4-cordiality of some regular graphs and the complete 4-partite graph

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Abstract

Suppose G is a graph with vertex set $V(G)$ and edge set $E(G)$, and let A be an additive Abelian group. A vertex labeling $f : V(G) \to A$ induces an edge labeling $f^* : E(G) \to A$ defined by $f^*(xy) = f(x) + f(y)$. For $a \in A$, let $n_a(f)$ and $m_a(f)$ be the number of vertices v and edges e with $f(v) = a$ and $f^*(e) = a$, respectively. A graph G is A – cordial if there exists a vertex labeling f such that $|n_a(f) - n_b(f)| \leq 1$ and $|m_a(f) - m_b(f)| \leq 1$ for all $a, b \in A$. When $A = \mathbb{Z}_k$, we say that G is k – cordial instead of $\mathbb{Z}_k$ – cordial. In this paper we investigate certain regular graphs and ladder graphs that are 4 – cordial and we give a complete characterization of the 4 – cordiality of the complete 4-partite graph. An open problem about which complete multipartite graphs are not 4-cordial is given.

Keywords: Regular graph, ladder, complete 4-partite graph, k – cordial labeling.

Mathematics Subject Classifications (2000): 05C78

1. Introduction

All graphs considered in this paper are finite, simple and undirected. For most of the graph theory terminology and notation used, we follow [8] and especially of graph labeling [10].

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$, and let A be an additive Abelian group. A vertex labeling $f: V(G) \rightarrow A$ induces an edge labeling $f^*: E(G) \rightarrow A$, defined by $f^*(xy) = f(x) + f(y)$, for all edges $xy \in E(G)$. For $a \in A$, let $n_a(f) = \left|f^{-1}(a)\right|$ and $m_a(f) = \left|f^{*-1}(a)\right|$. A labeling f of a graph G is said to be A-cordial labeling if $\left|n_a(f) - n_b(f)\right| \leq 1$ and $\left|m_a(f) - m_b(f)\right| \leq 1$ for all $a,b \in A$. A graph G is called A-*cordial* if it admits an A-cordial labeling. When $A = \mathbb{Z}_k$, we use the k-cordial labeling in stead of $\mathbb{Z}_k$-cordial labeling, and we say that G is k-cordial if G is $\mathbb{Z}_k$-cordial.

The notion of A-cordial labeling was first introduced by Hovey [14] who introduced a simultaneous generalization of harmonious [12] and cordial [3] labelings. Hovey [14] investigated many families of graphs that are k-cordial and gave the following conjectures: all trees are k-cordial for all k; all connected graphs are 3-cordial; and C_{2mk+j} is k-cordial if and only if $j \neq k$, where k and j are even and $0 \leq j < 2k$. The last conjecture was verified by Tao [16]. Youssef [17] proved some necessary conditions for a graph to be k-cordial and gave some new families of k-cordial graphs. See [1-2, 4-7, 9, 11, and 15] for other related topics. Gallian [10] surveys the current state of knowledge for k-cordial labeling and many other labelings.

We mention some known results which are referred to throughout the paper. The following theorem gives a necessary condition to certain regular graphs to be k-cordial when k is even. We use it many times throughout the paper.

Theorem 1 Let k be even and G be a d-regular (p,q) graph with $p,q \equiv 0 \pmod{k}$. If G is k-cordial, then $q \equiv 0 \pmod{2k}$.[17]

Lemma 1 If G is a (p,q) k-cordial graph with $p \equiv 0 \pmod{k}$, then $G+\bar{K}_n$ is k-cordial for all positive integer n.[17]

Lemma 2 If G is k-cordial, then $G+\bar{K}_n$ is k-cordial for every $n \equiv 0 \pmod{k}$.[18]

The following lemma shows that adding number of isolated vertices congruent to $0 \pmod{k}$ to one partition (then to many partitions) of complete multipartite k-cordial graph produce another complete multipartite k-cordial graph.

