Coloring the square of products of cycles and paths

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Abstract

The square G^2 of a graph G is a graph with the same vertex set as G in which two vertices are joined by an edge if their distance in G is at most two. For a graph G, $\chi(G^2)$, which is also known as the distance two coloring number of G is studied. We study coloring the square of grids $P_m \square P_n$, cylinders $P_m \square C_n$, and tori $C_m \square C_n$. For each m and n we determine $\chi((P_m \square P_n)^2)$, $\chi((P_m \square C_n)^2)$, and in some cases $\chi((C_m \square C_n)^2)$ while giving sharp bounds to the latter. We show that $\chi((C_m \square C_n)^2)$ is at most 8 except when $m = n = 3$, in which case the value is 9. Moreover, we conjecture that for every m ($m \geq 5$) and n ($n \geq 5$), we have, $5 \leq \chi((C_m \square C_n)^2) \leq 7$.

Key words: Distance coloring, square of graphs, grids, cylinders and tori.

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1 Introduction

We use standard terminology of graph theory. G is a simple graph. By $V(G)$ and $E(G)$ we denote the vertex set and the edge set of G, respectively. A (proper vertex) k-coloring of a graph G is a mapping $c : V(G) \to \{0, 1, \ldots, k-1\}$, with the property that $c(u) \neq c(v)$ whenever $\{u, v\} \in E(G)$. The smallest k for which there exists a k-coloring of G, is called the chromatic number of G and is denoted by $\chi(G)$. The square G^2 of a graph G is a graph with the same vertex set as G in which two vertices are joined by an edge if their distance in G is at most two. For a graph G, $\chi(G^2)$, which is also known as the distance two coloring number of G, is of great interest (see for example [6] and [7]). In [4] they consider the interaction between coding theory and distance k colorings of Hamming graphs $H(q,n) = \underbrace{K_n \Box K_n \Box \cdots \Box K_n}_{q}$ and find some bounds for $\chi(H^k(q,n))$. In [3], the d-dimensional infinite grid graph G_d is considered. They use a simple construction to show that $\chi((G_d)^2) = 2d + 1$ for all $d \geq 1$. They discuss an important application of this result in steganography.

Note that in a distance two coloring of G every vertex v of G is a rainbow, i.e. the set of all neighbors of v consists of distinct colors. Also we note that: $\Delta + 1 \leq \chi(G^2) \leq \Delta^2 + 1$, where Δ is the maximum degree of G.

Example 1 *For the path P_n and the cycle C_n we have,*

$$\chi(P_n^2) = \begin{cases} 1 & n = 1, \\ 2 & n = 2, \\ 3 & n \geq 3. \end{cases} \quad \chi(C_n^2) = \begin{cases} 3 & n = 3k, \\ 4 & n \neq 3k \text{ and } n \neq 5, \\ 5 & n = 5. \end{cases}$$

For the graphs G and H assume that $|V(G)| = m$ and $|V(H)| = n$. We refer to the vertices of $G \Box H$, the cartesian product of G and H, as an $m \times n$ array $[v_{ij}]$, where in each row we have a copy of H and in each column a copy of G.

Example 2 *The following is a 5-coloring of* $(C_5 \Box C_5)^2$.

$$\begin{array}{ccccc} 1 & 2 & 3 & 4 & 0 \\ 3 & 4 & 0 & 1 & 2 \\ 0 & 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 0 & 1 \\ 4 & 0 & 1 & 2 & 3 \end{array}$$

In 1977, G. Wegner [11] conjectured that:

Conjecture 1 *(Wegner 1977) Let G be a planar graph. Then*

$$\chi(G^2) \leq \begin{cases} \Delta(G)+5 & \text{if } 4 \leq \Delta(G) \leq 7, \\ \lfloor 3\Delta(G)/2+1 \rfloor & \text{if } \Delta(G) \geq 8. \end{cases}$$

This conjecture, in general, remains open. For a progress on this conjecture see ([10], [9], [7], [6], [2], [5]). In this paper we study coloring of the square of cartesian products of cycles and paths, namely grids $P_m \Box P_n$, cylinders $P_m \Box C_n$, and tori $C_m \Box C_n$. The obtained results are summarized in the following table.

