Characterizations of trees with unique minimum locating-dominating sets*

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Abstract

A set D of vertices in a graph $G = (V, E)$ is a locating-dominating set if for every two vertices u, v of $V \setminus D$ the sets $N(u) \cap D$ and $N(v) \cap D$ are non-empty and different. We establish two equivalent conditions for trees with unique minimum locating-dominating sets.

1 Introduction

In a graph $G = (V, E)$, the *open neighborhood* of a vertex $v \in V$ is the set $N(v) = \{u \in V,\ uv \in E\}$, the *closed neighborhood* is $N[v] = N(v) \cup \{v\}$, and the *degree* of v denoted by $\deg_G(v)$ is the size of its open neighborhood.

A set $D \subseteq V$ is a *locating-dominating set* if every two vertices x, y of $V \setminus D$ satisfy $N(x) \cap D \neq N(y) \cap D \neq \emptyset$. The *locating-domination number* $\gamma_L(G)$ is the minimum cardinality of a locating-dominating set. If D is any locating-dominating set minimum size, then we call D a $\gamma_L(G)$*-set*. Locating-domination was introduced by Slater [8, 9]. For more details on domination in graphs, see the monographs by Haynes, Hedetniemi, and Slater [4, 5].

The main goal of this paper is the characterization of trees with unique minimum locating-dominating sets. A graph G will be called a *unique locating-domination graph*, or *ULD-graph* for short, if it has a unique $\gamma_L(G)$-set. In [7], Lane gave a computer program that found all ULD-trees of

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order at most 15 and a construction of trees T with a unique $\gamma_L(T)$-set but no characterization of such trees. Much research has been done on the uniqueness of some variation of minimum dominating sets in the class of trees, see for example [1, 2, 3, 6].

A vertex of degree one is called a *leaf*, and its neighbor is called a *support vertex*. The set of leaves adjacent to a support vertex v is L_v and v is called *strong* if $|L_v| \geq 2$. We denote by $S(T)$ the set of support vertices of T.

We begin by giving two useful observations.

Observation 1 *If T is a nontrivial tree with a unique $\gamma_L(T)$-set D, then:*

a) every support vertex belongs to D,

b) every support vertex is adjacent to exactly one leaf,

c) D contains no leaf.

Proof. a) Assume that v is a support vertex such that $v \notin D$. Then D contains all leaves adjacent to v. By substituting any leaf of L_v by v in D we get a second $\gamma_L(T)$-set, a contradiction.

b) Assume now that v is adjacent to two or more leaves. Let v', v'' be any two leaves of L_v. Then the uniqueness of D implies that $v \in D$, and the minimality of D implies that D contains all leaves of v except one, say v', but then $\{v''\} \cup D \setminus \{v'\}$ is also a $\gamma_L(T)$-set, a contradiction.

c) Let u be the unique leaf adjacent to v and suppose that $u \in D$. By (a) $v \in D$. Since D is a $\gamma_L(T)$-set, the set $D \setminus \{u\}$ is not a locating-dominating set, so there is a vertex w such that $w \notin D$ and the only neighbor of w in D is v; but then $\{w\} \cup D \setminus \{u\}$ is also a $\gamma_L(T)$-set, a contradiction. □

Observation 2 *The path P_5 is the smallest nontrivial ULD-tree.*

Proof. Let T be the smallest nontrivial ULD-tree. By Observation 1 (b), T is not a star and so T has diameter at least three. The path P_4 is the only tree of diameter three with no strong support vertices, and P_4 admits at least two $\gamma_L(T)$-sets. It follows that T has diameter at least four. Clearly, among such trees P_5 is the smallest ULD-tree. □

2 Characterizations

Now we give a sufficient condition for a graph G to admit a unique $\gamma_L(G)$-set.