Lemma 3 If $K_{m_1,m_2,\cdots,m_r}$ is k-cordial so is $K_{m_1+t,m_2,\cdots,m_r}$ for all $t \equiv 0 \pmod{k}$.[18]

Theorem 2 K_n is 4-cordial if and only if $n \le 6$. [17]

Theorem 3 $K_{m,n,p}$ is 4-cordial if and only if $(m,n,p) \pmod 4 \not\equiv (0,2,2), (2,2,2)$, where $m \pmod 4 \le n \pmod 4 \le p \pmod 4$. [18]

In the next section of this paper we investigate certain regular graphs that are 4-cordial and in section 3, we determine exactly which of various kinds of ladders are 4-cordial. In section 4, we give a complete characterization of the 4-cordiality of the complete 4-partite graphs.

2. 4-cordiality of graphs related to cycles

In this section, we determine the 4-cordiality of the graphs mC_{4n} and C_n^3. The proofs of the following proposition and its corollary are immediate and we omit both of them.

Proposition 2.1 If G and H are (p_1,q_1) and (p_2,q_2) k-cordial graphs such that (p_1 or $p_2 \equiv 0 \pmod{k}$) and (q_1 or $q_2 \equiv 0 \pmod{k}$), then $G \cup H$ is k-cordial.

Corollary 2.2 If G is a (p,q) k-cordial graph such that p and $q \equiv 0 \pmod{k}$, then mG is k-cordial for every positive integer m.

Theorem 1 above gives the necessary condition for the 2-regular graph mC_{nk} to be k-cordial when k is even which is m or n is even. As a special case we determine which of mC_{4n} is 4-cordial.

Theorem 2.3 mC_{4n} is 4-cordial if and only if m or n is even

Proof. Necessity follows from Theorem 1. For sufficiency, if n is even, then C_{4n} is 4-cordial by Hovey [14] and hence mC_{4n} is 4-cordial for all positive integer m by Corollary 2.2. If n is odd, then the following 4-cordial labeling of $2C_{4n}$ and Corollary 2.2 completes the proof: We label the vertices of the first cycle successively by the labels $(0012)\prod_{i=1}^{\frac{n-1}{2}}(00112233)$ and $(1233)\prod_{i=1}^{\frac{n-1}{2}}(00112233)$ for the second cycle. □

Youssef [17] showed that C_n^2 is 4-cordial if and only if $n \not\equiv 2 \pmod 4$. However, we show the following.

Theorem 2.5 C_n^3 is 4-cordial if and only if $n \neq 8t+4, t \geq 1$ and $n \neq 7$

Proof. Necessity, if $n = 8t+4$, $t \geq 1$, then C_n^3 is not 4-cordial by Theorem 1. If $n = 7, C_7^3 = K_7$ which is not 4-cordial by Theorem 2. For sufficiency, if $3 \leq n \leq 6$, then $C_n^3 = K_n$, which is 4-cordial by Theorem 2. Now, let $n = 8t+s$, $t \geq 1$ and $0 \leq s \leq 7$, $s \neq 4$ and $V(C_n^3) = \{v_1, v_2, \ldots, v_n\}$. If $s = 0$, we label the vertices of the cycle by the successive labels:

$\prod_{i=1}^{t}(01231230)$. If $s=1$, we label the vertices of the cycle by the successive labels: $(\prod_{i=1}^{t}(01231230))(0)$. If $s=2$, we label the vertices of the cycle by the successive labels: $(\prod_{i=1}^{t}(01231230))(02)$. If $s=3$, we label the vertices of the cycle by the successive labels: $(\prod_{i=1}^{t}(01231230))(012)$. If $s=5$, we label the vertices of the cycle by the successive labels: $(\prod_{i=1}^{t}(01231230))\,(01230)$. If $s=6$, we label the vertices of the cycle by the successive labels: $(\prod_{i=1}^{t}(01231230))\,(012302)$. If $s=7$, we label the vertices of the cycle by the successive labels: $(\prod_{i=1}^{t}(01231230))(0221331)$. We show that the given labeling works in the first case when $n=8t$ and the reader may verify that in each case the labeling works: Let f be the described vertex labeling above and for $i\in\mathbb{Z}_4$, let