Graph	G	$\chi(G^2)$	Reference
Grid	$P_m \Box P_n$	4 if $m = 2$	Lemma 1
	$2 \leq m \leq n$	5 if $m \geq 3$	Theorem 1
Cylinder	$P_2 \Box C_n$	6 if $n = 3$ or 6	Lemma 4
		4 if $4 \mid n$	Lemma 3
		5 otherwise	Theorem 2
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	$P_m \Box C_n$	5 iff $5 \mid n$	Corollary 1
	$m \geq 3$	6 if $5 \nmid n$	Theorem 3,4,5
Torus	$C_m \Box C_n$	9 if $m = n = 3$	Subsection 4.1
		5 iff $5 \mid m \wedge 5 \mid n$	Corollary 4
		≤ 6 if $6 \mid mn$	Theorem 7,8
		≤ 7 if $3 \mid m \wedge 2 \nmid n$	Theorem 8
		≤ 8 for any m and n, $(m,n) \neq (3,3)$	Theorem 9

In [1], $\chi((P_m \square C_n)^2)$ is determined. In Section 2 and 3 we introduce some colorings of $(P_m \square P_n)^2$ and $(P_m \square C_n)^2$, and use them in subsequent section.

2 $P_m \square P_n$

In this section we determine $\chi((P_m \square P_n)^2)$. The pattern of colorings given in this section will be applied to other cases in next sections too.

Lemma 1 *For any n ($n \geq 2$), we have:* $\chi((P_2 \square P_n)^2) = 4$.

Proof. We have $\chi((P_2 \square P_n)^2) \geq \Delta(P_2 \square P_n) + 1 = 4$. The following is a 4-coloring of $(P_2 \square P_n)^2$.

$$c : V((P_2 \square P_n)^2) \rightarrow \{0, 1, 2, 3\}$$
$$c(v_{ij}) = (j + 2(i-1)) \pmod 4.$$ ■

Theorem 1 *For every m ($m \geq 3$) and n ($n \geq 3$) we have:*

$$\chi((P_m \square P_n)^2) = 5.$$

Proof. We have $\chi((P_m \square P_n)^2) \geq \Delta(P_m \square P_n) + 1 = 5$. The following is a 5-coloring of $(P_m \square P_n)^2$.

$$c : V((P_m \square P_n)^2) \rightarrow \{0, 1, 2, 3, 4\}$$
$$c(v_{ij}) = (j + 2(i-1)) \pmod 5.$$ ■

3 $P_m \square C_n$

In this section we find $\chi((P_m \square C_n)^2)$ by starting with some special cases which will be used throughout the paper.

Lemma 2 $\chi((P_3 \square C_n)^2) = 5$ *if and only if* $n \equiv 0 \pmod 5$. *Furthermore for every l, there exists a unique 5-coloring of $(P_3 \square C_{5l})^2$ up to some permutation of colors.*

Proof. Let c be a partial 5-coloring of $(P_3\Box C_n)^2$, where v_{22} and its neighbors are colored as in the following.

$$\begin{array}{cccccc} \bullet & 2 & \bullet & \bullet & \cdots & \bullet \\ 3 & 4 & 0 & \bullet & \cdots & \bullet \\ \bullet & 1 & \bullet & \bullet & \cdots & \bullet \end{array}$$

We must have, $c(v_{11}) \in \{1,0\}$, $c(v_{13}) \in \{1,3\}$, $c(v_{31}) \in \{2,0\}$, $c(v_{33}) \in \{2,3\}$. We show, that each choice of color for v_{33} from its list may be extended uniquely to a color function c on $V((P_3\Box C_n)^2)$, using a set of 5 colors.

