Fall coloring of graphs-II

R. Balakrishnan[1] and T. Kavaskar[2]

Srinivasa Ramanujan Centre, SASTRA University
Kumbakonam–612 001, India.
[1]: mathbala@satyam.net.in
[2]: t_kavaskar@yahoo.com

Abstract

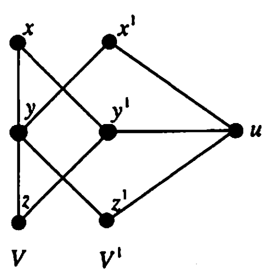
A fall coloring of a graph G is a color partition of the vertex set of G in such a way that every vertex of G is a colorful vertex in G (that is, it has at least one neighbor in each of the other color classes). The fall coloring number $\chi_f(G)$ of G is the minimum size of a fall color partition of G (when it exists). In this paper, we show that the Mycielskian $\mu(G)$ of any graph G does not have a fall coloring and that the generalized Mycielskian $\mu_m(G)$ of a graph G may or may not have a fall coloring. More specifically, we show that if G has a fall coloring, then $\mu_{3m}(G)$ has also a fall coloring for $m \geq 1$, and that $\chi_f(\mu_{3m}(G)) \leq \chi_f(G) + 1$.

Key Words: Fall coloring of graphs, Mycielskian of a graph.
2000 AMS Subject Classification: 05C15

1 Introduction

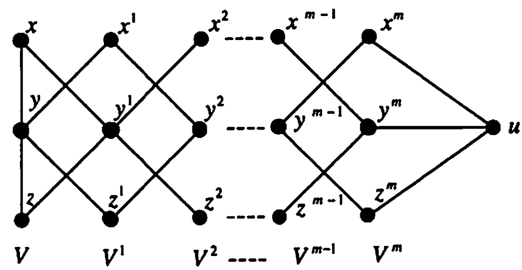
Let $G = (V, E)$ be a non-trivial simple graph which is undirected and connected. A proper coloring of a graph G is a partition $\Pi = \{V_1, V_2, \ldots, V_k\}$ of the vertex set V of G into independent subsets of V. Each V_i is called a color class of Π. A vertex $v \in V_i$ is a colorful vertex with respect to Π if it is adjacent to at least one vertex in each color class $V_j, j \neq i$. A k-coloring $\Pi = \{V_1, V_2, \ldots, V_k\}$ of G is a fall coloring of G [2] if each vertex of G is a colorful vertex with respect to Π. In this case, Π is called a k-fall coloring of G. The least positive integer k for which G has a k-fall coloring is the fall chromatic number of G and denoted by $\chi_f(G)$. A graph G may or may not have a fall coloring. For example, the cycle C_n has a fall coloring iff n is a multiple of 3 or n is even [2]. All complete graphs have a fall coloring; in fact, $\chi_f(K_n) = n$. It is clear that if G has a k-fall coloring, then $\delta(G) \geq k-1$ (where $\delta(G)$ denotes the minimum degree of G) and therefore $\delta(G)+1 \geq \chi_f(G) \geq \chi(G)$.

In the mid 20^{th} century there was a question regarding the construction of triangle-free k-chromatic graphs, where $k \geq 3$. In this search, Mycielski [4] developed an interesting graph transformation known as the *Mycielskian* as follows. For a graph $G = (V, E)$, the Mycielskain of G is the graph $\mu(G)$ with vertex set consisting of the disjoint union $V \cup V^1 \cup \{u\}$, where $V^1 = \{x^1 : x \in V\}$ and edge set $E \cup \{x^1y : xy \in E\} \cup \{x^1u : x^1 \in V^1\}$. Figure 1.1 gives the Mycielskian $\mu(P_3)$ of P_3, the path on three vertices. For $l \geq 2$, $\mu^l(G)$ is defined iteratively by setting $\mu^l(G) = \mu(\mu^{l-1}(G))$. The generalized Mycielskian [3] $\mu_m(G)$ of G is the graph whose vertex set is the disjoint union $V \cup (\bigcup_{i=1}^{m} V^i) \cup \{u\}$, where $V^i = \{x^i : x \in V\}$ is an independent set, $1 \leq i \leq m$, and edge set $E(\mu_m(G)) = E \cup (\bigcup_{i=1}^{m} \{y^{i-1}x^i : xy \in E\}) \cup \{x^m u : x^m \in V^m\}$, where $x^0 = x$ and $y^0 = y$. Figure 1.2 describes the construction of the generalized Mycielskian $\mu_m(P_3)$ of P_3. For $x \in V$ and for i, $1 \leq i \leq m$, x^i is the vertex of V^i that corresponds to the vertex x of V. For a subset S of V and for $i, 1 \leq i \leq m$, let S^i be the subset of V^i that corresponds to S. A graph G is chordal if every cycle C of length at least 4 in G has a chord, that is, an edge joining two nonconsecutive vertices of C. A graph G is split if its vertex set can be partitioned into a clique (a complete subgraph) and an independent set.