$$X_i=\left|\left\{v_jv_k:\quad j-k\equiv\pm1(\text{mod}\,n),\ f^*(v_jv_k)=i\right\}\right|,$$

$$Y_i=\left|\left\{v_jv_k:\quad j-k\equiv\pm2(\text{mod}\,n),\ f^*(v_jv_k)=i\right\}\right| \quad \text{and}$$

$$Z_i=\left|\left\{v_jv_k:\quad j-k\equiv\pm3(\text{mod}\,n),\ f^*(v_jv_k)=i\right\}\right|$$

. Clearly that $m_i=X_i+Y_i+Z_i$. Now, since $X_0=2t$, $X_1=X_3=3t$, $X_2=0$; $Y_i=2t$ for all $i\in\mathbb{Z}_4$ and $Z_0=2t$, $Z_1=Z_3=t$,

$Z_2 = 4t$, then $m_i = 6t$ for all $i \in \mathbb{Z}_4$ and f is a 4-cordial labeling of C_n^3. □

3. 4-cordiality of ladders

The ladder L_n $(n \geq 2)$ is defined to be the graph $P_n \times P_2$, while the Möbius ladder M_n, $(n \geq 3)$ is the graph obtained from the ladder L_n by joining the opposite vertices of the two copies of P_n. A circular ladder graph is a graph of the form $C_n \times P_2$, $(n \geq 3)$ and is denoted by CL_n, sometimes also called a prism graph . See Gross and Yellen [13]. A circular ladder graph CL_n may be obtained from the ladder L_n by joining the end vertices of each copy of P_n.

Let $V(L_n) = \{u_i, v_i: 1 \leq i \leq n\}$ and
$E(L_n) = \{u_i u_{i+1}, v_i v_{i+1}: 1 \leq i \leq n-1\} \cup \{u_i v_i : 1 \leq i \leq n\}$.
Note that $E(M_n) = E(L_n) \cup \{u_1 v_n, u_n v_1\}$, while $E(CL_n) = E(L_n) \cup \{u_1 u_n, v_1 v_n\}$.The following proposition shows that if we add 4 steps to a 4-cordial ladder L_n yields again a 4-cordial ladder L_{n+4}.

Proposition 3.1 If the ladder L_n has a 4-cordial labeling f such that $(f(u_n), f(v_n)) = (1,2)$ or $(2,3)$, then L_{n+4} is 4-cordial.

Proof. If $(f(u_n), f(v_n)) = (1,2)$, define a labeling $g: V(L_{n+4}) \to \mathbb{Z}_4$ as $g(V(L_n)) = f(V(L_n))$ and label the 4 vertices u_i, $i = n+1, n+2, n+3, n+4$ consecutively by the labels: 0,3,0, 2 and label the 4 vertices v_i, $i = n+1, n+2,$

$n+3, n+4$ consecutively by the labels:2,1,1,3. Then $n_i(g) = n_i(f) + 2$, $m_i(g) = m_i(f) + 3$ for all $i \in \mathbb{Z}_4$ and clearly g is a 4-cordial labeling of L_{n+4}.

If $(f(u_n), f(v_n)) = (2,3)$ then as above, define a labeling $g: V(L_{n+4}) \to \mathbb{Z}_4$ as $g(V(L_n)) = f(V(L_n))$ and label the 4 vertices u_i, $i = n+1, n+2, n+3, n+4$ consecutively by the labels: 0,3, 0,1 and label the 4 vertices v_i, $i = n+1, n+2, n+3, n+4$ consecutively by the labels: 1,3,2,2 . Again, g is a 4-cordial labeling of L_{n+4}.□

As an immediate consequence we have the following corollary.

