MAXIMAL STRONGLY INDEXABLE GRAPHS

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Abstract

Given any positive integer k, a (p,q)-graph $G=(V,E)$ is *strongly k-indexable* if there exists a bijection $f: V \to \{0,1,2,\ldots,p-1\}$ such that $f^+(E(G)) = \{k,k+1,k+2,\ldots,k+q-1\}$ where $f^+(uv)=f(u)+f(v)$ for any edge $uv \in E$; in particular, G is said to be *strongly indexable* when $k=1$. For any strongly k-indexable (p,q)-graph G, $q \leq 2p-3$ and if, in particular, $q=2p-3$ then G is called a *maximal strongly indexable graph*. In this paper, necessary conditions for an Eulerian (p,q)-graph G to be strongly k-indexable have been obtained. Our main focus is to initiate a study of maximal strongly indexable graphs and, on this front, we strengthen a result of G. Ringel on certain outerplanar graphs.

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1. Introduction

Unless mentioned otherwise, by a *graph* we shall mean in this paper a finite, undirected, connected graph without loops or multiple edges. Terms not defined here are used in the sense of Harary (12).

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Acharya and Hegde (2; 3) introduced the concept of an 'indexer' of a graph as a special case of *arithmetic numberings.* A *numbering* of a graph $G = (V, E)$ is an assignment f of distinct nonnegative integers to the vertices of G; it is an *additive numbering* of G if the induced 'edge function' $f^+ : E(G) \to \mathbb{N}$, from $E(G)$ into the set $\mathbb{N}$ of natural numbers, defined by the rule: $f^+(uv) = f(u) + f(v),\ \forall\ uv \in E(G)$, is also injective. It is known that every finite graph has an additive numbering (2); in fact, using Sidon sequences (see (11; 15)) it can be shown that every countable graph has an additive numbering. Hence, an additive numbering f is said to be *optimal* if $f[G] := max_{v \in V(G)}\{f(v)\}$ attains the least possible value $v(G)$ amongst all the additive numberings of G. Clearly, $v(G) \geq |V(G)|$ for any graph G with a countable number of vertices. For any given positive integer k, an additive numbering f of G is called an *arithmetic k-indexer* if $f^+(E(G)) := \{f^+(uv) : uv \in E(G)\} = \{k, k+1, k+2, \ldots,\}$ and G is *arithmetically k-indexable* if it admits an arithmetic k-indexer. Not every graph is arithmetically k-indexable as indicated by the following theorem for finite graphs.

Theorem 1.1. (2): *Let $G = (V, E)$ be any (p, q)-graph and f be any arithmetic k-indexer of G, where k is odd. Then, there exists an 'equitable partition' of V into two subsets V_o and V_e such that there are exactly $\lceil \frac{q+k-1}{2} \rceil$ edges each of which joins a vertex of V_o with one of V_e, where $\lceil . \rceil$ denotes the least integer function.*

In the above theorem, an *equitable partition* of a nonempty finite set X is defined as a partition $\{X_1, X_2\}$ of X such that the cardinalities of X_1 and X_2 differ by at most one, that is, if $||X_1| - |X_2|| \leq 1$.

Next, as in (2), if G is a (p, q)-graph then $v(G) \geq p - 1$ and, hence, an additive numbering f of G is called a *strong k-indexer* of G if $f(V(G)) := \{f(v) : v \in V(G)\} = \{0, 1, 2, \ldots, p - 1\}$ and $f^+(E(G)) = \{k, k + 1, k + 2, \ldots, k + q - 1\}$ for some positive integer k. Further, G is said to be *strongly k-indexable* if it admits a *strong k-indexer*. While, as mentioned already, every graph has an additive numbering, not every graph is arithmetically k-indexable for a given positive integer k. If $V(G)$ is countably infinite, then any bijection f from $V(G)$ onto the set $\mathbb{N} \cup \{0\}$ of nonnegative integers such that $f^+(E(G)) = \{k, k + 1, k + 2, \ldots, \}$ is defined as a strong k-indexer of G. In particular, if $k = 1$ in these definitions, then f is called a *strong indexer* of G and the graph G is said to be *strongly indexable* if it admits a strong indexer.

As in (13), a (p, q)-graph $G = (V, E)$ is *edge-magic* if it admits an *edge-magic labeling* of G, which is defined as a bijection $f : V(G) \cup E(G) \to \{1, 2, \ldots, p + q\}$ such that there exists a constant s, called the *magic number* of f, with $f(u) + f(v) + f(uv) = s$, $\forall uv \in E(G)$. An edge-magic labeling f of G is *super-edge-magic* if $f(V(G)) = \{1, 2, \ldots, p\}$ and $f(E(G)) = \{p+1, p+2, \ldots, p+q\}$ and G is *super-edge-magic* if it admits a super-edge-magic labeling (6) (also see (8; 14)). It has been recently proved in (4) that the class of strongly indexable graphs is a proper subclass of super-edge-magic graphs as also that every connected graph can be extended to be an induced subgraph of a strongly indexable graph, strengthening a result in (7).

