Super Edge-Graceful Labelings of Complete Tripartite Graphs

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Abstract

Let $[n]^*$ denote the set of integers $\{-\frac{n-1}{2}, \ldots, \frac{n+1}{2}\}$ if n is odd, and $\{-\frac{n}{2}, \ldots, \frac{n}{2}\} \setminus \{0\}$ if n is even. A super edge-graceful labeling f of a graph G of order p and size q is a bijection $f : E(G) \to [q]^*$, such that the induced vertex labeling f^* given by $f^*(u) = \sum_{uv \in E(G)} f(uv)$ is a bijection $f^* : V(G) \to [p]^*$. A graph is super edge-graceful if it has a super edge-graceful labeling. We prove that all complete tripartite graphs $K_{a,b,c}$, except $K_{1,1,2}$, are super edge-graceful.

Keywords: labeling in graphs; edge labeling; super edge-graceful labeling

1 Introduction

In this paper we consider only simple, finite, undirected graphs. We define the set of integers $[n]^*$ to be $\{-\frac{n-1}{2}, \ldots, \frac{n-1}{2}\}$ if n is odd, and $\{-\frac{n}{2}, \ldots, \frac{n}{2}\} \setminus \{0\}$ if n is even. Notice that the cardinality of $[n]^*$ is n, and $[n]^*$ contains 0 if and only if n is odd. A graph of order p and size q is said to be super edge-graceful (SEG) if there is a bijection $f : E(G) \to [q]^*$, such that the induced vertex labeling f^* given by $f^*(u) = \sum_{uv \in E(G)} f(uv)$ is a bijection $f^* : V(G) \to [p]^*$. We use [8] for terminology and notation not defined here.

A graph of order p and size q is *edge-graceful* [2] if the edges can be labeled by $1, 2, \ldots, q$ such that the vertex sums are distinct (mod p). A

*Research supported by a Faculty Research Grant, University of West Georgia

necessary condition for a graph with p vertices and q edges to be edge-graceful is that $q(q+1) \equiv \frac{p(p-1)}{2} \pmod{p}$.

Super edge-graceful labelings (SEGLs) were first considered by Mitchem and Simoson [7] who showed super edge-graceful trees are edge-graceful. In particular, Mitchem and Simoson noticed that if G is a super-edge graceful graph and $p|q$, if q is odd, or $p|(q+1)$, if q is even, then G is edge-graceful. Some families of graphs have been shown to be super-edge graceful by explicit labelings. It is known that, for example, paths of all orders except 2 and 4 and cycles of all orders except 4 and 6 are super edge-graceful [1], as are trees of odd order with three even vertices [6]. It was recently proved that [5] total stars and total cycles are also super edge-graceful. In addition,

Theorem 1. [3] All complete graphs of all orders except 1, 2 and 4 are super edge-graceful.

Theorem 2. [4] All complete bipartite graphs are super edge-graceful except for $K_{2,2}$, $K_{2,3}$, and $K_{1,n}$ if n is odd.

In this paper we prove that all complete tripartite graphs $K_{a,b,c}$, except $K_{1,1,2}$, are super edge-graceful. The following lemma is crucial in this paper. Throughout this paper $\ell(v)$ denotes the induced label of vertex v.

2 A SEGL of $K_{a,b,c}$, where a, b and c are even

In this section we prove that there exists a SEGL of $K_{a,b,c}$ for all positive even integers a, b and c.

Lemma 3. Let a, b and c be positive integers and $b+c$ even. Let $\theta = a$ if a is even and $\theta = a-1$ if a is odd. In addition, assume

$$\begin{cases} b+c > \frac{\theta}{4}+2 & \text{if } \theta \equiv 0 \pmod 4 \\ b+c > \frac{\theta+2}{4}+4 & \text{if } \theta \equiv 2 \pmod 4). \end{cases} \tag{1}$$

If there exists a SEGL of $K_{a-2,b,c}$ such that $\ell(w_1) = \ell \geq 0$, for some ℓ, and $\ell(w_2) = \ell+1$, where w_1 and w_2 are not in the partite set of size $a-2$, then there exists a SEGL of $K_{a,b,c}$ such that $\ell(w_1) = \ell+1$ and $\ell(w_2) = \ell$.

