ BMTDFs and $|\mathfrak{F}_{BT}(K_{m,n})| = mn$.

If $m = 1$ and $n \geq 2$, let $V(G) = \{v, v_1, v_2, \ldots, v_n\}$ and $d(v) = n$. Let $\mathfrak{F}_{BT}(K_{(1,n)}) = \{f_{(v,v_i)} : i = 1, 2, \ldots, n\}$. Then $B_{f_{(v,v_i)}} = V$ and $P_{f_{(v,v_i)}} = \{v, v_i\}$ for all $i$. Next let $S_1$ and $S_2$ be any two different nonempty subsets of $\mathfrak{F}_{BT}(K_{(1,n)})$ and the convex combinations of the elements of these sets be $h_1 = \sum_{f \in S_1} \lambda_f f$ and $h_2 = \sum_{g \in S_2} \lambda_g g$ respectively, where $\sum_{f \in S_1} \lambda_f = \sum_{g \in S_2} \lambda_g = 1$. The positive sets of $h_1$ and $h_2$ are always different. Also since $|\mathfrak{F}_T(K_{m,n})| < |V|$, we can conclude that the set of all MTDFs of $K_{1,n}$ is an $(n-1)$ - simplex. But when both $m$ and $n \geq 2$, the subsets $S_1 = \{f_{(u_1,v_1)}, f_{(u_1,v_2)}, f_{(u_2,v_1)}\}$ and $S_2 = \{f_{(u_1,v_1)}, f_{(u_1,v_2)}, f_{(u_2,v_1)}, f_{(u_2,v_2)}\}$ have the property $B_{S_1} = B_{S_2}$ and $P_{S_1} = P_{S_2}$. So $\mathfrak{F}_T(K_{m,n})$ is isomorphic to a complex polytope having maximum dimension $(m+n)$. □

**Theorem 2.5.** *$\mathfrak{F}_T(S(K_{1,n}))$ is a 1 - simplex.*

*Proof.* Let $V(S(K_{(1,n)})) = \{v, v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n\}$, $N(u_i) = \{v_i\}$, $N(v_i) = \{v, u_i\}$ and $N(v) = \{v_1, v_2, \ldots, v_n\}$. Let $g$ be an arbitrary MTDF of $G$. Since $N(u_i) = \{v_i\}$, we have $g(v_i) = 1$ for all $i$, where $1 \leq i \leq n$. Now if $g(v) = 0$, then we have $g(u_i) = 1$ for $i = 1, 2, \ldots, n$. If $g(u_i) = 0$ for $i = 1, 2, \ldots, n$, then $g(v) = 1$. Suppose $g(u_1) = r > 0$. Then $u_1 \in P_g$ and hence $v_1 \in B_g$. Consequently $g(v) = 1 - r$ and $g(u_i) = r$ for all $i = 1, 2, \ldots, n$. Now let $f_r : V \to [0, 1]$ be defined by

$$f_r(x) = \begin{cases} 1 & \text{if } x = v_i \text{ for } i = 1, 2, \ldots, n; \\ r & \text{if } x = u_i \text{ for } i = 1, 2, \ldots, n, \text{ where } r \in [0, 1]; \\ 1\text{-r} & \text{if } x = v. \end{cases}$$

Then the set of all MTDFs of $G$ is given by $\mathfrak{F}_T = \{f_r : 0 \leq r \leq 1\}$. The functions $f_0$ and $f_1$ take only the values 0 and 1 and hence are BMTDFs. The set $\mathfrak{F}_T$ can be divided into three equivalent classes, namely, $Y_1 = \{f_0\}$, $Y_2 = \{f_1\}$ and $Y_3 = \{f_r : 0 < r < 1\}$ and any function in $Y_3$ is a convex combination of $f_0$ and $f_1$. Thus $\mathfrak{F}_T(S(K_{1,n}))$ is a 1 - simplex. □

# 3 Characterization of graphs having $\mathfrak{F}_T$ isomorphic to one simplex.

Cockayne et al. obtained a characterization of the graphs having unique MTDF [3]. If $G$ has a unique MTDF, then $\mathfrak{F}_T(G)$ is isomorphic to a 0 - simplex. Reji Kumar [9] presented a characterization of the graphs having the set of all minimal dominating functions (the open neighbourhood analogue of MTDFs) isomorphic to a 1 - simplex. This motivated us to characterize the graphs having the set of all MTDFs isomorphic to a 1 - simplex. To proceed further we need the following definitions. For a graph $G$, let $C_2 = V - (C_0 \bigcup C_1)$. A $C_2$ path is a chain of vertices $\{v_1, v_2, \ldots, v_r\}$ ($r$ is odd), connected by edges, such that, $\{v_1, v_3, \ldots, v_{(r-2)}, v_r\} \subseteq C_2$ and $|N(v_i) \bigcap C_2| = 2$ and $|N(v_i) \bigcap C_1| = 0$ for all even values of $i$. Also $B(G) = \{v \in V : v = v_i$ where $i$ is even and $v_i$ is in some $C_2$ path$\}$

