Super edge-graceful labelings of total stars and total cycles*

Abdollah Khodkar
Department of Mathematics
University of West Georgia
Carrollton, GA 30118
akhodkar@westga.edu

Kurt Vinhage
Department of Mathematics
Florida State University
Tallahassee, FL 32306
kwv06@fsu.edu

Abstract

Let $[n]^*$ denote the set of integers $\{-\frac{n-1}{2},\ldots,\frac{n-1}{2}\}$ if n is odd, and $\{-\frac{n}{2},\ldots,\frac{n}{2}\}\setminus\{0\}$ if n is even. A super edge-graceful labeling f of a graph G of order p and size q is a bijection $f : E(G) \rightarrow [q]^*$, such that the induced vertex labeling f^* given by $f^*(u) = \sum_{uv\in E(G)} f(uv)$ is a bijection $f^* : V(G) \rightarrow [p]^*$. A graph is super edge-graceful if it has a super edge-graceful labeling. We prove that total stars and total cycles are super edge-graceful.

Keywords: labeling in graphs; edge labeling; super edge-graceful labeling

1 Introduction

In this paper we consider only simple, finite, undirected graphs. We define the set of integers $[n]^*$ to be $\{-\frac{n-1}{2},\ldots,\frac{n-1}{2}\}$ if n is odd, and $\{-\frac{n}{2},\ldots,\frac{n}{2}\}\setminus\{0\}$ if n is even. Notice that the cardinality of $[n]^*$ is n, and $[n]^*$ contains 0 if and only if n is odd. A graph of order p and size q is said to be *super edge-graceful* if there is a bijection $f : E(G) \rightarrow [q]^*$, such that the

*Research supported by NSF Grant DMS0648919, University of West Georgia

induced vertex labeling f^* given by $f^*(u) = \sum_{uv\in E(G)} f(uv)$ is a bijection $f^* : V(G) \to [p]^*$.

A graph of order p and size q is *edge-graceful* [2] if the edges can be labeled by $1, 2, \ldots, q$ such that the vertex sums are distinct (mod p). A necessary condition for a graph with p vertices and q edges to be edge-graceful is that $q(q+1) \equiv \frac{p(p-1)}{2}$ (mod p).

Super edge-graceful labelings (SEGL) were first considered by Mitchem and Simoson [7] showing that super edge-graceful trees are edge-graceful. In particular, Mitchem and Simoson noticed that if G is a super-edge graceful graph and $p|q$, if q is odd, or $p|q+1$, if q is even, then G is edge-graceful. Some families of graphs have been shown to be super-edge graceful by explicit labelings. It is known that, for example, paths of all orders except 2 and 4 and cycles of all orders except 4 and 6 are super edge-graceful [1], as are trees of odd order with three even vertices [5] and complete graphs of all orders except 1, 2 and 4 [3]. In [4] it is shown that all complete bipartite graphs are super edge-graceful except for $K_{2,2}$, $K_{2,3}$, and $K_{1,n}$ if n is odd.

For a graph $G = (V, E)$ we associate the *total graph* $T(G)$ as follows: $V(T(G)) = V(G) \cup E(G)$ and $E(T(G)) = E(G) \cup \{(v, (u, v)) \mid v \in V(G)$ and $\{u, v\} \in E(G)\}$. In this paper we deal with the total stars $T(\mathrm{St}(n))$ and the total cycles $T(C_n)$, where $\mathrm{St}(n)$ is the star graph with n vertices and C_n is the cycle graph with n vertices. We prove that the total stars and the total cycles are super edge-graceful. This confirms that Conjectures 2 and 3 of [6] are true.

2 Total stars

The total star $T(\mathrm{St}(2n+1))$ has $4n+1$ vertices and $6n$ edges. So the vertex labels required for a super edge-graceful labeling are $\{0, \pm1, \pm2, \ldots, \pm2n\}$ and the edge labels needed are $\{\pm1, \pm2, \ldots, \pm3n\}$. Similarly, the total star $T(\mathrm{St}(2n))$ has $4n-1$ vertices and $6n-3$ edges. So the vertex labels required for a super edge-graceful labeling are $\{0, \pm1, \pm2, \ldots, \pm(2n-1)\}$ and the edge labels needed are $\{0, \pm1, \pm2, \ldots, \pm(3n-2)\}$.

Theorem 1. *$T(\mathrm{St}(2n+1))$ is super-edge graceful for every $n \geq 1$.*

Proof. Because $2n+1$ is odd, there are an even number ($2n$) of edges in the original star. Thus, when the total star is taken, there are an even number of 3-cycles joined at a single point. Consider n of these 3-cycles independently from the rest. Label each of the edges not incident to the center vertex as $-n, -(n+1), -(n+2), \ldots, -(2n-1)$. On each of the other edges, then, label as follows (see Figure 1):

1. for the 3-cycle with edge labeled $-n$, label the other two edges $3n$ and $3n-1$;

2. for each other 3-cycle, if m represents the label already placed, label the other edges as $2n + m$ and $4n + m - 1$.