It has been noted in (2; 3) that for any strongly indexable (p, q)-graph G, $q \leq 2p - 3$, calling G a *maximal strongly indexable graph* if

$q = 2p - 3$. The main aim of this paper is to study maximal strongly indexable graphs, particularly, such outerplanar graphs. Our motivation for this comes from the well known fact that if G is a *maximal outerplanar* (p, q)-graph then $q = 2p - 3$ (see (12)). However, not all such graphs are strongly indexable as, for instance, the graph depicted in Figure 5 is not strongly indexable. In general, determination of maximal strongly indexable graphs is an open problem. Hence, we look for special classes of graphs satisfying $q \leq 2p - 3$, which are strongly indexable (*e.g., see* (4)).

As mentioned already, while the general question about maximal outerplanar graphs being strongly indexable is open, we report in this paper some progress made in this direction. We shall need the following known results for this purpose.

Theorem 1.2. **(2):** *Every strongly indexable finite graph has at most one nontrivial component which is either a star or has a triangle.*

Thus, according to Theorem 1.2, excepting stars $K_{1,n}$, no connected triangle-free (in particular, such outerplanar) graph is strongly indexable. However, there could be such super-edge-magic graphs.

Theorem 1.3. **(2):** *Let $G = (V, E)$ be any (p, q)-graph and f be any strong indexer of G. Then, there exists an equitable partition of V into two subsets V_o and V_e such that there are exactly $\lceil \frac{q}{2} \rceil$ edges each of which joins a vertex of V_o with one of V_e.*

2. Strongly k-indexable Eulerian graphs

The following is a new result for any 'additive labeling' of vertices of a graph, which will also provide us a tool for further progress in our investigation.

Theorem 2.1. *Let $G = (V, E)$ be any graph, not necessarily finite, f be an arbitrary assignment of integers to the vertices of G and let $f^+(uv) = f(u) + f(v)$ for each edge uv in G. Then in every cycle of G there are an even number of edges with odd f^+-values.*

Proof. Let $G = (V, E)$ be a graph not necessarily finite. Then any cycle Z of G is finite. Then, as pointed out in (2), $\Sigma_{uv \in E(Z)} f^+(uv) = 2\Sigma_{x \in V(Z)} f(x)$ and hence, $|\{f^+(uv) : uv \in E(Z) \wedge f^+(uv) \text{ is odd}\}|$ is even. □

Remark 2.2. The following alternative proof of Theorem 2.1 shows its connection with number theory and electrical network theory: The result follows from the number-theoretic congruence $a + b \equiv a - b$ for any two integers a and b, applied to f^+-values around any cycle Z in G and then by applying the well known Kirchhoff's Voltage Law (KVL) treating the set $\mathbb{Z}$ of integers as an additive 'voltage group' on the network $(G, f \cup f^+)$ (cf.: (1)).

Corollary 2.3. *Let $G = (V, E)$ be any Eulerian (p, q)-graph. If G is strongly k-indexable then $q \not\equiv 2 (mod\ 4)$. Further, exactly one of the following congruences holds*

(i) $q \equiv 0 (mod\ 4)$,

(ii) $q \equiv 1 (mod\ 4)$ *and* $k \equiv 0 (mod\ 2)$, *and*

(iii) $q \equiv 3 (mod\ 4)$ *and* $k \equiv 1 (mod\ 2)$.

Proof. Since G is Eulerian, its edge set $E(G)$ can be partitioned into edge-disjoint cycles $Z_1, Z_2, \ldots, Z_t$. Therefore, since $f^+(E(G)) = \{k, k+1, k+2, \ldots, k+q-1\}$ Theorem 2.1 yields

$$qk + \frac{q(q-1)}{2} = \sum_{x \in E(G)} f^+(x) = \sum_{j=0}^{t} f^+(E(Z_j)) \equiv 0 (mod\ 2),$$

from which the result follows. □

Remark 2.4. Corollary 2.3 can be used to rule out the possibility of certain classes of Eulerian graphs from their being strongly indexable. In this way, for instance, the cycles C_n for values of $n \equiv 1$ or $2 (mod\ 4)$ cannot be strongly indexable. Of course, in general, C_n is not strongly indexable for any value of $n \geq 4$ by virtue of Theorem 1.1, thus demonstrating that the converse of Corollary 2.1 does not hold. A more complex example of an Eulerian graph that is not strongly indexable by this argument is the complement of K_3^+ (which is also isomorphic to the 'shadow graph' of K_3; it is a maximal outerplanar graph too as seen from its depiction in Figure 5), where G^+ in general is the graph obtained by adjoining a new pendant vertex (or a 'leaf') v' to each vertex v of G.

An infinite class of Eulerian graphs that are not strongly indexable is the class of *Husimi trees* (viz., connected separable graphs in which every block is a triangle) in which the number of blocks is $m \equiv 3 (mod\ 4)$ since, in such a graph H, $|E(H)| \equiv 1 (mod\ 4)$. One such well known class is that of "friendship graphs" $F_t := tK_2 + K_1$, which consists of t triangles glued at one common vertex whence F_t consists of $q = 3t$ edges, so that $t \equiv 2$ or $3 (mod\ 4)$ yielding $q \equiv 2$ or $1 (mod\ 4)$ in the respective cases.

Conjecture 2.5. Any Husimi tree with $t \equiv 0 (mod\ 4)$ triangles is strongly indexable.

A particular case of Conjecture 2.5 is F_t with $t \equiv 0(mod\ 4)$; a strong indexer of F_4 is displayed in Figure 1 below; a solution of this problem is not yet known for higher admissible values of t.