Corollary 3.2 If each of the Möbius ladder M_n and the circular ladder CL_n has a 4-cordial labeling f such that $(f(u_n), f(v_n)) = (1,2)$, then so are M_{n+8} and CL_{n+8}.

Theorem 3.3 L_n is 4-cordial for all $n \geq 3$.

Proof. If $n = 2$, then $L_2 = C_4$ is not 4-cordial by Theorem 1. For $3 \leq n \leq 6$, we will give a 4-cordial labeling f such that $(f(u_n), f(v_n)) = (1,2)$ or $(2,3)$, then applying Proposition 3.1 completes the proof. We label the vertices of L_3 as follows: 0, 0, 2 for $f(u_i)$ and 1, 3, 3 for $f(v_i)$ respectively, $i = 1,2,3$. We label the vertices of L_4 as follows: 3,2,0,2 for $f(u_i)$ and 1,1,0,3 for $f(v_i)$ respectively, $i = 1,2,3,4$. We label the vertices of L_5 as follows: 3,2,0,1,1 for $f(u_i)$ and 1,3,0,2,2 for $f(v_i)$ respectively, $i = 1,2,3,4,5$. We label the vertices of L_6 as follows: 1,3,3,0,2,2 for $f(u_i)$ and 0,0,2,1,1,3 for $f(v_i)$ respectively, $i = 1,2,3,4,5,6$.□

Theorem 3.4 M_n is 4-cordial if and only if $n \not\equiv 4 \pmod 8$.

Proof. Necessity follows from Theorem 1. For sufficiency, we will give a 4-cordial labeling f of $M_n, 3 \leq n \leq 10, n \neq 4$, with $(f(u_n), f(v_n)) = (1,2)$, then applying Corollary 3.2 completes the proof. We label the vertices of M_n, successively as pattern: $(f(u_1), f(u_2), \ldots, f(u_n))$, $(f(v_1), f(v_2), \ldots, f(v_n))$

$n=3$: (3,0,1), (1,0,2)..

$n=5$: (3,2,0,1,1), (1,3,0,2,2).

$n=6$: (3,2,0,3,3,1), (1,0,0,2,1,2).

$n=7$: (3,0,2,0,3,0,1), (1,0,3,1,3,2,2).

$n=8$: (3,1,2,2,0,1,0,1), (1,3,3,0,3,2,0,2).

$n=9$: (3,0,1,3,1,3,2,1,1), (1,0,2,0,2,2,0,3,2).

$n=10$: (3,0,1,3,1,3,1,0,2,1), (1,0,2,0,2,2,0,3,3,2). □

In a similar argument to Theorem 3.4 we proof the following.

Theorem 3.5 CL_n is 4-cordial if and only if $n \not\equiv 4 \pmod 8$.

Proof. Necessity follows from Theorem 1. For sufficiency, we will give a 4-cordial labeling f of CL_n, $3 \le n \le 10$, $n \ne 4$, with $(f(u_n), f(v_n)) = (1,2)$, then applying Corollary 3.2 completes the proof. We label the vertices of CL_n, successively as pattern: $(f(u_1), f(u_2), \ldots, f(u_n))$, $(f(v_1), f(v_2), \ldots, f(v_n))$

$n=3$: (0,3,1), (0,3,2).

$n=5$: (2,0,1,3,1), (0,1,3,0,2).

$n=6$: (2,2,0,0,1,1), (0,3,1,3,3,2).

$n=7$: (0,0,3,3,2,1,1), (2,1,1,0,0,3,2).

$n=8$: (2,2,3,3,,0,0,1,1), (2,3,3,0,0,1,1,2).

$n=9$: (1,0,0,0,3,3,2,2,1), (1,1,1,0,0,3,3,2,2).

$n=10$: (1,2,2,3,3,0,0,1,0,1), (3,3,0,0,1,2,3,1,2,2). □

Another variation of a ladder graph is specified as follows. A ladder $\mathbb{L}_n, n \geq 2$ is a graph obtained by completing the ladder L_n by edges $u_i v_{i+1}$ for $1 \leq i \leq n-1$. See [2]. The following lemma shows that if the ladder $\mathbb{L}_n$ has a 4-cordial labeling then so is $\mathbb{L}_{n+2}$ under a restriction on the 4-cordial labeling of $\mathbb{L}_n$.