Lemma 3 *Let G be a connected nontrivial graph and D be a $\gamma_L(G)$-set. If for every $v \in D$ we have $\gamma_L(G \setminus v) > \gamma_L(G)$, then G is a ULD-graph.*

Proof. Assume that G has a second $\gamma_L(G)$-set D' and let v be a vertex of $D \setminus D'$. Then D' is a locating dominating set of $G \setminus v$ and so $\gamma_L(G \setminus v) \leq |D'| = |D|$, a contradiction. □

Next we will show that the converse of Lemma 3 is true for trees but not valid for all graphs. To see this, consider the graph obtained from two disjoint cycles C_5, $y_1y_2y_3y_4y_5$ and $x_1x_2x_3x_4x_5$ by identifying vertices x_1 and y_1. Let w denote the resulting new vertex. Add two new vertices u, v and edges ux_i, vy_i for $i = 2, 3, 4, 5$. Then $\{x_2, x_5, y_2, y_5\}$ is a unique $\gamma_L(G)$-set, but for every $z \in \{x_2, x_5, y_2, y_5\}$, $\gamma_L(G \setminus z) = \gamma_L(G)$.

Lemma 4 *Let T be a nontrivial ULD-tree and D be its unique $\gamma_L(T)$-set. Then for every vertex $v \in D$, $\gamma_L(T \setminus v) > \gamma_L(T)$.*

Proof. Let T be a ULD-tree with $\gamma_L(T)$-set D, and assume to the contrary that $\gamma_L(T \setminus v) \leq \gamma_L(T)$ for some vertex $v \in D$. By Observation 1 (c), v cannot be a leaf. Let $u_1, u_2, \ldots, u_p$ be the neighbors of v in T and T_{u_i} be the component of $T \setminus v$ that contains u_i. Clearly $p \geq 2$. Let D' be a $\gamma_L(T \setminus v)$-set and $D_i = D \cap T_{u_i}$ and $D'_i = D' \cap T_{u_i}$ for every i. By assumption $|D'| \leq |D|$. Note that D' contains at most one vertex of $\{u_1, u_2, \ldots, u_p\}$, for otherwise D' would be a second $\gamma_L(T)$-set since $v \notin D'$, contradicting the uniqueness of D. Also for every $i \in \{1, \ldots, p\}$, D'_i is a $\gamma_L(T_{u_i})$-set, $D_i \cup \{v\}$ is a locating dominating set of the subgraph $T_{u_i} + v$, and such a set is the smallest containing v. Since $D'_i \cup \{v\}$ is also a locating dominating set of $T_{u_i} + v$, it follows that $|D_i \cup \{v\}| \leq |D'_i \cup \{v\}|$, so $|D_i| \leq |D'_i|$ for every $1 \leq i \leq p$. Let $Q \subseteq \{1, 2, \ldots, p\}$ be such that $|D_\ell| < |D'_\ell|$ if and only if $\ell \in Q$. Thus $|D'_\ell| \geq |D_\ell| + 1$ for every $\ell \in Q$. Furthermore since $D_\ell \cup \{u_\ell\}$ is a locating dominating set of T_{u_ℓ}, we have $|D'_\ell| \leq |D_\ell \cup \{u_\ell\}| = |D_\ell| + 1$, and so $|D'_\ell| = |D_\ell| + 1$ for every $\ell \in Q$. Therefore $|D| \geq |D'| = \sum_{i=1}^{p} |D'_i| = \sum_{i=1}^{p} |D_i| + |Q| = |D| - 1 + |Q|$, so $|Q| \leq 1$. If $|Q| = 0$, then $|D'| = |D| - 1$ and, for some $u_\ell \notin D$, $D' \cup \{u_\ell\}$ is a second $\gamma_L(T)$-set, a contradiction. Thus $|Q| = 1$. Put $Q = \{j\}$. Since $|D'_j| = |D_j| + 1$, we have $u_j \notin D$, but then $(\bigcup_{i=1, i \neq j}^{p} D'_i) \cup D_j \cup \{u_j\}$ is a $\gamma_L(T)$-set different from D, a contradiction. Thus $\gamma_L(T \setminus v) > \gamma_L(T)$ for every $v \in D$. □

As an immediate consequence of Lemmas 3 and 4 we obtain our first characterization of ULD-trees.