Proof. By assumption, there exists a SEGL of $K_{a-2,b,c}$. Let $W = \{w_1, w_2, w_3, \ldots, w_{b+c}\}$ be the vertices of the partite sets of sizes b and c and suppose that the induced vertex labels for w_1 and w_2 are ℓ and $\ell+1$, respectively. Add two new vertices u and v to the partite set of size $a-2$ and join these two to every vertex in W to obtain a $K_{a,b,c}$. The labels we need to assign to the new edges are $\{\pm(\frac{m}{2}+1), \pm(\frac{m}{2}+2), \ldots, \pm(\frac{m}{2}+(b+c))\}$, where $m = (a-2)(b+c)+bc$ if bc is even and $m = (a-2)(b+c)+bc-1$ if bc is odd.

Label the edge uw_i with $(-1)^i(\frac{m}{2}+i)$ and the edge vw_i with $(-1)^{i+1}(\frac{m}{2}+i)$ for $i \in \{1,2,3,\ldots,b+c\}$. Note that with this labeling the vertex labels of $K_{a-2,b,c}$ do not change and $(\ell(u),\ell(v)) = ((b+c)/2, -(b+c)/2)$. To obtain a SEGL of $K_{a,b,c}$, we need to make $\{\ell(u),\ell(v)\} = \{\pm(\frac{\theta+b+c}{2})\}$.

Case 1: $a \equiv 0,1 \pmod 4$. Then $(b+c)/2$ and $(\theta+b+c)/2$ have the same parity. If $\theta/4$ is odd, swap the edge labels $\frac{m}{2}+2$ and $-(\frac{m}{2}+\frac{\theta}{4}+2)$ at u with the edge labels $-(\frac{m}{2}+2)$ and $\frac{m}{2}+\frac{\theta}{4}+2$ at v, respectively. Then $\ell(u) = \frac{\theta+b+c}{2}$ and $\ell(v) = -\frac{\theta+b+c}{2}$, as desired.

If $\theta/4$ is even, swap the edge labels $\frac{m}{2}+2$ and $-(\frac{m}{2}+\frac{\theta}{4}+3)$ at u with the edge labels $-(\frac{m}{2}+2)$ and $\frac{m}{2}+\frac{\theta}{4}+3$ at v, respectively. Then $\ell(u) = \frac{\theta+b+c}{2}+2$ and $\ell(v) = -\frac{\theta+b+c}{2}-2$. Now swap the edge labels $-(\frac{m}{2}+3)$ and $\frac{m}{2}+4$ at u with the edge labels $\frac{m}{2}+3$ and $-(\frac{m}{2}+4)$ at v, respectively, to obtain $\ell(u) = \frac{\theta+b+c}{2}$ and $\ell(v) = -\frac{\theta+b+c}{2}$.

Case 2: $a \equiv 2,3 \pmod 4$. Then $(b+c)/2$ and $(\theta+b+c)/2$ have different parity. Swap the edge label $\frac{m}{2}+2$ at u with the edge label $\frac{m}{2}+1$ at v. Now $\ell(w_1) = \ell+1$, $\ell(w_2) = \ell$, $\ell(u) = (b+c-2)/2$ and $\ell(v) = -(b+c-2)/2$.

If $(\theta+2)/4$ is odd, swap the edge labels $\frac{m}{2}+4$ and $-(\frac{m}{2}+\frac{\theta+2}{4}+4)$ at u with the edge labels $-(\frac{m}{2}+4)$ and $\frac{m}{2}+\frac{\theta+2}{4}+4$ at v, respectively. Then $\ell(u) = \frac{\theta+b+c}{2}$ and $\ell(v) = -\frac{\theta+b+c}{2}$, as desired.