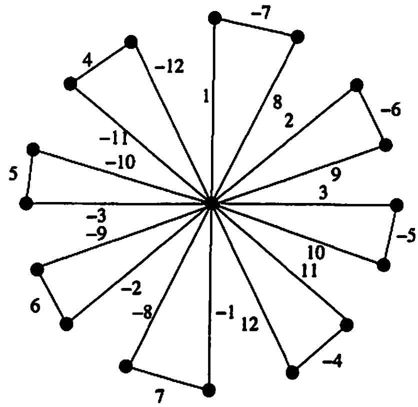

Figure 1: A SEGL of $T(St(9))$

Generating the associated vertex labels from these edge labelings, we see that for the first case, we get the vertex labels $2n$ and $2n - 1$. For the others, if m is the edge label not incident to the center vertex, then the associated vertex labels with it will be: $(2n + m) + m = 2(n + m)$ and $(4n + m - 1) + m = 2n + 2(n + m) - 1$. Note that when m ranges from $-(2n - 1)$ to $-(n + 1)$ each even number from $-2(n - 1)$ to -2 appears once from the first half of the labels, and each odd number from 1 to $2n - 3$ appears once from the other. The final 3-cycle produces the vertex labels $2n$ and $2n - 1$, making each of the vertex labels from 1 to $2n$ appears exactly once in the n 3-cycles in absolute value. In addition, each of the edge labels from 1 to $3n$ appears exactly once in this labeling.

To complete the labeling, copy the labeling produced above, replacing the values with their opposites, for the remaining n 3-cycles. Thus, since each edge and vertex label appeared exactly once in absolute value for the first n 3-cycles, their opposites will appear in the second n 3-cycles. The center vertex, by construction, will have the label of zero. This completes the proof. □

Theorem 2. *$T(\mathrm{St}(2n))$ is super-edge graceful for every $n \geq 1$.*

Proof. Since $2n$ is even, there are an odd number $(2n - 1)$ of edges in the original star. When the total graph is taken, there are $2n - 1$ 3-cycles connected at a single vertex. Select one such 3-cycle, and label its edges with zero as the edge not incident to the center vertex and the other two as $\pm(2n - 1)$. Pick, from the $2n - 2$ remaining 3-cycles, $n - 1$ of them. For each $0 \leq m \leq n - 2$, label an edge of a 3-cycle not incident to the center vertex with $-(n+m)$. Then label the remaining two edges on each 3-cycle as $n-m-1$ and $3n-m-2$ (see Figure 2).

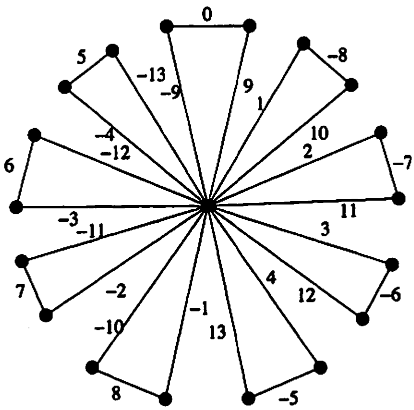

Figure 2: A SEGL of $T(St(10))$

Letting m range from 0 to $n-2$ shows that each of the integers from 1 to $3n-2$ appears exactly once in absolute value, with the exception of $2n-1$. Note that each of the two vertices that are not the center vertex in the 3-cycles will have labels $-(n+m)+n-m-1 = -2m-1$ and $-(n+m)+3n-m-2 = 2(n-m)-2$. The first of these produce, in absolute value, all odd integers from 1 to $2n-3$, and the others produce the even integers between 2 and $2n-2$, so that every integer from 1 to $2n-2$ appears exactly once in absolute value. Put the opposite edge labels on the remaining $n-1$ 3-cycles to make every edge label and vertex label appear exactly once, with the triangle singled out in the start containing the missing edge labels 0 and $\pm(2n-1)$, and generating the missing vertex labels $\pm(2n-1)$. The center vertex, again, produces label zero. This completes the proof. □

Now we are ready to state the main result of this section.

Theorem 3. *The total star $T(\mathrm{St}(n))$ is super-edge graceful for every $n \geq 2$.*

3 The total cycles

The total cycle $T(C_n)$ has a unique cycle of length n if $n \geq 4$ and we call it the inner cycle and a unique cycle of length $2n$ if $n \geq 3$, which is called the outer cycle. In $T(C_3)$ the original cycle is the inner cycle. In this section we assume $(u_1, u_2, \ldots, u_n)$ is the inner cycle and $(u_1, w_1, u_2, w_2, \ldots, u_n, w_n)$ is the outer cycle. The induced vertex label for a vertex v is denoted by $\ell(v)$ in this section. Consider $T(C_{16})$, displayed in Figure 3. We show how we can find a SEGL for this graph. Label the outer edges of $T(C_{16})$ as shown in Figure 3. With this labeling $\ell(w_i) = -18 + 2i$ for $1 \le i \le 8$ and $\ell(w_i) = -\ell(w_{17-i})$ for $9 \le i \le 16$. In addition, $\ell(u_1) = \ell(u_9) = 0$, $\ell(u_i) = -19 + 2i$ for $2 \le i \le 8$ and $\ell(u_i) = -\ell(u_{18-i})$ for $10 \le i \le 16$. The remaining edge labels are $\{\pm1, \pm2, \ldots, \pm7, \pm16\}$, which will be used to label the edges of the inner cycle. Note that labeling the edges of inner cycle will not change the labels of vertices $\{w_1, w_2, \ldots, w_{16}\}$. Figure 4 displays a SEGL for $T(C_{16})$. In this labeling the vertices of inner cycle have all odd labels and the other vertices have all even labels.