Theorem 5 *A nontrivial tree T is a ULD-tree with a unique $\gamma_L(T)$-set D if and only if for every vertex $v \in D$, $\gamma_L(T \setminus v) > \gamma_L(T)$.*

Let T_1 and T_2 be two vertex-disjoint ULD-trees, each of order at least 5. Let A_1 be the unique $\gamma_L(T_1)$-set and A_2 the unique $\gamma_L(T_2)$-set. We define two operations that can be used for linking T_1 and T_2 and producing a new ULD-tree.

- **Operation $\mathcal{O}_1$.** Let T be the tree obtained from T_1 and T_2 by adding an edge joining a vertex of A_1 in T_1 with a vertex of A_2 in T_2.
- **Operation $\mathcal{O}_2$.** Let T be the tree obtained from T_1 and T_2 by adding an edge joining a vertex of $V(T_1) \setminus A_1$ and a vertex of $V(T_2) \setminus A_2$.

Lemma 6 *The tree T obtained from T_1 and T_2 by performing Operation $\mathcal{O}_1$ or $\mathcal{O}_2$ is a ULD-tree and $A_1 \cup A_2$ is the unique $\gamma_L(T)$-set.*

Proof. Let D be a $\gamma_L(T)$-set, $D_1 = D \cap V(T_1)$ and $D_2 = D \cap V(T_2)$. Let v_1v_2 be the edge added between $v_1 \in T_1$ and $v_2 \in T_2$. It is clear that in each case $A_1 \cup A_2$ is a locating-dominating set of T and so $\gamma_L(T) \leq \gamma_L(T_1) + \gamma_L(T_2)$.

Suppose that T is produced by $\mathcal{O}_1$. If v_1, v_2 are both in D, then D_1 and D_2 are two locating-dominating sets of T_1 and T_2, respectively. Therefore $\gamma_L(T_1) + \gamma_L(T_2) \leq |D_1| + |D_2| = \gamma_L(T)$, and so $\gamma_L(T) = \gamma_L(T_1) + \gamma_L(T_2)$. It follows from the uniqueness of A_1 and A_2 that $A_1 = D_1$, $A_2 = D_2$ and $A_1 \cup A_2$ is the unique $\gamma_L(T)$-set, so T is a ULD-tree. If v_1, v_2 are not in D, then D_1 and D_2 are two locating-dominating sets of T_1 and T_2, respectively. Hence $\gamma_L(T_1) + \gamma_L(T_2) \leq |D_1| + |D_2| = \gamma_L(T)$, implying the equality $\gamma_L(T) = \gamma_L(T_1) + \gamma_L(T_2)$. It follows for $i \in \{1, 2\}$ that D_i is a $\gamma_L(T_i)$-set different from A_i, because $v_i \notin D_i$, a contradiction to the uniqueness of A_i. Thus let us assume without loss of generality that $v_1 \in D_1$ and $v_2 \notin D_2$. Then D_1 is a locating-dominating set of T_1 and so $|D_1| \geq |A_1|$. Also D_2 is a locating-dominating set of $T_2 \setminus v_2$; so, since A_2 is a unique $\gamma_L(T_2)$-set with $v_2 \in A_2$, then by Lemma 4, we have $|D_2| \geq \gamma_L(T_2 \setminus v_2) > |A_2|$. Then $\gamma_L(T) = |D_1| + |D_2| > |A_1| + |A_2| \geq \gamma_L(T)$, a contradiction.