If $(\theta+2)/4$ is even, swap the edge labels $\frac{m}{2}+4$ and $-(\frac{m}{2}+\frac{\theta+2}{4}+5)$ at u with the edge labels $-(\frac{m}{2}+4)$ and $\frac{m}{2}+\frac{\theta+2}{4}+5$ at v, respectively. Then $\ell(u) = \frac{\theta+b+c}{2}+2$ and $\ell(v) = -\frac{\theta+b+c}{2}-2$. Now swap the edge labels $-(\frac{m}{2}+3)$ and $\frac{m}{2}+4$ at u with the edge labels $\frac{m}{2}+3$ and $-(\frac{m}{2}+4)$ at v, respectively, to obtain $\ell(u) = \frac{\theta+b+c}{2}$ and $\ell(v) = -\frac{\theta+b+c}{2}$. □

Remark 4. A closer look at the proof of Lemma 3 shows that Condition (1) can be modified as follows:

$$\begin{cases} b+c > \frac{\theta}{4} & \text{if } \theta \equiv 0 \pmod 4 \text{ and } b+c \geq 10 \\ b+c > \frac{\theta+2}{4}+1 & \text{if } \theta \equiv 2 \pmod 4 \text{ and } b+c \geq 12. \end{cases} \tag{2}$$

In order to prove this, consider the case $\theta \equiv 0 \pmod 4$, $\theta/4$ is odd and $b+c = \frac{\theta}{4}+1$. Hence, there is no edge label $-(\frac{m}{2}+\frac{\theta}{4}+2)$ at u. We swap the edge labels $\frac{m}{2}+2$, $-(\frac{m}{2}+\frac{\theta}{4})$, $\frac{m}{2}+4$, $-(\frac{m}{2}+5)$, $\frac{m}{2}+6$ and $-(\frac{m}{2}+7)$ with their opposites to obtain $\ell(u) = \frac{\theta+b+c}{2}$ and $\ell(v) = -\frac{\theta+b+c}{2}$, as desired. The other cases are similar.

Theorem 5. Let a, b and c be positive even integers. Then there exists a SEGL of $K_{a,b,c}$.

Proof. Without loss of generality, we may assume $a \leq b \leq c$. By induction on the number of vertices $k = a+b+c$, we prove that there exists a

SEGL of $K_{a,b,c}$. The Appendix displays a SEGL of $K_{2,2,2}$. Hence, the statement is true for $k = 6$. Now assume every complete tripartite graph with $k \geq 6$ vertices is SEG. We prove that every complete tripartite graph $K_{a,b,c}$ with $k+2$ vertices is SEG. If $a = 2$, then $b + c \geq 6$, and hence, there is a SEGL of $K_{a-2,b,c}$ by Theorem 2. If $a \geq 4$, then by the inductive hypothesis, there exists a SEGL of $K_{a-2,b,c}$. Note that in both cases there exist two vertices w_1 and w_2 not in the partite set of size $a-2$ such that $\{\ell(w_1), \ell(w_2)\} = \{1, 2\}$. Apply Lemma 3 to construct a SEGL of $K_{a,b,c}$. This completes the proof. □

3 A SEGL of $K_{a,b,c}$, where a and b are even and c is odd

In this section we prove every $K_{a,b,c}$ is SEG if a and b are even and c is odd. We split our proof into three cases: $a \not\equiv b \pmod 4$, $a \equiv b \equiv 0 \pmod 4$, and $a \equiv b \equiv 2 \pmod 4$. When $b + c$ is even we make use of Lemma 3 in induction. Similarly, when $b + c$ is odd we apply the following lemma in induction.