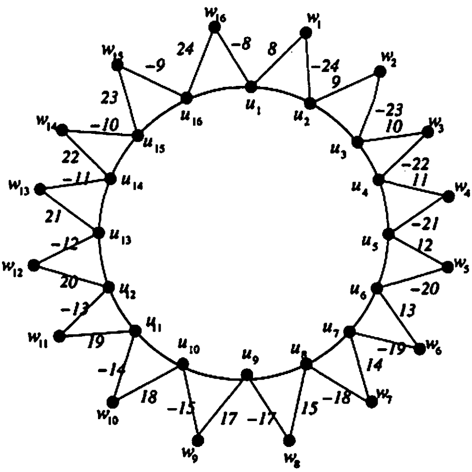

Figure 3: A Partial SEGL of $T(C_{16})$

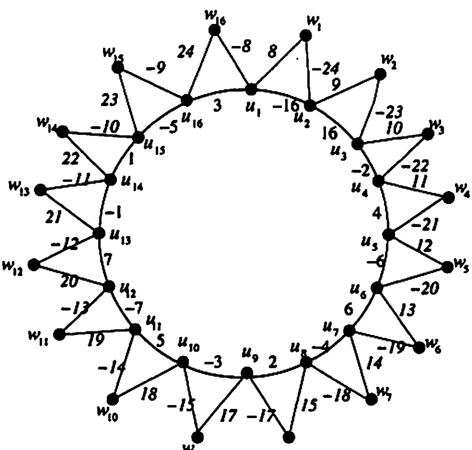

Figure 4: A SEGL of $T(C_{16})$

We write the SEGL shown in Figure 4 as follows:

Inner cycle:	−16	16	−2	4	−6	6	−4	2	−3	5
	−7	7	−1	1	−5	3				
Outer cycle:	8	−24	9	−23	10	−22	11	−21	12	−20
	13	−19	14	−18	15	−17	17	−15	18	−14
	19	−13	20	−12	21	−11	22	−10	23	−9
	24	−8								

A super edge-graceful labeling for $T(C_{16})$

The structure of the edge labeling described above can be generalized for $T(C_n)$ when $n \equiv 0 \pmod 8$. For $n \not\equiv 0 \pmod 8$ we need to use different modifications of this structure.

3.1 Case $n \equiv 0 \pmod 8$

The following labeling is a SEGL for $T(C_8)$:

Inner cycle:	−8	8	2	−2	1	−1	−3	3				
Outer cycle:	4	−12	5	−11	6	−10	7	−9	9	−7		
	10	−6	11	−5	12	−4						

Note that the edge u_1u_2 is labeled −8, the edge u_2u_3 is labeled 8 and so on. Similarly, the edge u_1w_1 is labeled 4, the edge w_1u_2 is labeled −12 and so on.

Now let $n \geq 16$. Define $f : E(T(C_n)) \to \{\pm1, \pm2, \ldots, \pm3n/2\}$ by (Inner cycle)

$$f(u_iu_{i+1}) = \begin{cases} -n & \text{if} \quad i = 1 \\ n & \text{if} \quad i = 2 \\ (-1)^i(2i-4) & \text{if} \quad 3 \leq i \leq n/4+1 \\ (-1)^i(n-2i+2) & \text{if} \quad n/4+2 \leq i \leq n/2 \\ (-1)^i(2i-n+1) & \text{if} \quad n/2+1 \leq i \leq 3n/4-1 \\ (-1)^i(2n-2i-1) & \text{if} \quad 3n/4 \leq i \leq n-4 \\ -1 & \text{if} \quad i = n-3 \\ 1 & \text{if} \quad i = n-2 \\ -5 & \text{if} \quad i = n-1 \\ 3 & \text{if} \quad i = n \end{cases}$$

(Outer cycle)

$$f(u_iw_i) = \begin{cases} n/2+i-1 & \text{if} \quad 1 \leq i \leq n/2 \\ n/2+i & \text{if} \quad n/2+1 \leq i \leq n \end{cases}$$

$$f(w_iu_{i+1}) = \begin{cases} -3n/2+i-1 & \text{if} \quad 1 \leq i \leq n/2 \\ -3n/2+i & \text{if} \quad n/2+1 \leq i \leq n \end{cases}$$

The SEGL for $T(C_{16})$ shown above is obtained from the edge labeling f when $n = 16$. The pattern in the edge labeling f can be observed better when $n = 24$. The function f produces the following SEGL for $T(C_{24})$.