Lemma 3.6 If $\mathbb{L}_n$ has a 4-cordial labeling f such that $(f(u_n), f(v_n)) = (1,2)$, then so is $\mathbb{L}_{n+2}$.

Proof. Define a labeling $g: V(\mathbb{L}_{n+2}) \rightarrow \mathbb{Z}_4$ as $g(V(\mathbb{L}_n)) = f(V(\mathbb{L}_n))$ and label the 2 vertices u_i, $i = n+1, n+2$ consecutively by the labels: 3,1 and label the 2 vertices v_i, $i = n+1, n+2$ consecutively by the labels: 0,2. Then $n_i(g) = n_i(f) + 1$, $m_i(g) = m_i(f) + 2$ for all $i \in \mathbb{Z}_4$ and clearly g is a 4-cordial labeling of $\mathbb{L}_{n+2}$. □

Theorem 3.7 $\mathbb{L}_n$ is 4-cordial for all $n \geq 2$.

Proof. We will give a 4-cordial labeling f of $\mathbb{L}_n, n = 2,3$, such that $(f(u_n), f(v_n)) = (1,2)$, then applying Lemma 3.6 completes the proof. We label the vertices of $\mathbb{L}_2$ as follows: 3,1 for $f(u_i)$ and 0,2 for $f(v_i)$ respectively, $i = 1,2$. We label the vertices of $\mathbb{L}_3$ as follows: 1,3,1 for $f(u_i)$ and 2,0,2 for $f(v_i)$ respectively, $i = 1,2,3$. □

4. 4-cordiality of complete 4-partite graph

Hovey [14] showed that the complete bipartite graph $K_{2,k-2}$ is not k-cordial for all $k \geq 4$. However we give a slight generalization to this result by showing that the graph obtained by appending any number of edges from the two vertices of degree $n \geq 2$ in $K_{2,n}$ is not k-cordial where k is equal to the order of the graph. We give a similar proof to the one due to Hovey.

Theorem 4.1 The graph obtained by appending m edges from the one vertex of degree $n \geq 2$ in $K_{2,n}$ and appending t edges from the other vertex of degree n is not p-cordial, where p is the order of the graph.

Proof. Suppose that the graph is p-cordial and let the two vertices of degree n of $K_{2,n}$ be u and v. As any vertex in a k-cordial labeling can be assigned the label 0, we can assume u labeled 0 and suppose v is labeled i. Then there is no way to get i as an edge label, a contradiction. □

Let $K_{m,n,p,r}$ be the complete 4-partite graph where u_i's (resp. v_j's, resp. w_k's, resp. x_t's) are the m-independent (resp. n-independent, resp. p-independent, resp. r-independent) vertices of $K_{m,n,p,r}$.

Youssef [17] showed that the complete bipartite graph $K_{m,n}$ is 4-cordial if and only if m or $n \not\equiv 2 \pmod 4$, while Youssef and Al-Kuleab [18] gave a complete characterization of the complete tripartite graphs which are 4-cordial. However, the following theorem extends these results to complete 4-partite graphs. Using the symmetry of the complete 4-partite graph $K_{m,n,p,r}$, we may assume that $m \pmod 4 \leq n \pmod 4 \leq p \pmod 4 \leq r \pmod 4$.

Theorem 4.2 $K_{m,n,p,r}$ is 4-cordial if and only if $(m,n,p,r) \pmod 4 \not\equiv (0,0,2,2)$.