Lemma 6. Let a, b and c be positive integers, b even, and c odd. Let

$$\begin{cases} c > 4(a+b) - 6 & \text{or} \\ b > 4(a+c) - 6 & \text{or} \\ c > 4(a+b) - 18 & \text{and } b \geq 4 \text{ or} \\ b > 4(a+c) - 18 & \text{and } c \geq 5. \end{cases} \tag{3}$$

If there exists a SEGL of $K_{a-2,b,c}$ such that $\ell(w_1) = \ell \geq 0$, for some ℓ, and $\ell(w_2) = \ell + 1$, where w_1 and w_2 are not in the partite set of size $a-2$, then there exists a SEGL of $K_{a,b,c}$ such that $\ell(w_1) = \ell + 1$ and $\ell(w_2) = \ell$.

Proof. By assumption, there exists a SEGL of $K_{a-2,b,c}$. We construct a SEGL for $K_{a,b,c}$. Assume a is even (the case a is odd is similar). Let $W = \{w_1, w_2, \ldots, w_{b+c}\}$ be the vertices in the partite sets of sizes b and c. In addition, suppose that the induced labels for vertices w_1 and w_2 are ℓ and $\ell + 1$ for some positive integer ℓ, respectively. Add two new vertices u and v to the partite set of size $a-2$ and join them to every vertex in W to obtain a $K_{a,b,c}$. The labels we need to assign to the new edges are $\{\pm(\frac{m}{2}+1), \pm(\frac{m}{2}+2), \ldots, \pm(\frac{m}{2}+(b+c))\}$, where $m = (a-2)(b+c) + bc$.

Label the edge uw_i with $(-1)^i(\frac{m}{2}+i)$ and the edge vw_i with $(-1)^{i+1}(\frac{m}{2}+i)$ for $i \in \{1, 2, 3, \ldots, b+c\}$. Note that with this labeling the vertex labels of $K_{a-2,b,c}$ do not change and $(\ell(u), \ell(v)) = (-\frac{m+b+c+1}{2}, \frac{m+b+c+1}{2})$. To obtain a SEGL of $K_{a,b,c}$, we need to make $\{\ell(u), \ell(v)\} = \{\pm(\frac{a+b+c-1}{2})\}$. Assume $\frac{m+b+c+1}{2}$ and $\frac{a+b+c-1}{2}$ have different parity (the case $\frac{m+b+c+1}{2}$

and $\frac{a+b+c-1}{2}$ have the same parity is similar). Swap the edge label $\frac{m}{2}+2$ at u with the edge label $\frac{m}{2}+1$ at v. Now $\ell(w_1) = \ell + 1$, $\ell(w_2) = \ell$, $\ell(u) = -\frac{m+b+c+3}{2}$ and $\ell(v) = \frac{m+b+c+3}{2}$.

Let $0 \le j \le \gamma = \lfloor (b+c-4)/4 \rfloor$ and $I = \{0, 1, 2, \ldots, \gamma\}$. If we swap the edge labels $m/2 + 2j + 4$ and $-(m/2 + b + c - 2j)$ at u with the edge labels $-(m/2 + 2j + 4)$ and $m/2 + b + c - 2j$ at v, then $\ell(u)$ increases ($\ell(v)$ decreases) by $2(b + c - 4j - 4)$. For $J \subseteq I$ define

$$S(J) = \sum_{j \in J} 2(b + c - 4j - 4).$$

Then

1. For $0 \le j \le \gamma - 1$, $2(b + c - 4j - 4) - 2(b + c - 4(j + 1) - 4) = 8$.

2. Let $J \subset I$ and $j \in I \setminus J$. If we swap the edge labels $m/2 + 2j + 4$ and $-(m/2 + 2j + 5)$ or the edge labels $(m/2 + b + c - 2j - 1)$ and $-(m/2 + b + c - 2j)$ at u with their opposites, then $\ell(u)$ increases ($\ell(v)$ decreases) by 2. Hence we can increase $\ell(u)$ and decrease $\ell(v)$ by $S(J) + 2$ or $S(J) + 4$.