$\mu(P_3)$

Figure 1.1



$\mu_m(P_3)$

Figure 1.2

In an earlier paper [1], the authors have settled two problems on fall coloring of graphs proposed by Dunbar et al. [2]: (i) Determination of smallest non-fall colorable graphs with prescribed minimum degree δ, and (ii) Existence of graphs G with $\chi_f(G) - \chi(G)$ equal to any preassigned number. The present paper deals exclusively with the fall coloring of Mycielskians of graphs and is independent of [1].

2 Fall coloring of Mycielskians

While considering the generalized Mycielskian, it is easy to observe that $\chi(G) \leq \chi(\mu_m(G)) \leq \chi(G) + 1$ and that $\delta(\mu_m(G)) = \delta(G) + 1$ for any $m \geq 1$. In this section, we show that $\mu(G)$ has no fall coloring for any graph G.

Theorem 2.1:
For any graph G, $\mu(G)$ has no k-fall coloring for any $k(\geq 2)$.

Proof. Suppose $\mu(G)$ has a k-fall coloring for some graph G and some integer $k(\geq 2)$, say, $V(\mu(G)) = \bigcup_{i=1}^{k} V_i$. Without loss of generality, let $u \in V_1$. Since u is a colorful vertex of $\mu(G)$, $V_i \cap V^1 \neq \emptyset$ for each i, $2 \leq i \leq k$ and $V_1 \cap V^1 = \emptyset$. Hence no vertex of V^1 receives color 1. Let x^1 be any vertex of V^1. Then $x^1 \in V_i$ for some i in $2 \leq i \leq k$.

Consider $x^1 \in V_i$, for some i where $2 \leq i \leq k$, then $x \in V_1 \cup V_i$ (because if $x \in V_j$, $j \neq i, 1$, then (as x^1 is a colorful vertex), x^1 must have a neighbor in V_j. But then x and x^1 have the same neighbors in V. This contradicts the fact that V_j is an independent set in $\mu(G)$).

Suppose $x \in V_1$. Since x is a colorful vertex in $\mu(G)$, x must be adjacent to a vertex in V_i. Clearly, this vertex can not belong to V, because x and x^1 have the same neighbors in V. Therefore this vertex must belong to V^1; call it z^1. Then (as seen in the case of x) $z \in V_1 \cup V_i$. If $z \in V_1$, then both x and z are in V_1. If $z \in V_i$, then both x^1 and z are in V_i. These are impossible since xz and x^1z are edges of $\mu(G)$. Thus x must be in V_i. As this is true for any x^1 in V^1, $V_1 \cap V = \emptyset$, and hence no vertex of V is adjacent to a vertex with color 1. Therefore no vertex of V is a colorful vertex. This contradicts the assumption that $\mu(G)$ has a fall coloring. ■

Corollary 2.2:
For any graph G, the iterated Mycielskian $\mu^l(G)$ of G has no fall coloring for any $l \geq 1$.

Suppose a and b are two positive integers with $3 \leq a < b$. Consider now a graph H with $\chi(H) = a - 1$ and $\delta(H) = b - 1$. Then if $G = \mu(H)$, G has no fall coloring with $\chi(G) = a < \delta(G) = b$. One way of constructing the graph H is to take $H = \underbrace{K_2 \square K_2 \square \ldots \square K_2}_{(b-a+1)-times} \square K_{a-1}$, where $\square$ is the Cartesian product. Then $\chi(H) = a - 1$ and $\delta(H) = (b - a + 1) + (a - 2) = b - 1$. We state it as a corollary.