Proof. Necessity, if $m,n \equiv 0(\mathrm{mod}\,4)$ and $p,r \equiv 2(\mathrm{mod}\,4)$, let $q = |E(K_{m,n,p,r})|$ and suppose that $K_{m,n,p,r}$ is 4-cordial with labeling f,

then $\sum_{e\in E(K_{m,n,p,r})} f^*(e) \equiv (\sum_{v\in V(K_{m,n,p,r})} \deg(v)f(v))(\mathrm{mod}\,4)$

$$\Rightarrow \frac{6q}{4} \equiv (n+p+r)\sum_{i=1}^{m} f(u_i) + (m+p+r)\sum_{j=1}^{n} f(v_j) +$$

$$(m+n+r)\sum_{k=1}^{p} f(w_k) + (m+n+p)\sum_{t=1}^{r} f(x_t)\ (\mathrm{mod}\,4)$$

Since $(n+p+r)$ and $(m+p+r) \equiv 0(\mathrm{mod}\,4)$,

then, $\frac{3q}{2} \equiv (m+n+r)\sum_{k=1}^{p} f(w_k) + (m+n+p)\sum_{t=1}^{r} f(x_t)\ (\mathrm{mod}\,4)$,

$$\frac{3q}{2} \equiv (m+n+r)\left[\frac{3(m+n+p+r)}{2} - \sum_{i=1}^{m} f(u_i) - \sum_{j=1}^{n} f(v_j) - \sum_{t=1}^{r} f(x_t)\right]$$

$$+(m+n+p)\sum_{t=1}^{r} f(x_t)\ (\mathrm{mod}\,4),$$

$$\frac{3q}{2} \equiv (m+n+r)\frac{3(m+n+p+r)}{2} - (m+n+r)[\sum_{i=1}^{m} f(u_i) + \sum_{j=1}^{n} f(v_j)]$$

$$+(p-r)\sum_{t=1}^{r} f(x_t)\ (\mathrm{mod}\,4)$$

And since, $(m+n+r)\frac{3(m+n+p+r)}{2} \equiv 0\,(\mathrm{mod}\,4)$ and $(p-r) \equiv 0(\mathrm{mod}\,4)$

$$\Rightarrow \frac{3q}{2} \equiv -(m+n+r)[\sum_{i=1}^{m} f(u_i) + \sum_{j=1}^{n} f(v_j)](\mathrm{mod}\,4)$$

And as, $\frac{3q}{2} \equiv 2(\mathrm{mod}\,4)$ and $(n+m+r) \equiv 2(\mathrm{mod}\,4)$

Then, $\sum_{i=1}^{m} f(u_i) + \sum_{j=1}^{n} f(v_j) \equiv 1 (\text{mod} 2)$

and since,

$$\frac{3(m+n+p+r)}{2} = \sum_{i=1}^{m} f(u_i) + \sum_{j=1}^{n} f(v_j) + \sum_{k=1}^{p} f(w_k) + \sum_{t=1}^{r} f(x_t)$$

and $\frac{3(m+n+p+r)}{2} \equiv 0 (\text{mod} 2)$,

then, $\sum_{k=1}^{p} f(w_k) + \sum_{t=1}^{r} f(x_t) \equiv 1 (\text{mod} 2)$.

Without any loss of generality we may assume that $\sum_{i=1}^{m} f(u_i) \equiv 1 (\text{mod} 2)$ and $\sum_{k=1}^{p} f(w_k) \equiv 1 (\text{mod} 2)$.

Now, let O_1 (resp. O_2,resp. O_3,resp. O_4) be the number of vertices whose its label is odd in the m-set (resp. n–set, resp. p–set, resp. r–set) and E_1(resp., E_2,resp., E_3,resp. E_4) be the number of vertices whose its label is even in the m-set (resp. n–set, resp. p–set, resp. r–set).

Since $\sum_{i=1}^{m} f(u_i) \equiv 1 (\text{mod} 2)$, then O_1 is odd and hence E_1 is odd too. Also, as $\sum_{k=1}^{p} f(w_k) \equiv 1 (\text{mod} 2)$, then O_3 is odd and hence E_3 is odd too. So, we can deduce that each of O_2 ,E_2 , O_4andE_4 is even.