Suppose for example $c(v_{33}) = 2$.

$$\begin{array}{cccccc} \bullet & 2 & \bullet & \bullet & \cdots & \bullet \\ 3 & 4 & 0 & \bullet & \cdots & \bullet \\ \bullet & 1 & 2 & \bullet & \cdots & \bullet \end{array}$$

It forces $c(v_{31}) = 0$, then $c(v_{11}) = 1$ and $c(v_{13}) = 3$. Then the color v_{24} is forced to be 1, and the color of other vertices are determined by the following function.

$$c(v_{ij}) = (j + 2(i-1)) \pmod 5.$$

It is clear to see that c is a 5-coloring of $(P_3\Box C_n)^2$, if and only if $n \equiv 0 \pmod 5$. The case of $c(v_{33}) = 3$ is similar.

Also in the first coloring of $(P_3\Box C_n)^2$ by interchanging the colors 1 and 2, and then the colors of row 1 and row 3, we obtain the second coloring. So there exists a unique 5-coloring of $(P_3\Box C_{5l})^2$ up to some permutation of colors. ∎

Corollary 1 *For every pair m $(m \geq 3)$ and n, $\chi((P_m\Box C_n)^2) = 5$ if and only if $n \equiv 0 \pmod 5$, and there exists a unique 5-coloring of $(P_m\Box C_{5l})^2$ up to some permutation of colors.*

Proof. If n is a multiple of 5, Lemma 2 results to a 5-coloring c of $(P_3\Box C_n)^2$. Now, we extend this coloring to a 5-coloring c' of $(P_m\Box C_n)^2$ as in the following.

For $i \geq 4$ let $c'(v_{ij}) = (j + 2(i-1)) \pmod 5$.

Conversely, if $\chi((P_m \square C_n)^2) = 5$, then $5 \leq \chi((P_3 \square C_n)^2) \leq \chi((P_m \square C_n)^2) = 5$, so we have $\chi((P_3 \square C_n)^2) = 5$, and by Lemma 2, $n \equiv 0 \pmod 5$. ■

Corollary 2 *For every m and n ($m \geq 3$, $5 \nmid n$) we have,*

$$\chi((P_m \square C_n)^2) \geq 6.$$

3.1 $P_2 \square C_n$

We note that if G is an r-regular graph with $\chi(G^2) = r + 1$, then $r + 1$ divides $|V(G)|$. So we have,

Corollary 3 *If n is an odd number, then* $\chi((P_2 \square C_n)^2) \geq 5$.

Lemma 3 $\chi((P_2 \square C_n)^2) = 4 \iff n \equiv 0 \pmod 4$.

Proof. We have $\chi((P_2 \square C_n)^2) \geq \Delta(P_2 \square C_n) + 1 = 4$. Let c_0 be a 4-coloring of $(P_2 \square P_2)^2$. This coloring may be extended uniquely to a color function c on $V((P_2 \square C_n)^2)$ only using colors of c_0. It is clear to see that c is a 4-coloring of $(P_2 \square C_n)^2$ if and only if $n \equiv 0 \pmod 4$. The following is a 4-coloring of $(P_2 \square C_n)^2$.

$$c : V((P_2 \square C_n)^2) \to \{0, 1, 2, 3\}$$
$$c(v_{ij}) = (j + 2(i - 1)) \pmod 4.$$ ■

Lemma 4 $\chi((P_2 \square C_3)^2) = \chi((P_2 \square C_6)^2) = 6$.

Proof. We know $(P_2 \square C_3)^2 = K_6$, so $\chi((P_2 \square C_3)^2) = \chi(K_6) = 6$. Since $\alpha((P_2 \square C_6)^2) = 2$, we have $\chi((P_2 \square C_6)^2) \geq 6$, The following is a 6-coloring of $(P_2 \square C_6)^2$.

1	2	3	1	2	3
4	5	0	4	5	0

Figure 1: A 6-coloring of $(P_2 \square C_6)^2$ ■

Theorem 2 *If $4 \nmid n$ and $n \neq 3, 6$, then,* $\chi((P_2 \square C_n)^2) = 5$.