Now suppose that T is produced by $\mathcal{O}_2$. If v_1, v_2 are both in D, or both not in D, then D_1 and D_2 are two locating-dominating sets of T_1 and T_2 respectively, and so $\gamma_L(T_1) + \gamma_L(T_2) \leq |D_1| + |D_2| = \gamma_L(T)$. Thus $\gamma_L(T) = \gamma_L(T_1) + \gamma_L(T_2)$. If $v_1, v_2 \in D$, then $v_i \in D_i$ ($i \in \{1, 2\}$), in which case D_i is a second $\gamma_L(T_i)$-set, a contradiction. So v_1, v_2 are both not in D. It follows from the uniqueness of A_1 and A_2 that $A_1 = D_1$, $A_2 = D_2$ and $A_1 \cup A_2$ is a unique $\gamma_L(T)$-set. Finally assume $v_1 \in D_1$ and $v_2 \notin D_2$. Then D_1 is a locating dominating set of T_1 and since A_1 is unique $|D_1| \geq |A_1| + 1$. Since $v_1 \notin A_1$, $A_1 \cup D_2 \cup \{v_2\}$ is a $\gamma_L(T)$-set; but since $D_2 \cup \{v_2\}$ is a locating dominating set of T_2 and T_2 is a ULD-tree, $|D_2| + 1 \geq |A_2| + 1$. It follows that D has size at least $|A_1| + |A_2| + 1$, a contradiction. □

Let $T_1, \ldots, T_k$ be k vertex-disjoint ULD-trees, each of order at least five, with A_i as a unique $\gamma_L(T_i)$-set for every $1 \le i \le k$. Let $u_i \in A_i$ for each i. We define the following two operations:

- **Operation $\mathcal{O}_3$:** Let T be the tree obtained from $T_1, \ldots, T_k$ $(k \ge 3)$ by adding a new vertex v and edges vu_i for every $1 \le i \le k$.
- **Operation $\mathcal{O}_4$:** Let T be the tree obtained from $T_1, \ldots, T_k$ $(k \ge 2)$ by adding a star $K_{1,k}$ $(k \ge 2)$ of center vertex v and leaves $w_1, \ldots, w_k$, and edges w_iu_i for every i, with the condition that for at least two values of $i \in \{1, \ldots, k\}$ vertex u_i has a private neighbor in T_i with respect of A_i.

Lemma 7 *The tree T obtained from $T_1, \ldots, T_k$ by performing Operation $\mathcal{O}_3$ is a ULD-tree and $A_1 \cup \cdots \cup A_k$ is its unique $\gamma_L(T)$-set.*

Proof. Let D be a $\gamma_L(T)$-set and let $D_i = D \cap V(T_i)$ for every $1 \le i \le k$. Since $k \ge 3$, $A_1 \cup \cdots \cup A_k$ is a locating-dominating set and so $\gamma_L(T) \le \sum_{i=1}^{k} \gamma_L(T_i)$. Assume for a contradiction that $v \in D$ and let u_i be a vertex such that $u_i \notin D$. Then $u_i \notin D_i$ and D_i is a locating dominating set of $T_i \setminus u_i$. Since A_i is a unique $\gamma_L(T_i)$-set and $u_i \in A_i$, by Theorem 5, we have $|D_i| > |A_i|$. Then $(D \setminus D_i) \cup A_i$ is a locating dominating set of T of size less than $|D|$, a contradiction. Thus $u_i \in D$ for every $1 \le i \le k$. It follows that $D \setminus \{v\}$ is a locating dominating set, which is again a contradiction. Thus D does not contain v, which implies that each D_i is a locating dominating set of T_i. Thus $\sum_{i=1}^{k} \gamma_L(T_i) \le \sum_{i=1}^{k} |D_i| = \gamma_L(T)$, which implies the equality. It follows from the uniqueness of A_i that $D_i = A_i$ and $A_1 \cup \cdots \cup A_k$ is a unique $\gamma_L(T)$-set. □