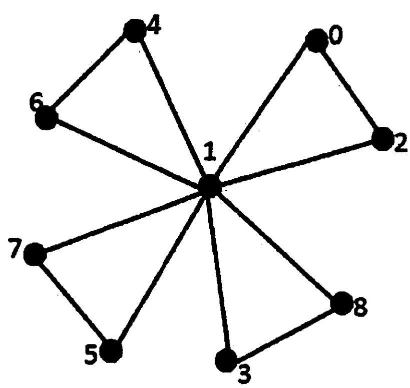

FIGURE 1

The following argument shows that F_5 is not strongly indexable: By the very definition of a strong indexer f of such a $(11, 15)$-graph F_5, $\{1, 2, \ldots, 15\}$ is the set $f^+(E(F_5))$ of edge labels. Here, the edge labels 1 and 2 are uniquely expressed as sums of two distinct nonnegative integers on an adjacent pair of vertices respectively as $1 = 0 + 1$; $2 = 0 + 2$; and the numbers $0, 1, 2$ should necessarily be assigned to the vertices of one of the triangles which in turn implies that the central vertex say, u at which the five triangles are glued together should receive one of these numbers, $0, 1$ or 2. Also, values in pairs are to be assigned to the non-central vertices that are adjacent and these pairs of numbers cannot be made adjacent to any other non-central vertex labels. For instance, the possible '2-partitions' of

Edge labellings	Possible pairs of numbers
15	(10,5), (9,6),(8,7)
14	(10,4),(9,5),(8,6)
13	(10,3),(9,4),(8,5),(7,6)
12	(10,2),(9,3),(8,4),(7,5)
11	(10,1),(9,2),(8,3),(7,4),(6,5)
10	(10,0), (9,1), (8,2), (7,3), (6,4)
9	(9,0),(8,1),(7,2(6,3), (5,4)
8	(8,0),(7,1),((6,2),(5,3)
7	(7,0), (6,1),(5,2),(4,3)
6	(6,0),(5,1),(4,2)
5	(5,0), (4,1),(3,2)
4	(4,0),(3,1)
3	(3,0), (2,1)
2	(2,0)
1	(1,0)

FIGURE 2

numbers in $f^+(F_5)$ into distinct 'parts' are tabulated in Figure 2. We prove that it is not possible to distribute these pairs of numbers among the vertices of F_5 so that F_5 is strongly indexable. There are three cases to be considered for the purpose.

Case (i) $f(u) = 0$**:** To obtain the highest edge weight 15, we may choose any pair of numbers of the 2-partition from the first row of the table. Let us start with $(10, 5)$. Now, to have the edge weight 14, the only possibility is to choose the pair $(8, 6)$ from the second row, since the other two pairs in the row contain the numbers 10 and 5. Now, to have the edge weight 13, the only possibility is to choose the pair $(9,4)$ from the third row. We have no choice to label any vertex to have the edge weight 12 as the maximum possible vertex value is 10 and $f(u) = 0$ (the maximum edge weight that is expected with the central vertex is 10). By a similar argument we can show that it is not possible to have a strong indexer by starting with any of the pairs $(9,6)$ and $(8,7)$ in the first row.

Case(ii) $f(u) = 1$: Again, let us start with $(10,5)$ in the first row of the table. Now, to have the edge weight 14 the only possibility is to choose the pair $(8,6)$ from the second row, since the two other pairs in the row contain the numbers 10 and 5 respectively. Now, to have the edge weight 13, the only possibility is to choose the pair $(9,4)$ from the third row. We have no choice to label any vertex to have the edge weight 12 as the maximum possible vertex value is 10 and $f(u) = 1$ (the maximum edge weight that is expected with the central vertex now is 11). By a similar argument we can show that it not possible to have a strong indexer by starting with any of the pairs $(9,6)$ and $(8,7)$ in the first row.

Case (iii) $f(u) = 2$: Lastly, let us start with $(10,5)$ in the first row of the table. Now, to have the edge weight 14, the only possibility is to choose the pair $(8,6)$ from the second row, since the other two pairs in the row contain the numbers 10 and 5 respectively. Now, to have the edge weight 13, the only possibility is to choose the pair $(9,4)$ from the third row. Then, the only choice to label any vertex to have the edge weight 12 is with the central vertex u. Hence, we need not choose any pair from the the row of edge weight 12. To have the edge weight 11, the only possibility is to choose the pair $(6,5)$ from the fifth row. The edge weight 10 already exists with the vertex labeled 8 and the central vertex. Now, we go for the edge weight 9. In this case, we are left with no pair to choose from the row of the weight 9 as at least one of the numbers of each other pairs in that row is already chosen.

Thus, we conclude that F_5 is not strongly indexable.

In general, it appears that the 'equitable partition theorem' (viz. Theorem 1.3) should be useful to establish that F_t is not strongly indexable for $t \equiv 1 (mod\ 4)$.