Corollary 2.3:
For any positive integers a, b with $a < b$, there exists a graph G such that G has no fall coloring and $\chi(G) = a$ and $\delta(G) = b$.

Next we prove that for any $k \geq 2$, $\mu_2(G)$ has no k-fall coloring for some families of graphs. But before establishing this result, we show that $\mu_2(G)$ has no 3-fall coloring for any graph G.

Theorem 2.4:
For any graph G, $\mu_2(G)$ has no 3-fall coloring.

Proof. Suppose $\chi(\mu_2(G)) \geq 4$, then $\mu_2(G)$ has no 3-fall coloring (because $\chi_f \geq \chi$). So we assume that $\chi(\mu_2(G)) \leq 3$. Since $\mu_2(G)$ is not bipartite, $\chi(\mu_2(G)) = 3$. Suppose $\mu_2(G)$ has a 3-fall coloring, say, $V(\mu_2(G)) = V_1 \cup V_2 \cup V_3$. Without loss of generality, let $u \in V_1$. Then no vertex of V^2 can receive color 1. Moreover, as u is a colorful vertex in $\mu_2(G)$, both the colors 2 and 3 must be present in V^2.

Let $x \in V$ and x^1, x^2 be the corresponding vertices of x in V^1 and V^2, respectively. Without loss of generality, let $x^2 \in V_2$. Since x and x^2 have the same neighbors in V^1, $x \in V_1 \cup V_2$. We now show that x^1 also belongs to $V_1 \cup V_2$. Suppose $x^1 \in V_3$; as x^2 is a colorful vertex, it is adjacent to a vertex in V_3, say, y^1. Then, as per the definition of $\mu_2(G)$, x^1 is adjacent to y^2 in $\mu_2(G)$. Hence y^2 cannot receive color 3 (as x^1 is also colored by 3). Therefore $y^2 \in V_2$, which implies that $x, y \in V_1 \cup V_2$ and $xy \in E(\mu_2(G))$. Without loss of generality, let $x \in V_1$ and $y \in V_2$. Since x^1 is a colorful vertex, x^1 should have a neighbor in V_1. Since $V_1 \cap V^2 = \emptyset$, this neighbor must be in V, a contradiction, because x and x^1 have same neighbors in V. Thus $x^1 \in V_1 \cup V_2$ (Already $x \in V_1 \cup V_2$).

Claim 1 $x \in V_2$.

If $x \in V_1$, x must be adjacent to a vertex w in V_2 and this vertex must belong either to V or V^1. If $w \in V$, then $w, w^1, w^2 \in V_2$, and $x^2 w^1 \in E(\mu_2(G))$, a contradiction, because $x^2, w^1 \in V_2$. Similarly, we get a contradiction when $w \in V^1$. Therefore $x \in V_2$.

Claim 2 $x^1 \in V_2$.

If $x^1 \in V_1$, x^1 must be adjacent to a vertex, say y, in V_2 and y can belong either to V or to V^2. If $y \in V$, then $xy \in E(\mu_2(G))$ and both x and y receive color 2, a contradiction. If $y = z^2 \in V^2$, then $xz \in E(\mu_2(G))$ and both x and z receive color 2 (by the above argument), again a contradiction. Thus $x^1 \notin V_1$ and so $x_1 \in V_2$.

A similar argument can be given when $x^2 \in V_3$. Since x is an arbitrary vertex in V, $V \cap V_1 = V^1 \cap V_1 = \emptyset$. Therefore no vertex of V is a colorful vertex, a contradiction. Therefore $\mu_2(G)$ has no 3-fall coloring. ∎

In 1985, Tuza and Rödl [6] observed that the graph $\mu_m(K_k)$ is $(k+1)$-critical for all $m \geq 1$. Thus, we have the lemma.