We calculate $m_0(f) + m_2(f)$ and $m_1(f) + m_3(f)$:

$$m_0(f) + m_2(f) = O_1(O_2 + O_3 + O_4) + O_2(O_3 + O_4) + E_1(E_2 + E_3 + E_4) + \\ + E_2(E_3 + E_4) + O_3O_4 + E_3E_4$$

$$m_1(f) + m_3(f) = O_1(E_2 + E_3 + E_4) + O_2(E_3 + E_4) + E_1(O_2 + O_3 + O_4) + \\ + E_2(O_3 + O_4) + O_3E_4 + E_3O_4$$

Put $x_1 = O_2 + O_3 + O_4, y_1 = E_2 + E_3 + E_4, x_2 = O_3 + O_4$ and $y_2 = E_3 + E_4$

Subtracting $m_0(f) + m_2(f)$ from $m_1(f) + m_3(f)$, we get:

$$m_1(f) + m_3(f) - (m_0(f) + m_2(f)) = (O_1 - E_1)(y_1 - x_1) +$$

$$+(O_2 - E_2)(y_2 - x_2) + (O_3 - E_3)(E_4 - O_4)$$

As $x_1 + y_1 = n + p + r \equiv 0 \pmod 4$ and since both of x_1 and y_1 are odd, then $y_1 - x_1 \equiv \pm 2 \pmod 8$. Similarly $x_2 + y_2 = p + r \equiv 0 \pmod 4$ and since both of x_2 and y_2 are odd, then $y_2 - x_2 \equiv \pm 2 \pmod 8$ and as:

$O_1 + E_1 = m \equiv 0 \pmod 4$ and both of O_1 and E_1 are odd, then $O_1 - E_1 \equiv \pm 2 \pmod 8$

$O_2 + E_2 = n \equiv 0 \pmod 4$ and both of O_2 and E_2 are even, then $O_2 - E_2 \equiv 0 \pmod 4$

$O_3 + E_3 = p \equiv 2 \pmod 4$ and both of O_3 and E_3 are odd, then $O_3 - E_3 \equiv 0 \pmod 4$

$O_4 + E_4 = r \equiv 2 \pmod 4$ and both of O_4 and E_4 are even, then $O_4 - E_4 \equiv \pm 2 \pmod 8$.

Hence $m_1(f) + m_3(f) - (m_0(f) + m_2(f)) \equiv 4 \pmod 8$, which is a contradiction, since $m_i = \frac{q}{4}$ for every $i \in \mathbb{Z}_4$.

Sufficiency, we have 34 cases to consider. If one of m, n, p or $r \equiv 0 \pmod 4$, say m, and the others satisfy that $(n,p,r) \not\equiv (0,2,2)$ and $(2,2,2)$, then $K_{n,p,r}$ is 4-cordial by Theorem 3 and hence $K_{m,n,p,r} = \overline{K}_m + K_{n,p,r}$ is 4-cordial by Lemma 2. This cover 18 cases. If $(n,p,r) \equiv (1,1,2), (2,3,3)$, then $K_{n,p,r}$ is 4-cordial by Theorem 3 and hence $K_{m,n,p,r} = \overline{K}_m + K_{n,p,r}$ is 4-cordial by Lemma 1. This cover the