Proof. For this n, by Corollary 3 and Lemma 3, $\chi((P_2 \square C_n)^2) \geq 5$. We can obtain a 5-coloring of $(P_2 \square C_{4l+1})^2$, $(P_2 \square C_{4l+2})^2 (l > 1)$ and $(P_2 \square C_{4l+3})^2$ by combining the coloring given in Lemma 3 and the colorings shown in Figure 2. ■

1	2	0	3	4	1	2	3	4	0
3	4	1	2	0	3	4	0	1	2

1	2	3	4	0
3	4	0	1	2

1	2	3	4	1	3	0
3	4	1	2	0	4	2

Figure 2: Some 5-colorings of $(P_2\square C_{10})^2$,$(P_2\square C_5)^2$, and $(P_2\square C_7)^2$

3.2 $P_m\square C_{3l}$

Theorem 3 *For each m $(m \geq 3)$ and l, we have:*

$$5 \leq \chi((P_m\square C_{3l})^2) \leq 6.$$

Proof. Clearly $\chi((P_m\square C_{3l})^2) \geq 5$. The following is a 6-coloring of $(P_m\square C_{3l})^2$.

$$c : V(P_m\square C_{3l}) \rightarrow \{0,1,2,3,4,5\}$$

$$c(v_{ij}) = \begin{cases} (1+(i-1)) \pmod 6 & j \equiv 1 \pmod 3 \\ (3+(i-1)) \pmod 6 & j \equiv 2 \pmod 3 \\ (5+(i-1)) \pmod 6 & j \equiv 0 \pmod 3). \end{cases}$$ ■

3.3 $P_m\square C_{3l+1}$

Theorem 4 *For every m $(m \geq 3)$ and l, we have:*

$$5 \leq \chi((P_m\square C_{3l+1})^2) \leq 6.$$

Proof. The lower bound is clear. The following is a 6-coloring of $(P_m\square C_{3l+1})^2$.

$$c : V(P_m\square C_{3l+1}) \rightarrow \{0,1,2,3,4,5\}$$

$$c(v_{ij}) = \begin{cases} (1+(i-1)) \pmod 6 & j = 1 \\ (2+(i-1)) \pmod 6 & j \equiv 1 \pmod 3) j \neq 1, 3l+1 \\ (4+(i-1)) \pmod 6 & j \equiv 2 \pmod 3 \\ (i-1) \pmod 6 & j \equiv 0 \pmod 3 \\ (3+(i-1)) \pmod 6 & j = 3l+1. \end{cases}$$ ■

3.4 $P_m \square C_{3l+2}$

Theorem 5 *For each m $(m \geq 3)$ and l, we have:*

$$5 \leq \chi((P_m \square C_{3l+2})^2) \leq 6.$$

Proof. The lower bound is clear. The following is a 6-coloring of $(P_m \square C_{3l+2})^2$.

$$c : V(P_m \square C_{3l+2}) \rightarrow \{0, 1, 2, 3, 4, 5\}$$

$$c(v_{ij}) = \begin{cases} (1 + (i-1)) \pmod 6 & j = 1 \\ (2 + (i-1)) \pmod 6 & j \equiv 1 \pmod 3 j \neq 1 \\ (4 + (i-1)) \pmod 6 & j \equiv 2 \pmod 3 j \neq 3l + 2 \\ (i-1) \pmod 6 & j \equiv 0 \pmod 3 \\ (5 + (i-1)) \pmod 6 & j = 3l + 2. \end{cases}$$

■

4 $C_m \square C_n$

First we prove a direct construction theorem.

Theorem 6 *If there exists a k-coloring of $(C_m \square C_n)^2$, then for every l and l' there exists a k-coloring of $(C_{ml} \square C_{nl'})^2$.*

Proof. Let c be a k-coloring of $(C_m \square C_n)^2$, then the following is a k-coloring of $(C_{ml} \square C_{nl'})^2$.

$$l \left\{ \overbrace{\begin{matrix} c & c & \cdots & c \\ c & c & \cdots & c \\ \vdots & \vdots & \vdots & \vdots \\ c & c & \cdots & c \end{matrix}}^{l'} \right.$$

■

Now we prove a necessary and sufficient condition for $\chi((C_m \square C_n)^2)$ to be equal to 5.