Inner cycle:	−24	24	−2	4	−6	8	−10	10	−8	6
	−4	2	−3	5	−7	9	−11	11	−9	7
	−1	1	−5	3						
Outer cycle:	12	−36	13	−35	14	−34	15	−33	16	−32
	17	−31	18	−30	19	−29	20	−28	21	−27
	22	−26	23	−25	25	−23	26	−22	27	−21
	28	−20	29	−19	30	−18	31	−17	32	−16
	33	−15	34	−14	35	−13	36	−12		

In order to prove that f defines a SEGL for $T(C_n)$ we first observe that every edge label appears at some edge. Now consider the edge labels of the outer cycle. Let $\ell'(x)$ denote the label of vertex x induced only by the edge labels of the outer cycle. We have $\ell'(w_i) = -n + 2i - 2$ for $1 \le i \le n/2$ and $\ell'(w_i) = -\ell'(w_{n-i+1})$ for $n/2 + 1 \le i \le n$. Hence, all even vertex labels appear at the vertices $\{w_1, w_2, \ldots, w_n\}$. In addition, as seen for case $n = 16$ above, $\ell'(u_1) = \ell'(u_{n/2+1}) = 0$, $\ell'(u_i) = -n + 2i - 3$ for $2 \le i \le n/2$ and $\ell'(u_i) = -\ell'(u_{n-i+2})$ for $n/2 + 2 \le i \le n$. Hence, every odd vertex label appears at some vertex of the inner cycle except ± 1. Now if we also consider the edge labels for the inner cycle, then $\ell(u_1) = -(n-3)$, $\ell(u_2) = -(n-1)$, $\ell(u_3) = 1$, $\ell(u_{n/4+2}) = -(n/2-1)$, $\ell(u_{n/2+1}) = -1$, $\ell(u_{3n/4}) = n/2 - 1$, $\ell(u_{n-3}) = n - 1$, $\ell(u_{n-2}) = n - 5$, $\ell(u_{n-1}) = n - 7$, and $\ell(u_n) = n - 3$. Recall that $\ell(x)$ denote the label of vertex x induced by the edge labels of the outer cycle and the inner cycle. Let

$$A = \{1, 2, 3, n/4 + 2, n/2 + 1, 3n/4, n - 3, n - 2, n - 1, n\}.$$

Then $\{\ell'(u_i) \mid i \in A \setminus \{1, n/2 + 1\}\} = \{\ell(u_i) \mid i \in A \setminus \{3, n/2 + 1\}\}$. Now partition the vertices of the inner cycle which are not in A into subsets of the form $\{u_i, u_{i+1}\}$ for some i. It is easy to see that if $\ell'(u_i) = r$ and $\ell'(u_{i+1}) = s$, then $\ell(u_i) = s$ and $\ell(u_{i+1}) = r$. Hence, f is a SEGL of $T(C_n)$.

3.2 Case $n \equiv 1 \pmod 8$

For $n = 9$ and 17 see the following SEGLs.

A super edge-graceful labeling for $T(C_9)$:

Inner cycle:	0	4	−4	−2	9	−3	1	−1	3	
Outer cycle:	5	−13	6	−12	7	−11	8	−10	2	−9
	10	−8	11	−7	12	−6	13	−5		

A super edge-graceful labeling for $T(C_{17})$:

Inner cycle:	−17	17	−5	−1	7	−7	1	−3	5	−8
	6	−4	2	−2	4	−6	8			
Outer cycle:	9	−25	10	−24	11	−23	12	−22	13	−21
	14	−20	15	−19	16	−18	3	0	18	−16
	19	−15	20	−14	21	−13	22	−12	23	−11
	24	−10	25	−9						

For $n \geq 25$ define $f : E(T(C_n)) \to \{0, \pm1, \pm2, \ldots, \pm(3n-1)/2\}$ by (Inner cycle)

$$f(u_iu_{i+1}) = \begin{cases} -n & \text{if } i = 1 \\ n & \text{if } i = 2 \\ -5 & \text{if } i = 3 \\ (-1)^i(n+5)/2 - 2i & \text{if } 4 \leq i \leq (n-9)/4 \\ 1 & \text{if } i = (n-5)/4 \\ -1 & \text{if } i = (n-1)/4 \\ -(n-3)/2 & \text{if } i = (n+3)/4 \\ (n-3)/2 & \text{if } i = (n+7)/4 \\ -(n-7)/2 & \text{if } i = (n+11)/4 \\ (-1)^{i+1}(2i-(n+9)/2) & \text{if } (n+15)/4 \leq i \leq (n+1)/2 \\ (-1)^{i+1}((3n+5)/2 - 2i) & \text{if } (n+3)/2 \leq i \leq (3n+1)/4 \\ (-1)^i((3n+1)/2 - 2i) & \text{if } (3n+5)/4 \leq i \leq n \end{cases}$$

(Outer cycle)

$$f(u_iw_i) = \begin{cases} (n+1)/2 + i - 1 & \text{if } 1 \leq i \leq n,\ i \neq (n+1)/2 \\ 3 & \text{if } i = (n+1)/2 \end{cases}$$

$$f(w_iu_{i+1}) = \begin{cases} (-3n+1)/2 + i - 1 & \text{if } 1 \leq i \leq n,\ i \neq (n+1)/2 \\ 0 & \text{if } i = (n+1)/2 \end{cases}$$

The edge labeling f produces the following SEGL for $T(C_{25})$.