3. $2(b + c - 4\gamma - 4) = 2$ if $b + c \equiv 1 \pmod 4$ and $2(b + c - 4\gamma - 4) = 6$ if $b + c \equiv 3 \pmod 4$.

4. If $b + c \equiv 3 \pmod 4$, since the edge labels $-(m/2 + 2\gamma + 5)$ and $(m/2 + 2\gamma + 6)$ are at u, we can increase $\ell(u)$ by $S(J) - 2$ and decrease $\ell(v)$ by $S(J) - 2$, where $J \subseteq I$.

5. $$\begin{aligned} S(I) &= \textstyle\sum_{j=0}^{\gamma} 2(b + c - 4j - 4) \\ &= -4\gamma^2 + 2(b + c - 6)\gamma + (2b + 2c - 8) \\ &= (((b + c)^2 + 3)/4) - b - c. \end{aligned}$$

6. By (3), it is straightforward to see that

$$\begin{aligned} S(I) + \ell(u) &= S(I) - \frac{(a-2)(b+c) + bc + b + c + 3}{2} \\ &\ge \frac{b^2 + c^2 - 2b - 2c - 2ab - 2ac - 3}{4} \\ &\ge -\frac{a + b + c - 1}{2}. \end{aligned}$$

Therefore, by proper edge labels swapping, we can make $\{\ell(u), \ell(v)\} = (-(a + b + c - 1)/2, (a + b + c - 1)/2)$. See Example 7. □

Example 7. Assume there exists a SEGL of $K_{2,4,59}$ and $\ell(w_1) = \ell$ and $\ell(w_2) = \ell+1$ for some positive integer ℓ. We extend this labeling to a SEGL of $K_{4,4,59}$. We use the notation in Lemma 6. Add the new edges uw_i and vw_i, $1 \le i \le 63$, to obtain a $K_{4,4,59}$. The edge labels required for the new edges are $\{\pm(182+i) \mid 1 \le i \le 63\}$. Label the edge uw_i with $(-1)^i(182+i)$ and vw_i with $(-1)^{i+1}(182+i)$. Then $\ell(u) = -214$ and $\ell(v) = 214$. Swap the edge labels 184 at u with the edge label 183 at v. Then $\ell(w_1) = \ell+1$, $\ell(w_2) = \ell$, $\ell(u) = -215$ and $\ell(v) = 215$. Now swap the edge labels uw_i with vw_i for $i \in \{4, 18, 34, 35, 49, 63\}$ to obtain $(\ell(u), \ell(v)) = (-33, 33)$, as required.

3.1 $a \equiv b \equiv 0 \pmod 4$ and c odd

Lemma 8. Let a, b and c be positive integers, $a \equiv b \equiv 0 \pmod 4$, and c odd.

1. Let $ab+ac-bc-3a-4 \ge 0$. Then there is an $a \times (b+c)$ array whose entries are precisely $\{\pm(\frac{bc}{2}+i) \mid 1 \le i \le \frac{ab+ac}{2}\}$ and whose column sums are all zero and row sums are $\{\pm(\frac{b+c-1}{2}+i) \mid 1 \le i \le \frac{a}{2}\}$.
2. Let $ab+ac-bc-3a-4 < 0$. Then there is a $b \times (a+c)$ array whose entries are precisely $\{\pm(\frac{ac}{2}+i) \mid 1 \le i \le \frac{ab+bc}{2}\}$ and whose column sums are all zero and row sums are $\{\pm(\frac{a+c-1}{2}+i) \mid 1 \le i \le \frac{b}{2}\}$.