3. Maximal strongly indexable graphs

In this section, we report results of our investigation on the maximal strongly indexable graphs. By the very definition of a strong indexer f of such a (p,q)-graph G, $\{1,2,\ldots,2p-3\}$ is the set $f^+(E(G))$ of edge labels. Here, the edge labels $1, 2, 2p-4$ and $2p-3$ are uniquely expressed as sums of two distinct nonnegative integers on an adjacent pair of vertices respectively as $1 = 0+1$; $2 = 0+2$; $2p-4 = (p-1)+(p-3)$ and $2p-3 = (p-1)+(p-2)$, since $f[G] = p-1$. That is, for any strongly indexable (p,q)-graph G, in any strong indexer f of G, the following pairs of vertex labels must be adjacent: $(0,1); (0,2); (p-1,p-2); (p-1,p-3)$. Every edge label from 3 to $2p-5$ can be obtained by more than one choice of adjacent vertex pairs and whether there exists a strong indexer in which the given pair of these numbers appear as end-vertex labels of an edge in G is an open question to examine. For instance, the possible '2-partitions' of $p = 4$ and $p = 5$ with distinct 'parts' are tabulated in Figure 3.

Weight	Pairs
1	(0,1)
2	(0,2)
3	(0,3), (1,2)
4	(0,4), (1,3)
5	(1,4), (2,3)
6	(2,4)
7	(3,4)

Weight	Pairs
1	(0,1)
2	(0,2)
3	(0,3), (1,2)
4	(0,4), (1,3)
5	(0,5)(1,4), (2,3)
6	(1,5) (2,4)
7	(2,5),(3,4)
8	(3,5)
9	(4,5)

FIGURE 3

For a maximal strongly indexable graph of order 5 and size 7 there are 8 ways to select the pairs to determine a strong indexer; these strong indexers are shown in Figure 4.

For order $p = 6$, one can see that there are 48 distinct such 2-partitions of $2p - 3 = 9$. How many of these are 'realizable' as labels of pairs of adjacent vertices in a maximal strongly indexable graph of order six? In general, it seems to be a hard problem to determine for a given integer $p \geq 5$ exactly how many 2-partitions of $2p - 3$ with distinct parts are realizable as pairs of labels on adjacent pairs of vertices in a maximal strongly indexable graph of order p.

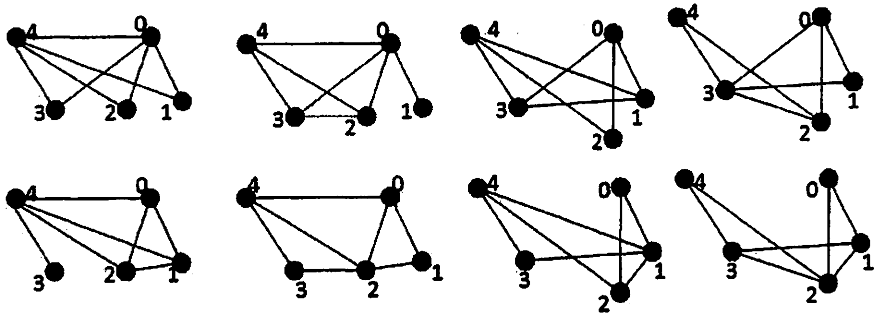

FIGURE 4

Remark 3.1. If G is a maximal strongly indexable graph of order p, then G can be extended to a maximal strongly indexable graph of order $p+1$. This can be done by adjoining a new vertex with label p and making it adjacent to the vertices with labels $p-2$ and $p-1$, so that we get a graph of order $p+1$ and the new edge labels will be $2p-2$ and $2p-1$, yielding a maximal strongly indexable graph of order $p+1$.

4. MAXIMAL OUTERPLANAR STRONGLY INDEXABLE GRAPHS

Since K_2 is a maximal outerplanar graph of order two, having a strong indexer that assigns 0 and 1 to its vertices, the construction mentioned in Remark 3.1, applied recursively, implies that for every integer $p \geq 2$ there exists a maximal outerplanar graph of order p that is strongly indexable. In this section, we embark on determining connected maximal outerplanar graphs, which are strongly indexable. We begin by determining the small order cases first.

Lemma 4.1. *Let $G = (V, E)$ be a (p, q)-graph which is a maximal outerplanar graph with $p \leq 7$. Then, G is strongly indexable if and only if G is not isomorphic to the graph given in Figure 5.*

Proof. That the graph given in Figure 5 cannot be strongly indexable is noted already in Remark 2.4. The other maximal outerplanar graphs of orders $p \leq 7$ are shown in Figure 6 along with a strong indexer for each of them. □

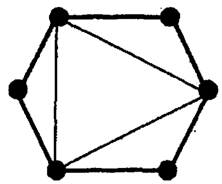

FIGURE 5

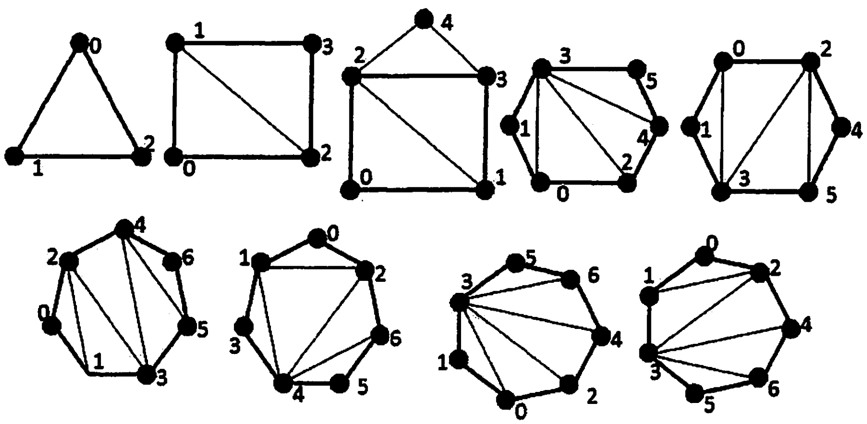

FIGURE 6

Lemma 4.2. *Let G be a maximal outerplanar (p, q)-graph with maximum degree $\Delta(G) = p - 1$. Then, G is strongly indexable if and only if $p \leq 7$.*

Proof. If $p \leq 7$ then all the maximal outerplanar graphs with maximum vertex degree $p - 1$ are contained in the set of graphs shown in Figure 6 along with a strong indexer in each case.