Lemma 2.5:
If G is a graph with $\chi(G) = \omega(G)$, then $\chi(\mu_m(G)) = \chi(G) + 1$ for any $m \geq 1$.

Proof. By hypothesis, $G \supseteq K_{\chi(G)}$, and hence $\mu_m(G) \supseteq \mu_m(K_{\chi(G)})$. Therefore $\chi(\mu_m(G)) \geq \chi(\mu_m(K_{\chi(G)})) = \chi(G)+1$, by the result of Tuza and Rodl [6]. Since $\chi(\mu_m(G)) \leq \chi(G)+1$ always, we have $\chi(\mu_m(G)) = \chi(G)+1$. ∎

Corollary 2.6:
If G is a graph with $\chi(G) = \omega(G)$, then $\mu_m(G)$ has no $\chi(G)$-fall coloring for any $m \geq 1$.

We next show that for graph G having a simplicial vertex, generalized Mycielskians $\mu_2(G)$ do not have a fall coloring. For this we need a lemma.

Lemma 2.7:
Let G be a graph with minimum degree $\delta(G)$. Then $\chi_f(\mu_2(G)) \leq \delta(G) + 1$

Proof. Suppose for contradiction that this is not the case, and there exists a k-fall coloring (where $k = \delta(G) + 2$), $V(\mu_2(G)) = \bigcup_{i=1}^{k} V_i$. Without loss of generality, let $u \in V_1$, and let v_1 be a vertex of minimum degree in G.

We first note two facts about each vertex in G. Suppose that $V_i^2 \in V_j$.
(1) The vertex $v_i \in V_1 \cup V_j$, so that if $v_i \notin V_1$, v_i and v_i^2 are in the same color class.

(2) If $v_i \in V_1$, then $v_i^1 \in V_1$, because no vertex in $N(v_i^1) \cap V^2$ is contained in V_1 and no vertex in $N(v_i^1) \cap V$ is contained in V_1.

If v_1 and v_1^2 are in the same color class, then v_1 must be adjacent to some vertex in $V \cup V^1$ which is contained in V_1. The second fact implies that at least one neighbor of v_1 in V^1 is in V_1. In this case v_1^2 is adjacent to two vertices in V_1, and can no longer be a colorful vertex.

If v_1 and v_1^2 are not in the same color class, then both v_1 and v_1^1 are in class V_1. This implies that every vertex $v_j \in N(v_1) \cap V$ is not in V_1, and so v_j and v_j^2 are in the same color class. In this case, v_1^1 is adjacent to at most $\delta(G)$ color classes and is therefore not a colorful vertex.

Therefore, there can be no $(\delta(G)+2)$-fall coloring of $\mu_2(G)$. ∎

Theorem 2.8:
If G is a graph having a simplicial vertex, then $\mu_2(G)$ has no k-fall coloring for any $k \geq 2$.

Proof. We prove by contradiction. Suppose $\mu_2(G)$ has a k-fall coloring. Let x_1 be a simplicial vertex of G and let $N[x_1] = \{x_1, x_2, \ldots, x_p\}$. Then $N[x_1]$ is a clique in G. Recall that $\chi(\mu_2(G)) = \chi(G)$ or $\chi(G)+1$.

Now consider the case when $\chi(\mu_2(G)) = \chi(G)+1$. Here $\delta(G)+2 \leq p+1 \leq \omega(G)+1 \leq \chi(G)+1 = \chi(\mu_2(G)) \leq \chi_f(\mu_2(G)) \leq k \leq \delta(\mu_2(G))+1 = \delta(G)+2$ so that $k = \delta(G)+2 = p+1$.

Next consider the case when $\chi(\mu_2(G)) = \chi(G)$. Then $\delta(G)+1 \leq p \leq \omega(G) \leq \chi(G) = \chi(\mu_2(G)) \leq \chi_f(\mu_2(G)) \leq k \leq \delta(\mu_2(G))+1 = \delta(G)+2$, and hence k is either $\delta(G)+1$ or $\delta(G)+2$. If $k = \delta(G)+1 = \chi(G)$, then by Corollary 2.6, $\mu_2(G)$ has no $(\delta(G)+1)$-fall coloring. Hence $k = \delta(G)+2$. If $p = \delta(G)+1$, then $k = p+1$. In case $\delta(G)+1 < p$, then $p = \delta(G)+2$ and this implies that $\chi(\mu_2(G)) = \chi(G)$, a contradiction to Lemma 2.5.