Inner cycle:	−25	25	−5	7	1	−1	−11	11	−9	−3
	5	−7	9	−12	10	−8	6	−4	2	−2
	4	−6	8	−10	12					
Outer cycle:	13	−37	14	−36	15	−35	16	−34	17	−33
	18	−32	19	−31	20	−30	21	−29	22	−28
	23	−27	24	−26	3	0	26	−24	27	−23
	28	−22	29	−21	30	−20	31	−19	32	−18
	33	−17	34	−16	35	−15	36	−14	37	−13

It is straightforward to check that f is a SEGL for $T(C_n)$ for $n \equiv 1 \pmod 8$, $n \geq 25$.

3.3 Case $n \equiv 2 \pmod 8$

Define the edge labeling $f : E(T(C_n)) \to \{\pm1, \pm2, \ldots, \pm 3n/2\}$ by

(Inner cycle)

$$f(u_iu_{i+1}) = \begin{cases} -(n+1) & \text{if } i=1 \\ n+1 & \text{if } i=2 \\ (-1)^{i+1}(n/2+2-2i) & \text{if } 3 \le i \le (n+2)/4 \\ (-1)^i(n-2i) & \text{if } (n+6)/4 \le i \le n/2-1 \\ -2 & \text{if } i=n/2 \\ -n/2+2 & \text{if } i=n/2+1 \\ (-1)^{i+1}((3n+4)/2-2i) & \text{if } n/2+2 \le i \le (3n+2)/4 \\ -n/2+1 & \text{if } i=(3n+6)/4 \\ (-1)^i((2n+4)-2i) & \text{if } (3n+10)/4 \le i \le n \end{cases}$$

(Outer cycle)

$$f(u_iw_i) = \begin{cases} n/2+i-1 & \text{if } 1 \le i \le n/2+1 \\ n/2+i & \text{if } n/2+2 \le i \le n \end{cases}$$

$$f(w_iu_{i+1}) = \begin{cases} -3n/2+i-1 & \text{if } 1 \le i \le n/2-1 \\ -3n/2+i & \text{if } n/2 \le i \le n \end{cases}$$

The edge labeling f produces the following SEGL for $T(C_{26})$.

Inner cycle:	−27	27	9	−7	5	−3	1	10	−8	6
	−4	2	−2	−11	11	−9	7	−5	3	−1
	−12	12	−10	8	−6	4				
Outer cycle:	13	−39	14	−38	15	−37	16	−36	17	−35
	18	−34	19	−33	20	−32	21	−31	22	−30
	23	−29	24	−28	25	−26	26	−25	28	−24
	29	−23	30	−22	31	−21	32	−20	33	−19
	34	−18	35	−17	36	−16	37	−15	38	−14
	39	−13								

It is straightforward to check that f is a SEGL for $T(C_n)$ for $n \equiv 2 \pmod 8$, $n \ge 10$.

3.4 Case $n \equiv 3 \pmod 8$

A super edge-graceful labeling for $T(C_3)$ is displayed below.

Inner cycle:	0	3	−4			
	4	−3	2	1	−1	−2

For $n \ge 11$ define $f : E(T(C_n)) \to \{0, \pm 1, \pm 2, \ldots, \pm(3n-1)/2\}$ by

(Inner cycle)

$$f(u_iu_{i+1}) = \begin{cases} 0 & \text{if} \quad i=1 \\ n & \text{if} \quad i=2 \\ (-1)^{i+1}((n+3)/2-2i) & \text{if} \quad 3 \le i \le (n+1)/4 \\ -1 & \text{if} \quad i=(n+5)/4 \\ -(n-5)/2 & \text{if} \quad i=(n+9)/4 \\ (-1)^i((n+4)-2i) & \text{if} \quad (n+13)/4 \le i \le (n+1)/2 \\ (3n+3)/2-2i & \text{if} \quad (n+3)/2 \le i \le (3n-1)/4 \\ -(2i-(3n-1)/2) & \text{if} \quad (3n+3)/4 \le i \le n-1 \\ -(n-1) & \text{if} \quad i=n \end{cases}$$

(Outer cycle)

$$f(u_iw_i) = \begin{cases} (n-1)/2+i-1 & \text{if} \quad 1 \le i \le (n+1)/2 \\ (n-1)/2+i & \text{if} \quad (n+3)/2 \le i \le n \end{cases}$$

$$f(w_iu_{i+1}) = \begin{cases} (-3n+1)/2+i-1 & \text{if} \quad 1 \le i \le (n+1)/2 \\ (-3n+1)/2+i & \text{if} \quad (n+3)/2 \le i \le n \end{cases}$$

The edge labeling f produces the following SEGL for $T(C_{27})$.