For the converse, let G be a maximal outerplanar graph with $p > 7$ and $\Delta(G) = p - 1$. Suppose, if possible, there exists a strong indexer f of G. By hypothesis, there exists a vertex $v_1 \in V(G)$ such that $d(v_1) = p - 1 = \Delta(G)$. Since G is maximal outerplanar, every triangle in G contains the vertex v_1 (*cf.* (12)) so that $f(v_1) \in \{0, 1, 2, 3\}$ since otherwise the induced edge labelling f^+ can be shown to produce nonconsecutive numbers.

Since G is maximal outerplanar, G contains a Hamilton cycle Z such that G has a plane embedding in which all the edges not on Z, called *chords* of Z, are within the region of the plane bound by Z (cf.: (12))). Now, suppose $f(v_1) = 0$. All the neighbors of v_1 are labeled in f with the consecutive numbers $1, 2, 3, \ldots, p - 1$ so that all the chords of Z together with the edges v_1v_2 and v_1v_p have the labels $1, 2, 3, \ldots, p - 1$ so that the remaining edges in Z should get $p, p+1, p+2, \ldots, 2p-3$ in a one-to-one manner. It is necessary that the labels $p - 1$ and $p - 2$ must be adjacent to get the label $2p - 3$. Without loss of generality, assume $f(v_i) = p-1$ and $f(v_{i+1}) = p-2$. Also, v_1 should be adjacent to the vertex with label $p-3$ so as to have an edge with label $2p - 4$. That is, $f(v_{i-1}) = p-3$. Now, because of the maximal outerplanarity of G, v_i cannot be adjacent to v_{i-1} and v_{i+1}. Also, v_{i+1} cannot be adjacent to v_{i-1} so that there cannot be an edge with label $2p - 5$. So, $f(v_1) \neq 0$, a contradiction. Therefore, $f(v_1) = 1$. Let v_i be the vertex such that $f(v_i) = 0$. That is, all the neighbors of v_i are labeled with the numbers $0, 2, 3, \ldots, p - 1$ so that all the chords together with v_1v_2 and v_1v_p have the edge labels from the set $\{1, 2, 3, \ldots, p-1, p\}$. It is necessary that the vertex labeled $p-1$ and $p-2$ are adjacent since otherwise there would be no edge with label $2p-3$. By a similar argument as above, there is no edge with label $2p-5$. That is, $f(v_1) \neq 1$, a contradiction. A similar contradiction may be arrived at when we assume $f(v_1) = 2$ or $f(v_1) = 3$. Thus, the proof follows by contradiction. □

The following theorem is crucial in the proof of the main result of this section, where $\Delta(G)$ denotes as usual the maximum vertex degree in G.

Theorem 4.3. *Let $G = (V, E)$ be a maximal outerplanar graph with $p > 7$. Let $H = (u_1, u_2, u_3, \ldots, u_p)$ be a Hamiltonian cycle in G. Then, there exists an equitable partition $\{V_1, V_2\}$ of $V(G)$ such that no chord of H has both its ends in V_1 or V_2 if and only if $\Delta(G) = \lfloor \frac{p}{2} \rfloor + 2$ and there exist exactly two vertices of degree 2.*

Proof. Let $V_1 = \{u_1, u_2, u_3, \ldots, u_{\lfloor \frac{p}{2} \rfloor}\}$ and $V_2 = \{u_{\lfloor \frac{p}{2} \rfloor+1}, u_{\lfloor \frac{p}{2} \rfloor+2}, u_{\lfloor \frac{p}{2} \rfloor+3}, \ldots, u_p\}$ constitute an equitable partition of the vertex set of G such that no chord of H has both its ends in V_1 or V_2. Clearly $u_{\lfloor \frac{p}{2} \rfloor+1}$ has degree 2 (*cf.* (12)) and $\Delta(G) = \lfloor \frac{p}{2} \rfloor + 2$. Since G is maximal outerplanar $u_{\lfloor \frac{p}{2} \rfloor+2}$ is adjacent to $u_{\lfloor \frac{p}{2} \rfloor}$. Now, let i be the least positive integer such that $1 \leq i \leq \lfloor \frac{p}{2} \rfloor$ and $u_{\lfloor \frac{p}{2} \rfloor+2}$ is adjacent to u_i. Since G is maximal outerplanar, $u_{\lfloor \frac{p}{2} \rfloor}$ is adjacent to u_j for each j with $i \leq j \leq \lfloor \frac{p}{2} \rfloor$. If $i = 1$, then u_i is adjacent to u_j for every j with $\lfloor \frac{p}{2} \rfloor + 3 \leq j \leq p-1$ and in this case u_p has degree 2. If $i > 1$, let i_1 be the least positive integer such that u_{i_1} is adjacent to u_j for each j with $\lfloor \frac{p}{2} \rfloor + 3 \leq j \leq p$. If $i_1 = p$ then u_i has degree 2. Otherwise let i_2 be the least positive integer such that $1 \leq i_1 \leq i-1$ and u_{i_1} is adjacent to u_{i_2}. Repeating this process, we conclude that either u_p

or u_1 has degree 2.