This shows that if $\mu_2(G)$ has a k-fall coloring, then $k = p+1 = \delta(G)+2$. But this violats Lemma 2.7. ∎

Since any chordal graph has a simplicial vertex, we have the following immediate corollary.

Corollary 2.9:
If G is a chordal graph, then $\mu_2(G)$ has no k-fall coloring for any $k \geq 2$.

3 Fall coloring of split graphs

In this section, we discuss the fall coloring of split graphs and their generalized Mycielskian. In addition, we show that if G has a fall coloring, then $\mu_{3m}(G)$, $m \geq 1$, has a fall coloring. To start with, we consider split graphs.

Theorem 3.1:
A split graph G has a fall coloring iff $\delta(G) = \chi(G) - 1$.

Proof. Assume that G is a split graph and that G has a k-fall coloring, then $\delta(G) + 1 \geq \chi_f(G) \geq \chi(G)$. Since G is split, $\delta(G) \leq \chi(G) - 1$. Thus $\delta(G) = \chi(G) - 1$.

Conversely, let us assume that G is a split graph with $\delta(G) = \chi(G) - 1$. Without loss of generality, let $G = K \cup I$, where $\langle K \rangle$ is a $\chi(G)$-clique and I is an independent set in G. Let $K = \{x_1, \ldots, x_k\}$, where $k = \chi(G)$, and $I = \{y_1, \ldots, y_l\}$. Since $\delta(G) = \chi(G) - 1$, every vertex $y_j \in I$ is adjacent to all vertices of K except one. Define $c : V(G) \to \{1, 2, \ldots, k\}$ by $c(x_i) = i$, $c(y_j) = c(x_i)$, if y_j is not adjacent to x_i. Then c is a proper coloring and every vertex of G is a colorful vertex. Thus G has a k-fall coloring. ∎

We now look at $\mu_m(G)$. Suppose that $\mu_m(G)$ has a fall coloring with $\chi(G) = \omega(G)$ and $\delta(G) < \chi(G) - 1$, then by Lemma 2.5, $\delta(\mu_m(G)) = \delta(G) + 1 < \chi(G) = \chi(\mu_m(G)) - 1 \leq \chi_f(\mu_m(G)) - 1 \leq \delta(\mu_m(G))$, a contradiction. Therefore, for any graph G with $\chi(G) = \omega(G)$ and $\delta(G) < \chi(G) - 1$, $\mu_m(G)$ has no fall coloring, for any $m \geq 1$. Using this observation, we prove the next theorem.

Theorem 3.2:
If G is a split graph and m is not a multiple of 3, then $\mu_m(G)$ has no fall coloring.

Proof. Let $m = 3l - 1$ or $3l - 2$, where $l \geq 1$. If $\delta(G) < \chi(G) - 1$, then there is nothing to prove. Therefore assume that $\delta(G) \geq \chi(G) - 1$. Since G is a split graph, $\delta(G) \leq \chi(G) - 1$ and hence $\delta(G) = \chi(G) - 1$. If $l = 1$, then by Theorems 2.1 and 2.8, the result follows (Recall that any split graph has a simplical vertex). Therefore assume that $l \geq 2$. First, we prove the result when $l = 2$ and then extend the argument for $l \geq 3$.

Suppose $\mu_m(G)$ has an r-fall coloring, say $V(\mu_m(G)) = \bigcup_{i=1}^{r} V_i$, then $r = \chi(G) + 1$ [because, $\chi(G) = \omega(G)$ and by Lemma 2.5, $\chi(\mu_m(G)) \leq \chi_f(\mu_m(G)) \leq r \leq \delta(\mu_m(G)) + 1 = \delta(G) + 1 + 1 = \chi(G) + 1 = \chi(\mu_m(G))$].