**Theorem 3.1.** *The set $\mathfrak{F}_T(G)$ is isomorphic to a 1 - simplex if and only if any two vertices $u, v \in V(G) - (C_0 \bigcup C_1)$ are connected by at least one $C_2$ path and $B(G) \subseteq B_f$ for any MTDF $f$ of $G$.*

*Proof.* Suppose, $\mathfrak{F}_T(G)$ is isomorphic to a 1 - simplex in $I^n$, where $n$ is the order of the graph. Then there exists two $n$ tuples $x_{i1}, x_{i2}, \ldots, x_{in}$ where $i = 1$ and 2, such that, all other $n$ tuples on the 1 - simplex are convex combinations of these two $n$ tuples. We define the functions $f_i(v_j) = x_{ij}$ for $i = 1, 2$ and $j = 1, 2, \ldots, n$. By our definition of the set $\mathfrak{F}_T(G)$, the functions $f_1$ and $f_2$ are MTDFs. Also all other MTDFs of $G$ are convex combinations of $f_1$ and $f_2$. So $f_1$ and $f_2$ are BMTDFs.

Next we claim that, $f_1$ and $f_2$ are $0-1$ MTDFs. To prove the claim, first suppose that $G$ has only one MTDS say $f$. Then we get $V(G) = C_0(G) \bigcup C_1(G)$ and $C_0(G) \bigcap C_1(G) = \emptyset$. Also $C_1 = R$. Let $g$ be any MTDF which is not a $0-1$ MTDF. Clearly $g(x) = 1$ for all $x \in C_1(G)$. So $g(x) > 0$ for some $x \in C_0((G)$. This contradicts the fact that $g$ is an MTDF, since there exists another MTDF $f < g$. So $G$ has at least two different MTDSs and hence at least two BMTDFs. But we know that $G$ has exactly two BMTDFs $f_1$ and $f_2$. Clearly $f_1$ and $f_2$ are the characteristic functions of two MTDSs of $G$.

Next let us partition $V$ in to three subsets $C_0$, $C_1$ and $C_2$. If $v \in C_2$, then either $f_1(v) = 0$ and $f_2(v) = 1$ or $f_1(v) = 1$ and $f_2(v) = 0$. Both cannot be either 0 or 1 simultaneously. So we can further partition the set $C_2$ into $C_{21}$ and $C_{22}$, such that

$$f_1(v) = \begin{cases} 1 & \text{if } v \in C_{21}; \\ 0 & \text{if } v \in C_{22}. \end{cases}$$

$$f_2(v) = \begin{cases} 1 & \text{if } v \in C_{22}; \\ 0 & \text{if } v \in C_{21}. \end{cases}$$

Let $f$ be a convex combination of $f_1$ and $f_2$. Next we proceed to prove the following claims about the corresponding system of linear equations of $f$.

1. Each equation contains exactly two variables.
2. If the system has $r$ variables, then its rank is $r-1$.
3. In the solution, the values of the variables are either $\triangle$ or $-\triangle$.

Proof of the claim (1): Each equation in the system represents a vertex in the boundary of the corresponding MTDF. We shall prove that for all $v \in B_f$, $|N(v) \bigcap P'_f| = 2$. At this moment, it is useful to note that $P'_f = C_2$. Suppose $v \in B_f$ contains three or more vertices of $C_2$. Then one of the sets $C_{21}$ and $C_{22}$ contains at least two of these vertices. For these vertices, either $f_1$ values or $f_2$ values are all 1, which implies either $v \notin B_{f_1}$ or $v \notin B_{f_2}$. Hence $v \notin (B_{f_1} \bigcap B_{f_2}) = B_f$. This is a contradiction.

Proof of the claim (2) : Suppose the rank of the system is less than $r-1$. Then the number of variables assigned arbitrary values is at least 2. These independent variables cannot occur together in one equation. Also there does not exist a subset of the equations $e_1, e_2, \ldots, e_r$ such that, one independent variable is present in $e_1$, another independent variable is in $e_r$ and $e_i$ and $e_{i+1}$ have one variable in common, for $i = 1, 2, \ldots, r$. This gives more than two $0-1$ MTDFs of $G$, which is a contradiction.

Proof of the claim (3) : Since the system has only one independent variable and each equation contains exactly two variables, the system has a solution in which the variables get the values either $\triangle$ or $-\triangle$. This proves the claim 3.