Conversely, let G be a maximal outerplanar graph such that $\Delta(G) = \lfloor \frac{p}{2} \rfloor + 2$ and let there exist exactly two vertices of degree 2. Let u_1 be a vertex of degree 2 so that u_1 is adjacent to u_2 and u_p. Let i be the largest integer such that $i \leq \lfloor \frac{p-1}{2} \rfloor$ and u_p is adjacent to u_i. Then, since G is a maximal outerplanar graph we see that u_p is adjacent to each of the vertices $u_2, u_3, \ldots, u_i$. If $i \leq \lfloor \frac{p}{2} \rfloor + 1$ and u_i is adjacent to u_j for some j with $i + 1 \leq j \leq \lfloor \frac{p}{2} \rfloor + 1$, then we get a vertex u_k, $i \leq k \leq \lfloor \frac{p}{2} \rfloor$, of degree 2 and in this case u_i must be adjacent to $u_{j+1}, u_{j+2}, \ldots, u_{p-1}$ whence $d(u_i) > \lfloor \frac{p}{2} \rfloor + 2$, a contradiction. Hence, $V_1 = \{u_1, u_2, u_3, \ldots, u_{\lfloor \frac{p}{2} \rfloor}\}$ and $V_2 = \{u_{\lfloor \frac{p}{2} \rfloor + 1}, u_{\lfloor \frac{p}{2} \rfloor + 2}, u_{\lfloor \frac{p}{2} \rfloor + 3}, \ldots, u_p\}$ gives the required partition of the vertex set of G. □

Remark 4.4. G is a maximal outerplanar graph of order p with exactly two vertices x and y of degree 2 and $\Delta(G) = \lfloor \frac{p}{2} \rfloor + 2$, then the distance $d_H(x, y)$ between x and y on the Hamiltonian cycle H in G is either $\lfloor \frac{p}{2} \rfloor$ or $\lfloor \frac{p}{2} \rfloor - 1$.

Ringel (16) has established the following theorem.

Theorem 4.5. *(16): If G is a maximal outerplanar graph of order p with exactly two vertices a, b of degree 2 and whose distance $d_H(a, b)$ on the Hamiltonian cycle H in G is either $\lfloor \frac{p}{2} \rfloor$ or $\lfloor \frac{p}{2} \rfloor - 1$, then G is supermagic.*

Theorem 4.3, Remark 4.4 and Theorem 4.5 imply the following result which, of course, is subsumed by the more general result that every strongly indexable graph is super-edge-magic established in (4).

Corollary 4.6. *Let $G = (V, E)$ be a maximal outerplanar (p, q)-graph with $p > 7$. If G is strongly indexable then G is super-edge-magic.*

Proof. Let G be strongly indexable and let f be a strong indexer of G. Let $H = (u_1, u_2, u_3, \ldots, u_p)$ be a Hamiltonian cycle in G. Then, by Theorem 1.3, there exists an equitable partition of $V(G)$ into two subsets V_o and V_e such that there are exactly $\lceil \frac{q}{2} \rceil$ edges each of which joins a vertex of V_o with one of V_e. This very partition of $V(G)$ satisfies the condition on chords of H as specified in the statement of Theorem 4.3. Therefore, G is a maximal outerplanar graph of order p with exactly two vertices a, b of degree 2 and whose distance $d_H(a, b)$ on the Hamiltonian cycle H in G is either $\lfloor \frac{p}{2} \rfloor$ or $\lfloor \frac{p}{2} \rfloor - 1$ and hence by Theorem 4.5 the result follows. □

As noted in (4), not every super-edge-magic graph is strongly indexable in general. Therefore, the following main result of this section may be seen as stronger than Theorem 4.5.

Proposition 4.7. Let G be any maximal outerplanar graph of order $p > 7$ with $\Delta(G) = \lfloor \frac{p}{2} \rfloor + 2$ and exactly two vertices of degree 2. Then G is strongly indexable.

Proof. Let $H = (u_1, u_2, u_3, \ldots, u_p)$ be a Hamiltonian cycle in G. Then, by Theorem 4.3, we have the sets $V_1 = \{u_1, u_2, u_3, \ldots, u_{\lfloor \frac{p}{2} \rfloor}\}$ and $V_2 = \{u_{\lfloor \frac{p}{2} \rfloor+1}, u_{\lfloor \frac{p}{2} \rfloor+2}, u_{\lfloor \frac{p}{2} \rfloor+3}, \ldots, u_p\}$ constituting an equitable partition of the vertex set of G such that no chord of H has both vertices in V_1 or V_2. Let the function $f : V \to \mathbb{N}$ be defined as follows:

$f(u_1) = 0$

$f(u_i) = 2(i - 1);\ 2 \leq i \leq \lceil \frac{p}{2} \rceil$

$f(u_j) = p - 2i + 1$; if p is even, $j = \frac{p}{2} + i,\ 1 \leq i \leq \frac{p}{2}$.