Lemma 8 *The tree T obtained from $T_1, \ldots, T_k$ by performing Operation $\mathcal{O}_4$ is a ULD-tree and $A_1 \cup \cdots \cup A_k \cup \{v\}$ is its unique $\gamma_L(T)$-set.*

Proof. Since $k \ge 2$, $A_1 \cup \cdots \cup A_k \cup \{v\}$ is a locating-dominating set, and so $\gamma_L(T) \le \sum_{i=1}^{k} \gamma_L(T_i) + 1$. Let D be a $\gamma_L(T)$-set and let $D_i = D \cap V(T_i)$ for every $1 \le i \le k$. Let $Q \subseteq \{1, \ldots, k\}$ such that $u_i \notin D$ if and only if $i \notin Q$. Then D contains $|Q|$ vertices of $\{v, w_i,\ i \in Q\}$. Since for each $i \in Q$, A_i is a unique $\gamma_L(T_i)$-set, $u_i \notin D$ and D_i is a locating dominating set of $T_i \setminus u_i$, it follows from Theorem 5 that $|D_i| > |A_i|$. But then $(\bigcup_{i \notin Q} D_i) \cup (\bigcup_{i \in Q} A_i) \cup \{v\}$ is a locating dominating set of T of size less than $|D|$, a contradiction. Thus $u_i \in D$ for every i. It follows that each D_i is a locating dominating set of T_i and D contains at least one vertex from $N[v]$ to dominate v. Hence $\sum_{i=1}^{k} \gamma_L(T_i) \le \sum_{i=1}^{k} |D_i| \le |D| - 1$, which implies equality. Note that uniqueness of A_i implies that the private neighbor of u_r with respect to A_r remains a private neighbor of u_r with respect of D in T (else D_r would be a second $\gamma_L(T_r)$-set), and similarly for the private neighbor of u_s. Thus D contains v, for otherwise $w_r, w_s \in D$ and then $\{v\} \cup D \setminus \{w_r, w_s\}$ is a locating dominating set smaller than D. Consequently $A_1 \cup \cdots \cup A_k \cup \{v\}$ is a unique $\gamma_L(T)$-set. □

The *corona* of a tree T is the tree constructed from a copy of T by adding, for each vertex $v \in V(T)$, a new vertex v' and an edge vv'. Let $\mathcal{F}$ be the family of trees obtained from coronas of order at least four by subdividing once each edge between support vertices. For instance $P_5 \in \mathcal{F}$ since it is obtained from a path P_4 (which is a corona of P_2) by subdividing the edge linking the support vertices.

The following observation is easy to establish.

Observation 9 *If $T \in \mathcal{F}$, then the set of support vertices of T is the unique $\gamma_L(T)$-set.*

We now are ready to establish our main result.

Theorem 10 *Let T be a tree of order $n \geq 2$. Then the following conditions are equivalent:*

(a) T is a ULD-tree.

(b) T has a $\gamma_L(T)$-set D such that $\gamma_L(T \setminus v) > \gamma_L(T)$ for every $v \in D$.

(c) T is either in $\mathcal{F}$ or can be constructed from disjoint trees of $\mathcal{F}$ by a finite sequence of operations $\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3$ or $\mathcal{O}_4$.

Proof. By Theorem 5, we have (a)⇔(b). By Observation 9 and Lemmas 6, 7 and 8, we have (c)⇒(a). Hence it remains to show that (a)⇒(c). Let T be a ULD-tree and D its unique $\gamma_L(T)$-set. We use induction on the order n of T. By Observation 2, T has diameter at least four and $n \geq 5$. If $n = 5$, then $T = P_5$ and $P_5 \in \mathcal{F}$, establishing the base case. Assume that every nontrivial ULD-tree T' of order $n' < n$ is in $\mathcal{F}$ or can be constructed from disjoint trees of $\mathcal{F}$ by a finite sequence of operations $\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3$ or $\mathcal{O}_4$. Let T be a ULD-tree of order n. By Observation 1, every support vertex is adjacent to exactly one leaf, so D contains every support vertex and does not contain any leaf.