Proof. **Part 1.** Define an $a \times (b+c)$ array $A = [a_{i,j}]$ as follows.

$$a_{i,j} = \begin{cases} \frac{bc}{2}+k & \text{if } i = 2k-1, j = 1 \\ -(\frac{bc}{2}+k) & \text{if } i = 2k, j = 1 \\ \frac{bc}{2}+a+1-k & \text{if } i = 2k-1, j = 2 \\ -(\frac{bc}{2}+a+1-k) & \text{if } i = 2k, j = 2 \\ -(\frac{3a}{2}+bc+3-k) & \text{if } i = 2k-1, j = 3 \\ \frac{3a}{2}+bc+3-k & \text{if } i = 2k, j = 3, \end{cases}$$

for $1 \le i \le a$. So far, we have filled the first three columns of A. Note that

1. by construction, no entry is repeated in the first three columns;
2. since $ab+ac-bc-3a-4 \ge 0$, the entries in the first three columns are all in $\{\pm(\frac{bc}{2}+i) \mid 1 \le i \le \frac{ab+ac}{2}\}$;
3. the column sums for the first three columns are zero and the row sums are $-(\frac{a}{2}+2-k)$ if $i = 2k-1$ and $\frac{a}{2}+2-k$ if $i = 2k$.

The remaining entries are

$$\begin{aligned} L_1 &= \{\pm(\tfrac{2a+bc}{2}+i) \mid 1 \le i \le \tfrac{bc}{2}+2\} \\ L_2 &= \{\pm(\tfrac{3a}{2}+bc+2+i) \mid 1 \le i \le \tfrac{ab+ac-bc-3a-4}{2}\}. \end{aligned}$$

Since $a, b \equiv 0 \pmod 4$, it follows that $L_1 \cup L_2$ can be partitioned into $a(b+c-3)/4$ 4-subsets of the form $\{\pm\ell, \pm(\ell+1)\}$ for some positive integer ℓ. Partition the empty cells of A into $a(b+c-3)/4$ 2×2 sub-arrays and fill each 2×2 sub-array with a 4-subset in such a way that the column sums are zero and the row sum for the first row is -1 and for the second row is 1 in each sub-array. Now the row sum for row $i = 2k-1$ is $-(\frac{a+b+c+1}{2} - k)$ and for row $i = 2k$ is $\frac{a+b+c+1}{2} - k$. Hence, A is the required array. Example 12 displays an 8×13 array A when $a = b = 8$ and $c = 5$.

Part 2. We first note that if $ab + ac - bc - 3a - 4 < 0$, then $ab + bc - ac - 3b - 4 \geq 0$. Hence, by Part (1), the required $b \times (a+c)$ array exists. □

Example 9. The construction given in Part (1) of Lemma 8 with $a = b = 8$ and $c = 5$ provides the following 8×13 array with column sums zero and row sums $\{\mp 10, \mp 9, \mp 8, \mp 7\}$. This array is employed to extend a SEGL of $K_{8,5}$ to a SEGL of $K_{8,8,5}$ (see Theorem 10).

21	28	−54	29	−30	37	−38	45	−46	57	−58	65	−66
−21	−28	54	−29	30	−37	38	−45	46	−57	58	−65	66
22	27	−53	31	−32	39	−40	47	−48	59	−60	67	−68
−22	−27	53	−31	32	−39	40	−47	48	−59	60	−67	68
23	26	−52	33	−34	41	−42	49	−50	61	−62	69	−70
−23	−26	52	−33	34	−41	42	−49	50	−61	63	−69	79
24	25	−51	35	−36	43	−44	55	−56	63	−64	71	−72
−24	−25	51	−35	36	−43	44	−55	56	−63	64	−71	72

Theorem 10. Let a, b and c be positive integers, $a \equiv b \equiv 0 \pmod 4$, and c odd. Then there exists a SEGL of $K_{a,b,c}$.