Let $\chi(G) = k$ so that $r = k+1$. Since G is a split graph, $V = K \cup I$, where $\langle K \rangle$ is a k- clique and I is an independent set in G. As K is a clique in $\mu_m(G)$, all vertices of K receive distinct colors. Without loss of generality, we assume that no vertex of K receives the color $k+1$. Let $x \in K$ with $x \in V_i$, then $x^1 \in V_i \cup V_{k+1}$. Clearly, no vertex of I belongs to V_{k+1}. Thus $V \cap V_{k+1} = \emptyset$. As every vertex of V is a colorful vertex, $V^1 \cap V_{k+1} \neq \emptyset$. Moreover, as $I \cup I^1$ is an independent set and each vertex of I is a colorful vertex, we conclude that $K^1 \cap V_{k+1} \neq \emptyset$.

Claim: $V^1 \subset V_{k+1}$.

We first consider the case when $|K^1 \cap V_{k+1}| = 1$ and the common vertex is x^1. In this case, there exists $y^1 \in I^1$ such that $y^1 \in V_{k+1}$ and x adjacent to y^1. This is because x is a colorful vertex with color, say, s. Since y^1 is adjacent to all vertices of K except one, say z, with $z \in V_t$, $t \neq s, k+1$, y^1 is not a colorful vertex. Therefore, at least two vertices x^1, y^1 of K^1 must belong to V_{k+1}, and as a consequence $V^2 \cap V_{k+1} = \emptyset$. If $z^1 \in V^1$ with $z^1 \notin V_{k+1}$, then z^1 is not a colorful vertex, a contradiction. Therefore, $V^1 \subset V_{k+1}$ and hence $V^2 \cap V_{k+1} = \emptyset$.

Fact 1. Suppose $x^2, y^2 \in K^2$ with $x^2, y^2 \in V_j$, then $V_j \cap V^3 = \emptyset$ and if there is a vertex $z^2 \in V^2$ with $z^2 \notin V_j$, then z^2 is not a colorful vertex. Therefore $V^2 \subset V_j$, which implies that at least one vertex of V^1 is not a colorful vertex. Thus no two vertices of K^2 receive the same color (that is, no two vertices of K^2 belong to V_j, for any j, $1 \leq j \leq k$). $\rightarrow$ (a).

Hence, if $x^2 \in K^2$ with $x^2 \in V_j$, then $x^3 \in V_j \cup V_{k+1}$.
We now prove that $V^3 \cap V_{k+1} = \emptyset$. Suppose $V^3 \cap V_{k+1} \neq \emptyset$.

Case 1. There is a vertex $x^3 \in K^3$ such that $x^3 \in V_{k+1}$.
Then we have $x^2 \in V_j$, for some $j, j \neq k+1$. We now consider two subcases.

Subcase 1. There exists a vertex $y^2 \in I^2$ such that y^2 is not adjacent to x^3.
Thus x^2 is not adjacent to y^3, and hence $y^3 \in V_j \cup V_{k+1}$. If $y^3 \in V_j$, then y^3 is not a colorful vertex, because y^3 is not adjacent to any vertex of $V_{k+1} \cap K^4$. Therefore $y^3 \in V_{k+1}$. Since y^3 is a colorful vertex and y^3 is not adjacent to x^2, there is a vertex $z^4 \in K^4$ such that $z^4 \in V_j$ and y^3 is adjacent to z^4. By (a), $z^2 \in V_i, i \neq j, k+1$. As z^2 is a colorful vertex, there is a vertex $w^3 \in I^3$ with $w^3 \in V_j$ and z^2 is adjacent to w^3, which implies that w^3 is adjacent to z^4, a contradiction to the fact that $w^3, z^4 \in V_j$.

Subcase 2. Every vertex of I^2 is adjacent to x^3.
Suppose there is a vertex $y^2 \in I^2$ with $y^2 \in V_j$, then y^2 is adjacent to x^3 and y^2 is adjacent to all vertices of K^3 except one, say z^3, with $z^3 \in V_s$,

$s \neq j, k+1$ and hence y^2 is not a colorful vertex. Hence $I^2 \cap V_j = \emptyset$. But then no vertex of I^2 is a colorful vertex, because $V_j \cap V^3 = \emptyset$. Thus $K^3 \cap V_{k+1} = \emptyset$ and therefore, if $x^2 \in K^2$ with $x^2 \in V_i$, then $x^3 \in V_i$.