Inner cycle:	0	27	9	−7	5	−3	1	−1	−11	11
	−9	7	−5	3	12	10	8	6	4	2
	−2	−4	−6	−8	−10	−12	−26			
Outer cycle:	13	−40	14	−39	15	−38	16	−37	17	−36
	18	−35	19	−34	20	−33	21	−32	22	−31
	23	−30	24	−29	25	−28	26	−27	28	−25
	29	−24	30	−23	31	−22	32	−21	33	−20
	34	−19	35	−18	36	−17	37	−16	38	−15
	39	−14	40	−13						

It is straightforward to check that f is a SEGL for $T(C_n)$ for $n \equiv 3 \pmod 8$, $n \ge 11$.

3.5 Case $n \equiv 4 \pmod 8$

A SEGL for $T(C_4)$ is displayed below.

Inner cycle:	−6	6	−5	5				
Outer cycle:	4	−2	−1	−3	3	1	2	−4

For $n \ge 12$ define $f : E(T(C_n)) \to \{\pm 1, \pm 2, \ldots, \pm 3n/2\}$ by

(Inner cycle)

$$f(u_iu_{i+1}) = \begin{cases} -n & \text{if} \quad i=1 \\ n & \text{if} \quad i=2 \\ (-1)^{i+1}(2i-4) & \text{if} \quad 3 \le i \le (n/4)+1 \\ (-1)^{i+1}(n-2i+2) & \text{if} \quad n/4+2 \le i \le n/2 \\ -1 & \text{if} \quad i = n/2+1 \\ 1 & \text{if} \quad i = n/2+2 \\ 3 & \text{if} \quad i = n/2+3 \\ -3 & \text{if} \quad i = n/2+4 \\ (-1)^i(2i-n-5) & \text{if} \quad n/2+5 \le i \le 3n/4+2 \\ (-1)^i(2n-2i+5) & \text{if} \quad 3n/4+3 \le i \le n \end{cases}$$

(Outer cycle)

$$f(u_iw_i) = \begin{cases} n/2+i-1 & \text{if} \quad 1 \le i \le n/2 \\ n/2+i & \text{if} \quad n/2+1 \le i \le n \end{cases}$$

$$f(w_iu_{i+1}) = \begin{cases} -3n/2+i-1 & \text{if} \quad 1 \le i \le n/2 \\ -3n/2+i & \text{if} \quad n/2+1 \le i \le n \end{cases}$$

The edge labeling f produces the following SEGL for $T(C_{20})$.

Inner cycle:	−20	20	2	−4	6	−8	8	−6	4	−2
	−1	1	3	−3	−5	7	−9	9	−7	5
Outer cycle:	10	−30	11	−29	12	−28	13	−27	14	−26
	15	−25	16	−24	17	−23	18	−22	19	−21
	21	−19	22	−18	23	−17	24	−16	25	−15
	26	−14	27	−13	28	−12	29	−11	30	−10

It is straightforward to check that f is a SEGL for $T(C_n)$ for $n \equiv 4 \pmod 8$, $n \geq 12$.

3.6 Case $n \equiv 5 \pmod 8$

For $n = 5$ and 13 see the following SEGLs.

A super edge-graceful labeling for $T(C_5)$:

Inner cycle:	−5	5	−1	−4	4					
Outer cycle:	2	−7	3	−6	0	1	6	−3	7	−2

A super edge-graceful labeling for $T(C_{13})$:

Inner cycle:	−13	13	−5	−1	−3	3	1	−6	4	2
	−2	−4	6							
Outer cycle:	7	−19	8	−18	9	−17	10	−16	11	−15
	12	−14	5	0	14	−12	15	−11	16	−10
	17	−9	18	−8	19	−7				

For $n \geq 21$ define $f : E(T(C_n)) \to \{0, \pm1, \pm2, \ldots, \pm(3n-1)/2\}$ by (Inner cycle)

$$f(u_iu_{i+1}) = \begin{cases} -n & \text{if} \quad i = 1 \\ n & \text{if} \quad i = 2 \\ 3 & \text{if} \quad i = 3 \\ (n-3)/2 & \text{if} \quad i = 4 \\ (-1)^{i+1}((n+9)/2 - 2i) & \text{if} \quad 5 \leq i \leq (n-1)/4 \\ -(n-7)/2 & \text{if} \quad i = (n+3)/4 \\ 1 & \text{if} \quad i = (n+7)/4 \\ -3 & \text{if} \quad i = (n+11)/4 \\ (-1)^i((n+6) - 2i) & \text{if} \quad (n+15)/4 \leq i \leq (n+1)/2 \\ 2 & \text{if} \quad i = (n+3)/2 \\ -2 & \text{if} \quad i = (n+5)/2 \\ (-1)^{i+1}((3n+13)/2 - 2i) & \text{if} \quad (n+7)/2 \leq i \leq (3n+5)/4 \\ (-1)^{i+1}(2i - (3n+1)/2) & \text{if} \quad (3n+9)/4 \leq i \leq n \end{cases}$$