1	2	5	7
3	4	6	0
6	5	1	2
0	7	3	4

Figure 3: An 8-coloring of $(C_4 \square C_4)^2$

Corollary 4 *For all m and n, $\chi((C_m \square C_n)^2) = 5$ if and only if both m and n are multiple of 5.*

Proof. If both m and n are multiple of 5, then by Example 2 and Theorem 6 we can generate a 5-coloring of $(C_m \square C_n)^2$. For the converse, let c be a 5-coloring of $(C_m \square C_n)^2$. Then c is also a 5-coloring of $(P_m \square C_n)^2$ and by Corollary 1, $n \equiv 0 \pmod 5$, and c is also a 5-coloring of $(C_m \square P_n)^2$, thus again by Corollary 1, $m \equiv 0 \pmod 5$. ■

G. Matthews informed us that they also have the result of Corollary 4 [8].

4.1 $\chi((C_n \square C_n)^2)$ for small n

In this subsection we find $\chi((C_n \square C_n)^2)$, for some small values of n.

- $\chi((C_3 \square C_3)^2) = 9$, for $(C_3 \square C_3)^2 = K_9$.
- $\chi((C_4 \square C_4)^2) = 8$.
  We have $\alpha((C_4 \square C_4)^2) = 2$, so $\chi((C_4 \square C_4)^2) \geq 8$. In Figure 3, we present an 8-coloring of $(C_4 \square C_4)^2$.
- $\chi((C_5 \square C_5)^2) = 5$. See Example 2.
- $\chi((C_6 \square C_6)^2) = 6$. By Corollary 4, $\chi((C_6 \square C_6)^2) \geq 6$. In Figure 4, we present a 6-coloring of $(C_6 \square C_6)^2$.
- $\chi((C_7 \square C_7)^2) = 7$. By Corollary 4, $\chi((C_7 \square C_7)^2) \geq 6$. Assume c is a 6- coloring of $(C_7 \square C_7)^2$, so there exists a color

1	2	3	4	5	0
3	4	5	0	1	2
5	0	1	2	3	4
1	2	3	4	5	0
3	4	5	0	1	2
5	0	1	2	3	4

Figure 4: A 6-coloring of $(C_6 \square C_6)^2$

class of size at least $\lceil \frac{49}{6} \rceil = 9$ in c. If in a row of $(C_7 \square C_7)^2$ two independent vertices are chosen in a class, then there can be at most one vertex from the previous or the next chosen row in that class. So in any color class of $(C_7 \square C_7)^2$ there are 4 consecutive rows with at most 4 independent vertices. But in the 3 remaining rows of $(C_7 \square C_7)^2$, we can not choose other 5 independent vertices. In Figure 5, we present a 7-coloring of $(C_7 \square C_7)^2$.

- $\chi((C_8 \square C_8)^2) = 7$. By Corollary 4, $\chi((C_8 \square C_8)^2) \geq 6$. Assume c is a 6-coloring of $(C_8 \square C_8)^2$, so there exists a color class of size at least $\lceil \frac{64}{6} \rceil = 11$ in c. Also in any color class of $(C_8 \square C_8)^2$ there are 4 consecutive rows with at most 6 independent vertices. But in the 4 remaining rows of $(C_8 \square C_8)^2$, we can not choose other 5 independent vertices. In Figure 5, we present a 7-coloring of $(C_8 \square C_8)^2$.

- $\chi((C_9 \square C_9)^2) = 7$. By Corollary 4, $\chi((C_9 \square C_9)^2) \geq 6$. Let c be a 6-coloring of $(C_9 \square C_9)^2$, so there exists a color class of size at least $\lceil \frac{81}{6} \rceil = 14$ in c. Also in any color class of $(C_9 \square C_9)^2$ there are 4 consecutive rows with at most 6 independent vertices. But in the 5 remaining rows of $(C_9 \square C_9)^2$, we can not choose 8 independent vertices. In Figure 5, we present a 7-coloring of $(C_9 \square C_9)^2$.