Case 2. There is a vertex $x^3 \in I^3$ such that $x^3 \in V_{k+1}$.
Then we have x^3 is not a colorful vertex, as x^3 is adjacent to all vertices of V^2 except one, say, y^2 with color $s \neq k+1$. Therefore $I^3 \cap V_{k+1} = \emptyset$.

Hence by Cases 1 and 2, we have $V^3 \cap V_{k+1} = \emptyset$.

Now if $l = 2$, $m = 4$ or 5.

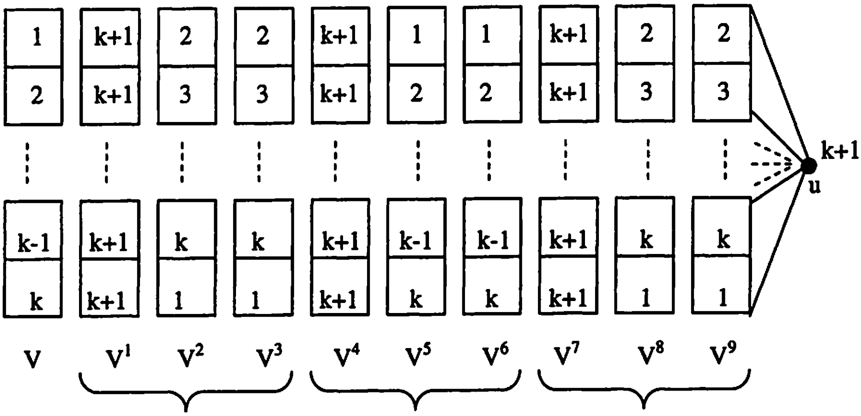
Fact 2. Let $m = 4$. As $V^2 \cap V_{k+1} = \emptyset$ and $V^3 \cap V_{k+1} = \emptyset$, we conclude that $V^4 \cap V_{k+1} \neq \emptyset$, and hence $u \notin V_{k+1}$. If there is a vertex $y^4 \in V^4$ such that $y^4 \notin V_{k+1}$, then y^4 is not a colorful vertex, because $u \notin V_{k+1}$ and $V^3 \cap V_{k+1} = \emptyset$. Therefore $V^4 \subset V_{k+1}$, which implies that u is not a colorful vertex. Hence $\mu_4(G)$ has no fall coloring.

Fact 3. Let $m = 5$. As $V^2 \cap V_{k+1} = \emptyset$, $V^3 \cap V_{k+1} = \emptyset$ and every vertex of V^3 is a colorful vertex, $V^4 \cap V_{k+1} \neq \emptyset$. Suppose $K^4 \cap V_{k+1} = \emptyset$, then no vertex of I^3 is a colorful vertex, because $V^2 \cap V_{k+1} = \emptyset$. Therefore, $K^4 \cap V_{k+1} \neq \emptyset$. By an argument similar to that given for Claim 1, $V^4 \subset V_{k+1}$. Thus $V^5 \cap V_{k+1} = \emptyset$, and no vertex V^5 is a colorful vertex, a contradiction. Hence $\mu_5(G)$ has no fall coloring.

If $m = 3l - 2$, $l \geq 3$, we apply the same arguments as in Fact 1 and Fact 2, and get a contradiction and if $m = 3l - 1$, $l \geq 3$, we apply the same arguments as in Fact 1 and Fact 3, and get a contradiction. Therefore, for any split graph and any m that is not a multiple of 3, $\mu_m(G)$ has no fall coloring. ■

Finally, we look at the case when m is a multiple of 3. In this case, we show that if G has a fall coloring, then so does $\mu_{3m}(G)$.