$f(u_j) = p - 2i$; if p is odd, $j = \lceil \frac{p}{2} \rceil + i, \ 1 \leq i \leq \lfloor \frac{p}{2} \rfloor$.

Clearly, $f(V(G)) = \{0, 1, 2, \ldots, p-1\}$.

Now, $f^+(x) \neq f^+(y)$ for any two edges x and y for, otherwise, we have

$$f(u_i u_j) = f(u_\ell u_k), \text{ where } i, j, k \text{ and } \ell \text{ are all distinct}$$

and $u_i, u_j \in V_1$ and $u_\ell, u_k \in V_2$, contrary to our first derivation above. Let $j < k$ and $i < \ell$. Then, $f(u_j) < f(u_k)$ and $f(u_i) < f(u_\ell)$ whence

$$f(u_i u_j) = f(u_i) + f(u_j) < f(u_k) + f(u_\ell) = f(u_k u_\ell).$$

The result is obvious when $i = k$ or $j = k$.

Thus, it follows that f is indeed a strong indexer of G and the proof is complete. □

5. Connection with Fibonacci sequences

For the vertices of a clique in a graph to be labeled so that no edge label is repeated, the labels must be chosen from a set of positive integers in which the sums of the pairs of distinct vertex labels are all distinct. Such a set is called a *weak Sidon set* (see (14), Ch.6), but we shall call it here just a *Sidon set*. When the members of the Sidon set are placed in ascending order, the resulting sequence is called a *Sidon sequence*(see (9; 10; 11; 15) for various directions of research on these sequences). Choose a Sidon sequence $(s_1, s_2, \ldots, s_r)$ in which the largest element s_r is as small as possible. If $s(r)$ denotes the smallest possible value of s_r taken over all Sidon sequences of length r then, a Sidon sequence of length r with largest element

$s(r)$ must have 0 as the smallest element. Note that the Fibonacci sequence (f_n), defined by $f_1 = 1, f_2 = 2$ and $f_n = f_{n-1} + f_{n-2}$, is a Sidon sequence. Hence, Fibonacci numbers provide a reasonably good upper bound for the function σ whose values are all elements of a Sidon sequence.

Let U_3 denote the set of all unicyclic graphs with C_3 as their unique cycle (5). Acharya and Germina (4), while establishing the characterization of strongly indexable unicyclic graphs, noted that a strong indexer f generated by the algorithm contains the first n terms f_j of a Fibonacci sequence augmented with an extra term $f_0 = 0$ before its first term also brings out the following converse part of this statement.

Corollary 5.1. *For any integer $n \geq 3$, given the set $A_n(\mathbb{F})$ of the first n terms $f_1, f_2, f_3, \ldots, f_n$ of a given Fibonacci sequence $\mathbb{F}$ with $f_1 = 1$ and $f_2 = 2$ and a zero term augmented before its first term (i.e., $f_o = 0$), there exists a graph in U_3 together with a strong indexer which uses all the elements of $A_n(\mathbb{F})$. However, a graph in U_3 having a strong indexer f such that $A_n(\mathbb{F}) \subset f(G)$ may not be unique.*

They raised the following interesting problem.

Problem 1. (4): Given the set $A_n(\mathbb{F})$ of the first n terms of a given Fibonacci sequence $\mathbb{F}$ with an augmented zero before its first term, determine the class of all minimal non-isomorphic strongly indexable graphs in U_3 for which $A_n(\mathbb{F}) \subset f(G)$ for some strong indexer f.

We can now prove the following:

Theorem 5.2. *Given a Fibonacci sequence* (f_r) *with* $f_1 = 1, f_2 = 2$ *there exists a connected maximal strongly indexable graph of order* $f_r + 1$ *and size* $2f_r - 1$ *that contains a clique of order* r.

Proof. Label the vertices of the clique with the members of the Fibonacci sequence with $f_1 = 1$, $f_2 = 2$. Add a new vertex with label zero, make it adjacent to the vertices labeled 1 and 2 on the clique, and other new vertices labeled with the remaining positive integers smaller than f_r that are not appearing as edge labels in the clique and join them to the vertices on the clique as follows: All the vertices whose labels lie between f_{r-1} and f_r are joined to the vertex labeled f_r on the clique. All the vertices whose labels lie between f_{i-1} and f_i are joined to f_j whenever $j > i$. Clearly, this construction gives a connected maximal strongly indexable graph, of order $f_r + 1$ and size $2f_r - 1$ that contains a clique of order r, as claimed (see Figure 7). □

Remark 5.3. From the construction of the maximal strongly indexable graph in Theorem 5.2 it is clear that there could exist a disconnected strongly indexed graph G of smaller size with *clique number* $\omega(G) = r$, but then it would not be a maximal one (see Figure 8).