1	2	3	4	5	6	0
3	4	5	6	0	1	2
5	6	0	1	2	3	4
0	1	2	3	4	5	6
2	3	4	5	6	0	1
4	5	6	0	1	2	3
6	0	1	2	3	4	5

1	2	4	6	3	4	0	6
4	0	1	5	0	6	5	2
3	5	6	4	2	3	4	1
6	1	3	0	6	1	2	5
4	0	5	2	4	5	6	3
2	3	6	1	0	3	4	0
6	4	0	5	6	2	1	5
0	5	3	2	1	5	3	4

1	0	3	4	5	6	0	2	3
5	6	1	2	3	4	5	1	4
2	4	5	6	1	2	3	6	0
1	0	3	4	5	6	0	2	3
5	6	1	2	3	4	5	1	4
2	4	5	6	1	2	3	6	0
1	0	3	4	5	6	0	2	3
5	6	1	2	3	4	5	1	4
2	4	5	6	1	2	3	6	0

Figure 5: Some 7-colorings of $(C_7\Box C_7)^2$, $(C_8\Box C_8)^2$ and $(C_9\Box C_9)^2$

- $\chi((C_{10}\Box C_{10})^2) = 5$. By Corollary 4.
- $\chi((C_{12}\Box C_{12})^2) = 6$. By Corollary 4, $\chi((C_{12}\Box C_{12})^2) > 5$. Now it follows by $\chi((C_6\Box C_6)^2) = 6$ and Theorem 6.

4.2 Some bounds for $\chi((C_m\Box C_n)^2)$

In this subsection, we give more precise bounds for products of some particular cycles.

Theorem 7 *For all k and $n (n \geq 3)$, we have:*

$$5 \leq \chi((C_{6k}\Box C_n)^2) \leq 6.$$

Proof. The lower bound is trivial. For the upper bound, all of the colorings given in Subsections 3.2, 3.3 and 3.4 for $(P_m \Box C_{3l+i})^2$, $i = 0, 1, 2$, also works for $(C_{6k} \Box C_{3l+i})^2$ $i = 0, 1, 2$. ■

Theorem 8 *For all k and l, We have:*

1. $\chi((C_3 \Box C_5)^2) = 8$,
2. $5 \leq \chi((C_{3k} \Box C_{2l})^2) \leq 6$,
3. $\chi((C_3 \Box C_{2l+1})^2) > 6$,
4. $5 \leq \chi((C_{3k} \Box C_{2l+1})^2) \leq 7$, $(l \geq 3)$.

Proof.

1. We have $\alpha(C_3 \Box C_5)^2 = 2$, so $\chi((C_3 \Box C_5)^2) > 7$. The following is an 8-coloring of $(C_3 \Box C_5)^2$.

1	4	6	2	7
2	5	7	3	6
0	3	1	5	4

2. We consider two cases for l.

   - l is even, $l = 2l'$ for some l'. By Theorem 6, it is sufficient to find a 6-coloring of $(C_3 \Box C_4)^2$. The following is a 6-coloring of $(C_3 \Box C_4)^2$.

1	4	2	5
2	5	3	0
3	0	1	4

   - l is odd, $l = 2l' + 1$ for some l'. Let c be a 6-coloring of $(C_{3k} \Box C_6)^2$, obtained by combining the following 6-coloring of $(C_3 \Box C_6)^2$.

1	4	2	5	3	0
2	5	3	0	1	4
3	0	1	4	2	5

Also let c' be a 6-coloring of $(C_{3k}\Box C_{4(l'-1)})^2$ obtained in the previous case. The following is a 6-coloring of $(C_{3k}\Box C_{2l})^2$.

$c'((C_{3k}\Box C_{4(l'-1)})^2)$	$c((C_{3k}\Box C_6)^2)$

3. If there exists a 6-coloring of $(C_3\Box C_{2l+1})^2$, then there exists at least one color class of size at least $\lceil\frac{3(2l+1)}{6}\rceil = l+1$. We can choose at most 1 independent vertex from two consecutive columns of $(C_3\Box C_{2l+1})^2$. So the maximum size of each color class of $(C_3\Box C_{2l+1})^2$ is l. Therefore $\chi((C_3\Box C_{2l+1})^2) > 6$.