(Outer cycle)

$$f(u_iw_i) = \begin{cases} (n+1)/2 + i - 1 & \text{if} \quad 1 \leq i \leq (n-1)/2 \\ -1 & \text{if} \quad i = (n+1)/2 \\ (n-1)/2 + i & \text{if} \quad (n+3)/2 \leq i \leq n \end{cases}$$

$$f(w_iu_{i+1}) = \begin{cases} (-3n+1)/2 + i - 1 & \text{if} \quad 1 \leq i \leq (n-1)/2 \\ 0 & \text{if} \quad i = (n+1)/2 \\ (-3n-1)/2 + i & \text{if} \quad (n+3)/2 \leq i \leq n \end{cases}$$

The edge labeling f produces the following SEGL for $T(C_{29})$.

Inner cycle:	−29	29	3	13	9	−7	5	−11	1	−3
	−13	11	−9	7	−5	2	−2	−14	12	−10
	8	−6	4	−4	6	−8	10	−12	14	
Outer cycle:	15	−43	16	−42	17	−41	18	−40	19	−39
	20	−38	21	−37	22	−36	23	−35	24	−34
	25	−33	26	−32	27	−31	28	−30	−1	0
	30	−28	31	−27	32	−26	33	−25	34	−24
	35	−23	36	−22	37	−21	38	−20	39	−19
	40	−18	41	−17	42	−16	43	−15		

It is straightforward to check that f is a SEGL for $T(C_n)$ for $n \equiv 5 \pmod 8$, $n \geq 21$.

3.7 Case $n \equiv 6 \pmod 8$

A super edge-graceful labeling for $T(C_6)$:

Inner cycle: 7 −8 9 −9 8 −7

Outer cycle: 1 −3 5 −2 6 −4 −1 −5 2 4 −6 3

A super edge-graceful labeling for $T(C_{14})$:

Inner cycle:	−15	15	1	−3	3	5	−4	−1	−5	2
	−2	4	−6	6						
Outer cycle:	7	−21	8	−20	9	−19	10	−18	11	−17
	12	−16	13	−14	14	−13	16	−12	17	−11
	18	−10	19	−9	20	−8	21	−7		

Now let $n \geq 22$. Define $f : E(T(C_n)) \to \{\pm 1, \pm 2, \ldots, \pm 3n/2\}$ by (Inner cycle)

$$f(u_i u_{i+1}) = \begin{cases} -(n+1) & \text{if} \quad i=1 \\ n+1 & \text{if} \quad i=2 \\ (-1)^{i+1}(2i-5) & \text{if} \quad 3 \leq i \leq (n-6)/4 \\ n/2-8 & \text{if} \quad i=(n-2)/4 \\ (-1)^{i+1}(2i-n/2) & \text{if} \quad (n+2)/4 \leq i \leq n/2-5 \\ n/2-2 & \text{if} \quad i=n/2-4 \\ -(n/2-4) & \text{if} \quad i=n/2-3 \\ (n/2-6) & \text{if} \quad i=n/2-2 \\ -(n/2-6) & \text{if} \quad i=n/2-1 \\ -(n/2-2) & \text{if} \quad i=n/2 \\ (n/2-4) & \text{if} \quad i=n/2+1 \\ (-1)^i((3n/2+1-2i) & \text{if} \quad n/2+2 \leq i \leq (3n-2)/4 \\ (-1)^i(2i-(3n-2)/2) & \text{if} \quad (3n+2)/4 \leq i \leq n-1 \\ (n-2)/2 & \text{if} \quad i=n \end{cases}$$

(Outer cycle)

$$f(u_i w_i) = \begin{cases} n/2+i-1 & \text{if} \quad 1 \leq i \leq n/2+1 \\ n/2+i & \text{if} \quad n/2+2 \leq i \leq n \end{cases}$$

$$f(w_i u_{i+1}) = \begin{cases} -3n/2+i-1 & \text{if} \quad 1 \leq i \leq n/2-1 \\ -3n/2+i & \text{if} \quad n/2 \leq i \leq n \end{cases}$$

The edge labeling f produces the following SEGL for $T(C_{30})$.

Inner cycle:	−31	31	1	−3	5	−7	7	−1	3	−5
	13	−11	9	−9	−13	11	−12	10	−8	6
	−4	2	−2	4	−6	8	−10	12	−14	14
Outer cycle:	15	−45	16	−44	17	−43	18	−42	19	−41
	20	−40	21	−39	22	−38	23	−37	24	−36
	25	−35	26	−34	27	−33	28	−32	29	−30
	30	−29	32	−28	33	−27	34	−26	35	−25
	36	−24	37	−23	38	−22	39	−21	40	−20
	41	−19	42	−18	43	−17	44	−16	45	−15

It is straightforward to check that f is a SEGL for $T(C_n)$ for $n \equiv 6 \pmod 8$, $n \geq 22$.