Corollary 5.4. *There exist maximal strongly indexable graphs of arbitrarily high chromatic numbers.*

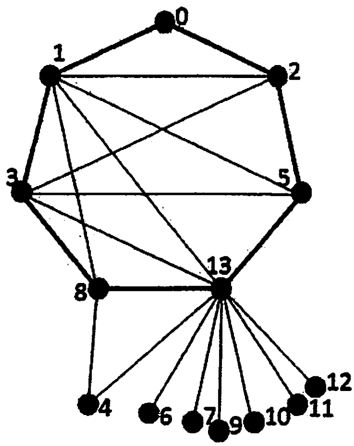

FIGURE 7

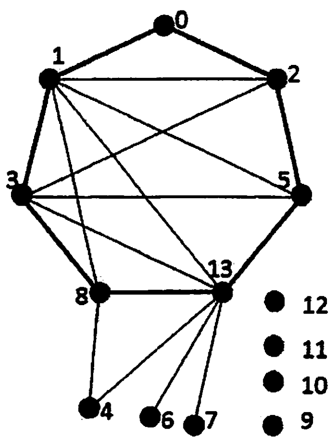

FIGURE 8

6. CONCLUSIONS AND SCOPE

Acharya & Hegde (3) proved in the case of finite graphs that the only regular graphs that are strongly indexable are either K_3 or $K_3 \times K_2$. It is not known whether there are regular infinite outerplanar graphs that are strongly indexable. However, there are infinite

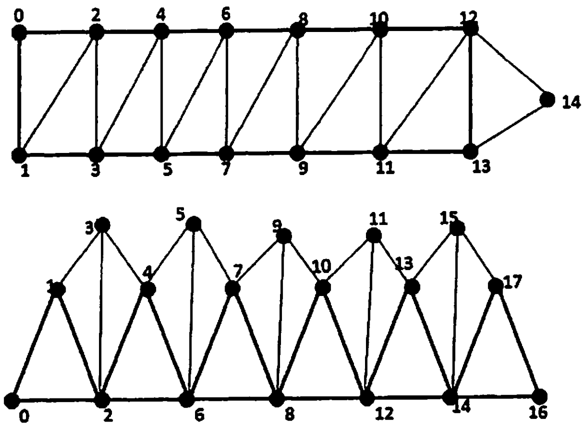

FIGURE 9

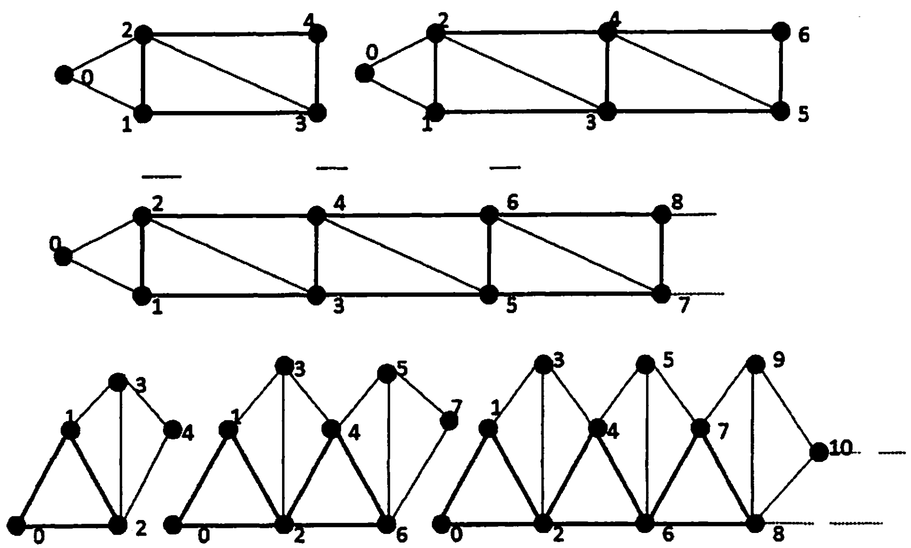

FIGURE 10

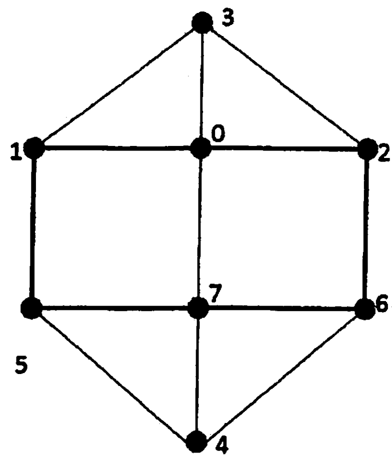

FIGURE 11

classes of strongly indexable finite nonregular maximal outerplanar (p, q)-graphs as illustrated in Figure 9. Also, there are such infinite graphs as is evident from the example shown in Figure 10. Further, there are planar nonregular graphs which are not outerplanar but strongly indexable as depicted in Figure 11.

Next, in view of Corollary 5.4, given an arbitrary integer $n \geq 2$, it might be of specific interest to find maximal strongly indexable graphs having chromatic number n and with least possible order (i.e., the number of vertices) $P(n)$ or with least possible size (i.e., the number of edges) $Q(n)$; such connected graphs might be useful for specific applications. Since such graphs must be finite in number, apart from their enumeration, their determination for specific values of n could be of further interest.

Lastly, if not least, in general it may now be seen that given a strongly indexed graph (G, f) one can look at various Sidon sequences generated by the cliques in G and hence seek to minimize the 'Sidon numbers' $s_f(r_Q)$ over the set of all strong indexers f of G, where Q varies in the set $\mathcal{K}_G$ of all cliques of G, in the sense that $\Sigma_{f\in\mathcal{I}_G}\Sigma_{Q\in\mathcal{K}_G}s_f(r_Q)$ is minimum, where $\mathcal{I}_G$ denotes the set of all strong indexers of G. This optimization problem, even if restricted to the class of maximal strongly indexable (planar or outerplanar) graphs, appears quite challenging.

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