4. For $l = 3$ by Theorem 6, it is sufficient to consider the following 7-coloring of $(C_3\Box C_7)^2$.

1	2	3	4	5	6	0
3	4	5	6	0	1	2
5	6	0	1	2	3	4

For $l \geq 4$, let c' be a 6-coloring of $(C_{3k}\Box C_{2(l-2)})^2$ obtained in the second item of this theorem. The following is a 7-coloring of $(C_{3k}\Box C_{2l+1})^2$.

$c'((C_{3k}\Box C_{2(l-2)})^2)$	$c(C_{3k}\Box C_5)$

where, c is obtained by combining the following 7-coloring of $C_3\Box C_5$ as follows:

1	4	6	2	0
2	5	0	3	6
0	3	1	5	4

■

Based on the results above we make the following conjecture:

Conjecture 2 *For all m and n, we have:*

$$5 \leq \chi((C_m\Box C_n)^2) \leq 6 \iff 6|mn\,.$$

Finally in this section we find an upper bound for $\chi((C_m \square C_n)^2)$. For this purpose we use some 4-colorings of K_4. First we introduce 6 of them as follows.

$$\begin{matrix} 1 & 2 \\ 3 & 4 \\ & A \end{matrix} \quad \begin{matrix} 3 & 4 \\ 1 & 2 \\ & R \end{matrix} \quad \begin{matrix} 2 & 1 \\ 4 & 3 \\ & C \end{matrix} \quad \begin{matrix} 5 & 6 \\ 7 & 0 \\ & A' \end{matrix} \quad \begin{matrix} 7 & 0 \\ 5 & 6 \\ & R' \end{matrix} \quad \begin{matrix} 6 & 5 \\ 0 & 7 \\ & C' \end{matrix}$$

Theorem 9 *For all m and n, $(m,n) \neq (3,3)$, we have:*

$$5 \leq \chi((C_m \square C_n)^2) \leq 8.$$

Proof. The lower bound is trivial. We prove the upper bound with introducing an 8-coloring in each of the possible cases and subcases.

(1) $C_{4k} \square C_n$

- $C_{4k} \square C_{4l}$

  An 8-coloring of $(C_{4k} \square C_{4l})^2$ can be obtained by repetition of the following pattern: $\begin{matrix} A & A' \\ A' & A \end{matrix}$, i.e.:

$$\begin{matrix} A & A' & \cdots & A & A' \\ A' & A & \cdots & A' & A \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ A & A' & \cdots & A & A' \\ A' & A & \cdots & A' & A \end{matrix}$$

- $C_{4k} \square C_{4l+1}$

  An 8-coloring of $(C_{4k} \square C_{4l+1})^2$ can be obtained by repetition of the following pattern:

$A \quad A'$	$\cdots$	$A \quad A'$	4 2
$R' \quad R$	$\cdots$	$R' \quad R$	6 0

- $C_{4k}\square C_{4l+2}$
  An 8-coloring of $(C_{4m}\square C_{4n+2})^2$ can be obtained by repetition of the following pattern:

A	A'	$\cdots$	A	A'	R'
A'	A	$\cdots$	A'	A	R

- $C_{4k}\square C_{4l+3}$
  An 8-coloring of $(C_{4k}\square C_{4l+3})^2$ can be obtained by repetition of the following pattern:

A	A'	$\cdots$	A	A'	R	0 6
A'	A	$\cdots$	A'	A	R'	4 2

(2) $C_{4k+1}\square C_n$

- $C_{4k+1}\square C_{4l+1}$
  An 8-coloring of $(C_{4k+1}\square C_{4l+1})^2$ can be obtained by repeating the following pattern:

A	A'	$\cdots$	A	A'	A	A'	4 2
A'	A	$\cdots$	A	A'	A'	R	1 3

  and adding them to the top of the following:

A	A'	$\cdots$	A	A'	A	A'	4 2
6 5 2 7	C	$\cdots$	6 5 2 7	C	C'	3 4 6 5	1 3
5 3	1 7	$\cdots$	5 3	1 7	5 3	1 7	0