Next we claim that for any two variables $x_i$ and $x_j$ in the system, there exists a chain of equation $e_1, e_2, \ldots, e_r$ such that, $x_i$ is present in $e_1$, $x_j$ is in $e_r$ and $e_i$ and $e_{i+1}$ have one variable in common, for $i = 1, 2, \ldots, r$. If not, the system of equations can be partitioned into more than one set of equations (say $S_1, S_2, \ldots, S_p$), such that each set has the above mentioned property. If the number of equations in the set $S_i$ is $n_i$, then it contains $n_i + 1$ variables. Thus the whole system contains $\sum_i (n_i + 1)$ variables, all together in its $\sum_i n_i$ equations. Then the rank of the system must be at most $\sum_i n_i$ and the number of variables which are assigned independent values is at least $p > 1$. This is a contradiction to the claim 2.

Next we shall show that, in the graph, there exists a $C_2$ path between every pair of vertices $u, v \in C_2$. As a contrary suppose that there exists no such path between $u, v \in C_2$. Let $f$ be an MTDF of $G$. Rename $u$ as $u_1$.

Let $w_{11}, w_{12}, \ldots, w_{1r_1} \in B_f$ such that $u_1 \in N(w_{1i})$, where $i = 1, 2, \ldots, r_1$. Then $N(w_{1i}) \bigcap C_2 = 2$ for all $i$. Otherwise we get a contradiction to the claim 1. Let $N(w_{1i}) \bigcap C_2 = \{u_1, u_{1i}\}$. The corresponding equations are $x_1 + x_{1i}$, where $i = 1, 2, \ldots, r_1$. We can apply the same argument to each $u_{1i}$ again and include more and more equations to the system. Whenever a new variable is introduced in the system, both the the number of equations and the number of variables increase by one and whenever a variable repeats in the system, the number of equations alone increases by one. Since $C_2$ is a finite set, this procedure must end after some time. If the vertex $v$ is represented somewhere in the equations, then we are done. Otherwise we can repeat the above steps replacing the vertex $u$ by $v$ and get another set of equations, such that none of its variables occurs in the first set. Suppose there are $n_1$ and $n_2$ variables in the first and second set of equations respectively. Then the rank of the whole system, which contains $n_1 + n_2$ equations is $n_1 + n_2 - 2$. Thus we get a contradiction to the claim 2.

To prove the converse, assume that $G$ is a graph with the specified properties. Take an arbitrary MTDF $f$ of $G$. Then $B(G) \subseteq B_f$. The corresponding system of linear equations will give only $\Delta$ or $-\Delta$ as solutions for the variables corresponding to the elements of $C_2$. So there exist two $0-1$ BMTDFs $f_1$ and $f_2$ such that, $f$ is a convex combination of $f_1$ and $f_2$. □

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# References

[1] E. J. Cockayne, G. MacGillivray and C. M. Mynhardt, Convexity of minimal dominating functions and universal in graphs, *Bull. Inst. Combin. Appl.*, **5** (1992), 37 - 48.

[2] E. J. Cockayne and C. M. Mynhardt, A characterization of universal minimal total dominating functions in trees, *Discrete Mathematics,* **141** (1995), 75 - 84.

[3] E. J. Cockayne, C. M. Mynhardt and B. Yu, Universal minimal total dominating functions in graphs, *Networks,* **24** (1994), 83 - 90.

[4] E. J. Cockayne, C. M. Mynhardt and B. Yu, Total dominating functions in trees: minimality and convexity, *Journal of Graph Theory,* **1** (1995), 83 - 92.

[5] E. J. Cockayne, G. Fricke, S. T. Hedetniemi and C. M. Mynhardt, Properties of minimal dominating functions of graphs, *ARS Combinatoria,* **41** (1995), 107 - 115.

[6] B. Grunbaum, Convex Polytopes. With the cooperation of V. Klee, M. A. Perles and G. C. Shephard. *Pure and Applied Mathematics,* Vol. 16, Interscience Publishers, John Wiley and Sons, Inc., New York, 1967.

[7] S. M. Hedetniemi, S. T. Hedetniemi and T. V. Wimer, Linear time resource allocation algorithm for trees, Technical Report URI - 014, Department of Mathematical Sciences, Clemson University, 1987.

[8] K. Reji Kumar, Studies in graph theory - dominating functions, Ph. D thesis, Manonmaniam Sundaranar University, Tirunelveli, India, 2004.

[9] K. Reji Kumar, Topological properties of the set of all minimal dominating functions of graphs, *Lecture note series of RMS, Proc. of ICDM 2006*, 249 - 254.

[10] K. Reji Kumar and G. MacGillivray, Structure of the set of all minimal total dominating functions of some classes of graphs, *Submitted.*

[11] D. B. West, Graph Theory : an introductory course, Prentice Hall, New York, 2002.

[12] B. Yu, Convexity of minimal total dominating functions in graphs, *Journal of Graph Theory,* **4** (1997), 313 - 321.