Suppose G has a k-fall coloring, with $V(G) = \bigcup_{i=1}^{k} V_i$. Let V_j^i denote the subset of V^i that corresponds to V_j in V. We show that $\mu_{3m}(G)$ has a $(k+1)$-fall coloring. We divide the $3m$ sets of $\mu_{3m}(G)$ that correspond to V, namely, $V^1, V^2, \ldots, V^{3m}$ into blocks of size 3 as in Figure 3.1 (which corresponds to the case when $m = 3$). In V^1 of the first block, we give the color $k+1$ to all the subsets V_j^1, $1 \leq j \leq k$, and the colors $2, 3, \ldots, k, 1$ to the sets $V_1^2, V_2^2, \ldots, V_k^2$ in order, and to the sets $V_1^3, V_2^3, \ldots, V_k^3$ in order (that is, repeat the colors of V^2 for V^3). For the next block, we give the color $k+1$ to all subsets V_i^4 corresponding to V_i, and the colors $1, 2, \ldots, k$ in order to the sets $V_1^5, V_2^5, \ldots, V_k^5$ and the sets $V_1^6, V_2^6, \ldots, V_k^6$, in order (that is, repeat the colors of V^5 for V^6). We repeat this coloring until we exhaust all the $3m$ blocks in order.



$\mu_9(G)$
Figure 3.1

In symbols: Define $c : V((\mu_{3m}(G))) \to \{1, 2, \ldots, k, k+1\}$ as follows: For $1 \leq j \leq k$ and $s \geq 1$, set $c(V_j) = j$ (that is, all the vertices of V_j in $\mu_{3m}(G)$ are given the color j), $c(V_j^{6s-5}) = k+1$ and for each i, $1 \leq i \leq k-1$, $c(V_j^{6s-4}) = i+1$, $c(V_k^{6s-4}) = 1$ and for each i, $1 \leq i \leq k$, $c(V_j^{6s-3}) = c(V_j^{6s-4})$.

Next for j, $1 \leq j \leq k$ and $s \geq 1$, $c(V_j^{6s-2}) = k+1$, $c(V_j^{6s-1}) = c(V_j^{6s}) = c(V_j)$, and $c(u) = k+1$.

Then c is a proper coloring of $\mu_{3m}(G)$ and every vertex of $\mu_{3m}(G)$ is a colorful vertex. Therefore we have the following theorem.

Theorem 3.3:
If G has a fall coloring, then $\mu_{3m}(G)$ has a fall coloring and $\chi_f(\mu_{3m}(G)) \leq \chi_f(G) + 1$.

Equality holds in the statement of Theorem 3.3 whenever G satisfies the condition that $\omega(G) = \chi(G) = \chi_f(G)$. Our results lead naturally to the following two problems.

Problem 3.4
Does there exist a fall colorable graph G for which $\chi_f(\mu_{3m}(G)) < \chi_f(G)+1$?

Problem 3.5
Find some new families of graphs G for which the generalized Mycielskian $\mu_m(G)$ has a fall coloring for some m.

Acknowledgement

This research was supported by the Department of Science and Technology, Government of India grant DST SR / S4 / MS: 234 / 04 dated March 31, 2006. The authors thank the two referees for their helpful comments. Lemma 2.7 is due to one of the referees who also pointed out that our Theorem 2.8 could be deduced from this Lemma.

References

[1] R. Balakrishnan and T. Kavaskar, *Fall Coloring of Graphs -I*, submitted.

[2] J.E. Dunbar, S.M. Hedetniemi, S.T. Hedetniemi, D.P. Jacobs, J. Knisely, R.C. Laskar and D.F. Rall, *Fall coloring of graphs*, J. Combin. Math. and Combin. Comput. **33** (2000), 257-273.

[3] P.C.B. Lam, G. Gu, W. Lin and Z. Song, *Circular chromatic number and a generalization of the construction of Mycielski*, J. Combin. Theory, Ser. B **89** (2003), 195-205.

[4] J. Mycielski, *Sur le coloriage des graphs*, Colloq. Math., **3**, (1955), 161-162.

[5] R.C. Laskar, and Jeremy Lyle, *Fall coloring of bipartited graphs and cartesian products of graphs*, Discrete Applied Mathematics **157** (2009), 330-338.

[6] Zs. Tuza and V. Rödl, *On color critical graphs*, J. Combin. Theory, Ser. B **38** (1985), 204-213.