- $C_{4k+1}\square C_{4l+2}$
  An 8-coloring of $(C_{4k+1}\square C_{4l+2})^2$ can be obtained by repeating the following coloring:

$$\begin{array}{|cc|c|cc|cc|c} A & A' & \cdots & A & A' & A & A' & G \\ A' & C & \cdots & A' & C & A' & H & I \end{array}$$

and adding them to the top of the following:

$$\begin{array}{|c|c|c|c|c} A \quad A' & \cdots & A \quad A' & A \quad A' & G \\ A' \quad C & \cdots & A' \quad C & A' \quad M & N \\ 4\ 3\ 1\ 2 & \cdots & 4\ 3\ 1\ 2 & 4\ 3\ 1\ 2 & 0\ 5 \end{array}$$

where G, H, I, M and N are the following colorings:

$$\begin{array}{ccccc} \begin{matrix} 3 & 7 \\ 1 & 2 \end{matrix} & \begin{matrix} 3 & 5 \\ 1 & 2 \end{matrix} & \begin{matrix} 7 & 0 \\ 4 & 6 \end{matrix} & \begin{matrix} 3 & 5 \\ 4 & 7 \end{matrix} & \begin{matrix} 4 & 0 \\ 6 & 3 \end{matrix} \\ G & H & I & M & N \end{array}$$

- $C_{4k+1} \square C_{4l+3}$
  An 8-coloring of $(C_{4k+1} \square C_{4l+3})^2$ can be obtained by repeating the following:

$$\begin{array}{|cc|c|cc|cc|cc} A & A' & \cdots & A & A' & A & A' & G & \begin{matrix} 5 \\ 6 \end{matrix} \\ \hline A' & C & \cdots & A' & C & A' & H & I & \begin{matrix} 1 \\ 2 \end{matrix} \end{array}$$

and adding them to the top of the following:

$$\begin{array}{|c|c|c|c|cc} A \quad A' & \cdots & A \quad A' & A \quad A' & G & \begin{matrix} 5 \\ 6 \end{matrix} \\ \hline A' \quad C & \cdots & A' \quad C & A' \quad M & N & \begin{matrix} 1 \\ 2 \end{matrix} \\ \hline 4\ 3\ 1\ 2 & \cdots & 4\ 3\ 1\ 2 & 4\ 3\ 1\ 2 & 5\ 1 & 0 \end{array}$$

(3) $C_{4k+2} \square C_n$

- $C_{4k+2} \square C_{4l+2}$
  An 8-coloring of $(C_{4k+2} \square C_{4l+2})^2$ can be obtained by repeating the following:

$$\begin{array}{|cc|c|cc|c} A & A' & \cdots & A & A' & R' \\ A' & A & \cdots & A' & A & R \end{array}$$

and adding them to the top of the following:

A	A'	$\cdots$	A	A'	A	A'	R'
5 6 0 7	A	$\cdots$	5 6 0 7	A	5 6 0 7	A	R
C	C'	$\cdots$	C	C'	C	6 5 0 1	0 7 3 5

- $C_{4k+2}\square C_{4l+3}$
  An 8-coloring of $(C_{4k+2}\square C_{4l+3})^2$ can be obtained by repeating

A	A'	$\cdots$	A	A'	A	6 0
A'	A	$\cdots$	A'	A	A'	2 4

and adding them to the top of the following:

C'	C	$\cdots$	C'	C	C'	1 3

(4) $C_{4k+3}\square C_n$

- $C_{4k+3}\square C_{4l+3}$
  An 8-coloring of $(C_{4k+3}\square C_{4l+3})^2$ can be obtained by repeating

A	A'	$\cdots$	A	A'	A	6 7
A'	A	$\cdots$	A'	A	A'	1 3

and adding them to the top of the following:

A	A'	$\cdots$	A	A'	1 2 3 5	6 7
5 0	1 2	$\cdots$	5 0	1 2	7 0	4

■

Conjecture 3 *For every* m $(m \geq 5)$ *and* n $(n \geq 5)$, *we have:*

$$5 \leq \chi((C_m \square C_n)^2) \leq 7.$$

Acknolegments.

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