3.8 Case $n \equiv 7 \pmod 8$

A super edge-graceful labeling for $T(C_7)$:

Inner cycle:	0	7	1	−1	−6	−2	2						
Outer cycle:	3	−10	4	−9	5	−8	6	−7	8	−5	9	−4	
	10	−3											

A super edge-graceful labeling for $T(C_{15})$:

Inner cycle:	0	15	−5	3	−3	−1	1	5	−4	6
	2	−2	4	−6	−14					
Outer cycle:	7	−22	8	−21	9	−20	10	−19	11	−18
	12	−17	13	−16	14	−15	16	−13	17	−12
	18	−11	19	−10	20	−9	21	−8	22	−7

Let $n \geq 23$. Define $f : E(T(C_n)) \to \{0, \pm 1, \pm 2, \ldots, \pm(3n-1)/2\}$ by (Inner cycle)

$$f(u_iu_{i+1}) = \begin{cases} 0 & \text{if } i = 1 \\ n & \text{if } i = 2 \\ (-1)^i((n+7)/2 - 2i) & \text{if } 3 \leq i \leq (n+1)/4 \\ 1 & \text{if } i = (n+5)/4 \\ -3 & \text{if } i = (n+9)/4 \\ -(n-9)/2 & \text{if } i = (n+13)/4 \\ (n-5)/2 & \text{if } i = (n+17)/4 \\ -1 & \text{if } i = (n+21)/4 \\ (-1)^i(n+6-2i) & \text{if } (n+25)/4 \leq i \leq (n+1)/2 \\ -(n-7)/2 & \text{if } i = (n+3)/2 \\ (n-3)/2 & \text{if } i = (n+5)/2 \\ (-1)^{i+1}((3n+3)/2 - 2i) & \text{if } (n+7)/2 \leq i \leq (3n-1)/4 \\ (-1)^{i+1}(2i - (3n-1)/2) & \text{if } (3n+3)/4 \leq i \leq n-1 \\ -(n-1) & \text{if } i = n \end{cases}$$

(Outer cycle)

$$f(u_iw_i) = \begin{cases} (n-1)/2 + i - 1 & \text{if } 1 \leq i \leq (n+1)/2 \\ (n-1)/2 + i & \text{if } (n+3)/2 \leq i \leq n \end{cases}$$

$$f(w_iu_{i+1}) = \begin{cases} (-3n+1)/2 + i - 1 & \text{if } 1 \leq i \leq (n+1)/2 \\ (-3n+1)/2 + i & \text{if } (n+3)/2 \leq i \leq n \end{cases}$$

The edge labeling f produces the following SEGL for $T(C_{31})$.

Inner cycle:	0	31	−13	11	−9	7	−5	3	1	−3
	−11	13	−1	9	−7	5	−12	14	10	−8
	6	−4	2	−2	4	−6	8	−10	12	−14
	−30									
Outercycle:	15	−46	16	−45	17	−44	18	−43	19	−42
	20	−41	21	−40	22	−39	23	−38	24	−37
	25	−36	26	−35	27	−34	28	−33	29	−32
	30	−31	32	−29	33	−28	34	−27	35	−26
	36	−25	37	−24	38	−23	39	−22	40	−21
	41	−20	42	−19	43	−18	44	−17	45	−16
	46	−15								

It is straightforward to check that f is a SEGL for $T(C_n)$ for $n \equiv 3 \pmod 8$, $n \geq 23$.

Now we are ready to state the main result of this section.

Theorem 4. *The total cycle $T(C_n)$ is super edge-graceful for $n \geq 3$.*

References

[1] S. Cichacz, D. Froncek, A. Khodkar and W. Xu, *Super edge-graceful paths and cycles*, Bulletin of the Institute of Combinatorics and its Applications **57** (2009), 79–90.

[2] S.P. Lo, *On edge-graceful labelings of graphs*, Congressus Numerantium **50** (1985), 231–241.

[3] A. Khodkar, R. Rasi and S.M. Sheikholeslami, *Super edge-gracefulness of complete graphs*, Utilitas Mathematica (to appear).

[4] A. Khodkar, S. Nolen and J.T. Perconti, *Super Edge-Graceful Labelings of Complete Bipartite Graphs*, Australasian Journal of Combinatorics (to appear).

[5] S.-M. Lee and Y.-S. Ho, *All trees of odd order with three even vertices are super edge-graceful*, JCMCC **62** (2007), 53–64.

[6] S.-M. Lee, L. Wang and E.R. Yera, *On super edge-graceful Eulerian graphs*, Congressus Numerantium **174** (2005), 83–96.

[7] J. Mitchem and A. Simoson, *On edge-graceful and super-edge-graceful graphs*, Ars Combin. **37** (1994), 97–111.