Assume first that T contains two adjacent vertices u and v that are both in D or both not in D. Let T_u and T_v be the subtrees of T obtained by removing the edge uv. Clearly $D_u = D \cap T_u$ and $D_v = D \cap T_v$ are two locating dominating sets of T, and so $\gamma_L(T_u) + \gamma_L(T_v) \leq \gamma_L(T)$. Equality is obtained from the fact that the union of any $\gamma_L(T_u)$-set and any $\gamma_L(T_v)$ is a locating dominating set of T. Since D is the unique $\gamma_L(T)$-set, it follows that D_u is the unique $\gamma_L(T_u)$-set and D_v is the unique $\gamma_L(T_v)$-set.

By the induction hypothesis, T_u (resp. T_v) is in $\mathcal{F}$ or can be constructed from disjoint trees of $\mathcal{F}$ by a finite sequence of operations $\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3$ or $\mathcal{O}_4$. Thus T can be obtained from T_u and T_v by using Operation $\mathcal{O}_1$ or $\mathcal{O}_2$. Thus we may now assume that each of D and $V \setminus D$ is independent.

Suppose that there exists a vertex $z \in V \setminus D$ of degree at least 3. Let $t = \deg_T(z)$ and $y_1, \ldots, y_t$ be the neighbors of z in D. Consider the forest obtained by removing z. Clearly, $|D| = \sum_{i=1}^{t} |D_i|$, where $D_i = D \cap T_{y_i}$. Also the uniqueness of D implies that D_i is a unique $\gamma_L(T_{y_i})$-set for each $1 \leq i \leq t$. By the induction hypothesis, T_{y_i} is in $\mathcal{F}$ or can be constructed from disjoint trees of $\mathcal{F}$ by a finite sequence of operations $\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3$ or $\mathcal{O}_4$. Thus T can be obtained from $T_{y_1}, \ldots, T_{y_t}$ by Operation $\mathcal{O}_3$. Thus from now on we suppose that each vertex of $V \setminus D$ has degree at most two.

Suppose now that D contains a vertex x that is not a support vertex. Then x has degree at least two. Let $w_1, \ldots, w_p$ ($p \geq 2$) be the neighbors of x in $V \setminus D$. Since as assumed above $\deg_T(w_i) = 2$ for every i, let u_i denote the second neighbor of w_i for every i. Assume for a contradiction that at most one vertex of $\{u_1, \ldots, u_p\}$, say u_1, has a private neighbor with respect to D (note that such a private neighbor will be a leaf). Then $\{w_1\} \cup D \setminus \{x\}$ is a second $\gamma_L(T)$-set, a contradiction. Thus at least two vertices from $\{u_1, \ldots, u_p\}$ have private neighbors with respect to D. Let $T' = T \setminus \{x, w_1, \ldots, w_p\}$. It can be seen easily that $|D| = \sum_{i=1}^{p} \gamma_L(T_{u_i}) + 1 = \sum_{i=1}^{p} |D_i| + 1$ and D_i is a unique $\gamma_L(T_{u_i})$-set for each $1 \leq i \leq p$, where u_j has a private neighbor with respect to its $\gamma_L(T_{u_j})$-set for at least two values of j. By the induction hypothesis, every T_{u_i} either is in $\mathcal{F}$ or can be constructed from disjoint trees of $\mathcal{F}$ by a finite sequence of operations $\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3$ or $\mathcal{O}_4$. Thus T can be obtained from $T_{u_1}, \ldots, T_{u_p}$ by Operation $\mathcal{O}_4$.

Finally we assume that every vertex of D is a support vertex. Since every vertex of $V \setminus D$ has degree at most two, we have $T \in \mathcal{F}$. This completes the proof. □

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