cases $(m,n,p,r) \equiv (1,1,1,2), (1,2,3,3)$ and $(2,2,3,3)$. If $(m,n,p) \equiv (1,1,2), (2,3,3)$, then $K_{m,n,p}$ is 4-cordial by Theorem 3 and $K_{m,n,p,r} = K_{m,n,p} + \overline{K}_r$ is 4-cordial by Lemma 1. This cover the cases $(1,1,2,2), (1,1,2,3)$ and $(2,3,3,3)$. Finally, as $K_{4,2,2,2}$ is 4-cordial by the following pattern: $(0,0,1,2;1,2;0,3;2,3)$, where the labels of each independent vertex set is separated by a semicolon ,then $K_{m,n,p,r}$, $(m,n,p,r) \equiv (0,2,2,2)$, is 4-cordial by Lemma 3 . Similarly, we label the remaining other 9 cases in same above pattern such that the graph is beside its labeling and again Lemma 3 completes the proof: $K_{1,1,1,1}$ $(0;1;2;3)$, $K_{1,1,1,3}$ $(0;1;2;0,2,3)$, $K_{1,1,3,3}$ $(0;1;0,2,3;0,1,2)$, $K_{1,2,2,2}$ $(3;0,2;2,1;0,1)$, $K_{1,2,2,3}$ $(2;0,1;0,3;1,2,3)$, $K_{1,3,3,3}$ $(2;0,1,2;0,1,3;0,2,3)$, $K_{2,2,2,2}$ $(0,1;0,3;2,3;1,2)$,
$K_{2,2,2,3}$ $(0,1;0,3;2,3;0,1,2)$, $K_{3,3,3,3}$ $(0,1,2;0,1,3;0,2,3;1,2,3)$.
□

The above theorem along with the 4-cordiality of complete bipartite graph in [17] and of complete tripartite graph in [18] raises the question of which complete multipartite graphs are not 4-cordial. Therefore we propose the following open problem.

Open Problem Which of complete multipartite graphs are not 4-cordial.

References

[1] **B. D. Acharya and S. M. Hegde**, Arithmetic graphs, *J. Graph Theory*, **14** (1990) 275-299.

[2] **M. Baca, B. Baskoro, M. Miller, J. Ryan, R. Simanjuntack and K. Sugeng**, Survey of edge antimagic labelings of graphs, *J. Indonesian Math. Soc.*, **12** (2006) 113-130.

[3] **I. Cahit**, Cordial graphs : a weaker version of graceful and harmonious graphs, *Ars Combin.*, **23** (1987) 201-207.

[4] **I. Cahit**, On cordial and 3-equitable labelings of graphs, *Utilitas Math.*,**37** (1990) 189-198.

[5] **I. Cahit**, H-cordial graphs, *Bull. Inst. Combin. Appl.*, **18** (1996) 87-101.

[6] **I. Cahit and R. Yilmaz,** E_k -cordial graphs, *Ars Combin.* **54** (2000)119-127.

[7] **G. J. Chang, D. F. Hsu and D. G. Rogers**, Additive variations on a graceful theme: some results on harmonious and other related graphs, *Congress. Numer.*, **32** ((1981) 181-197.

[8] **G. Chartrand and L. Lesniak-Foster**, Graphs and Digraphs (3rd Edition) CRC Press, 1996.

[9] **H. Enomoto, A. S. Llado, T. Nakaamigawa and G. Ringel**, Super edge-magic graphs, *SUT J. Math.* **34** (1998)105-109.

[10] **J. A. Gallian**, A dynamic survey of graph labeling, *The Electronic J. of Combin.***16** (2009), # DS6, 1-219.

[11] **T. Grace**, On sequential labelings of graphs, *J. Graph Theory* **7** (1983) 195-201.

[12] **R. L. Graham and N. J. A. Sloane**, On additive bases and harmonious graphs, *SIAM J. Alg. Discrete Method*, **1** (1980) 382-404.

[13] **J. T. Gross and J. Yellen**, Graph Theory and Its Applications. Boca Raton, FL: CRC Press, 1999.

[14] **M. Hovey**, A-cordial graphs, *Discrete Math.*, **93** (1991) 183-194.

[15] **S. M. Lee, E. Schmeichel, and S. C. Shee**, On felicitous graphs, *Discrete Math.*, **93** (1991) 201-209.

[16] **R. Tao**, On k-cordiality of cycles, crowns and wheels, *System Sci. Math. Sci.* **11** (1998) 227-229.

[17] **M. Z. Youssef**, On k-cordial labeling, *Australas. J. Combin.* **43** (2009) 31-37.

[18] **M. Z. Youssef and N. A. AL-Kuleab**, Further results on k-cordial labeling, preprint.

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