Proof. If $ab + ac - bc - 3a - 4 \geq 0$, by Part (1) of Lemma 8, there is an $a \times (b+c)$ array $A = [a_{i,j}]$ whose entries are precisely $\{\pm(\frac{bc}{2} + i) \mid 1 \leq i \leq \frac{ab+ac}{2}\}$ and whose column sums are all zero and row sums are $\{\pm(\frac{b+c-1}{2} + i) \mid 1 \leq i \leq \frac{a}{2}\}$. (The case $ab + ac - bc - 3a - 4 < 0$ is similar.) Consider the graph $K_{a,b,c}$ with partite sets $U = \{u_1, u_2, \ldots, u_a\}$, $W_1 = \{w_1, w_2, \ldots, w_b\}$ and $W_2 = \{w_{b+1}, w_{b+2}, \ldots, w_{b+c}\}$. By Theorem 2, there is a SEGL of $K_{b,c}$. Use this labeling to label the edges between W_1 and W_2. In addition, label the edge $u_i w_j$ with $a_{i,j}$ for $1 \leq i \leq a$ and $1 \leq j \leq b+c$. The resulting labeling is a SEGL of $K_{a,b,c}$. □

3.2 $a \not\equiv b \pmod 4$ and c odd

Similar to Lemma 8 we have the following lemma for this case.

Lemma 11. Let a, b and c be positive integers, a and b even, $a \not\equiv b \pmod 4$, and c odd.

1. Let $ab+ac-bc-2a+2 \geq 0$ and $a < bc$. Then there exists an $a \times (b+c)$ array whose entries are precisely $\{\pm(\frac{bc}{2}+i) \mid 1 \leq i \leq \frac{ab+ac}{2}\}$ and whose column sums are all zero and row sums are $\{\pm(\frac{b+c-1}{2}+i) \mid 1 \leq i \leq \frac{a}{2}\}$.
2. Let $ab+ac-bc-2a+2 < 0$ and $b < ac$. Then there exists a $b \times (a+c)$ array whose entries consists of $\{\pm(\frac{ac}{2}+i) \mid 1 \leq i \leq \frac{ab+bc}{2}\}$ and whose column sums are all zero and row sums are $\{\pm(\frac{a+c-1}{2}+i) \mid 1 \leq i \leq \frac{b}{2}\}$.

Proof. **Part 1.** Define an $a \times (b+c)$ array $A = [a_{i,j}]$ as follows.

$$a_{i,j} = \begin{cases} \frac{bc}{2}+k & \text{if } i = 2k-1, j = 1 \\ -(\frac{bc}{2}+k) & \text{if } i = 2k, j = 1 \\ \frac{a+bc}{2}+k & \text{if } i = 2k-1, j = 2 \\ -(\frac{a+bc}{2}+k) & \text{if } i = 2k, j = 2 \\ -(\frac{a+2bc}{2}+k-1) & \text{if } i = 2k-1, j = 3 \\ \frac{a+2bc}{2}+k-1 & \text{if } i = 2k, j = 3, \end{cases}$$

for $1 \leq i \leq a$. So far, we have filled the first three columns of A. Note that

1. since $a < bc$, no entry is repeated in the first three columns;
2. since $ab+ac-bc-2a+2 \geq 0$, the entries in the first three columns are all in $\{\pm(\frac{bc}{2}+i) \mid 1 \leq i \leq \frac{ab+ac}{2}\}$;
3. the column sums for the first three columns are zero and the row sums are $k+1$ if $i = 2k-1$ and $-(k+1)$ if $i = 2k$.

The remaining entries are

$$\begin{aligned} L_1 &= \{\pm(\tfrac{2a+bc}{2}+i) \mid 1 \leq i \leq \tfrac{bc-a}{2}-1\} \\ L_2 &= \{\pm(a+bc+i-1) \mid 1 \leq i \leq \tfrac{ab+ac-bc}{2}-a+1\}. \end{aligned}$$

Since a and b are even and $a \not\equiv b \pmod 4$, it follows that $L_1 \cup L_2$ can be partitioned into $a(b+c-3)/4$ 4-subsets of the form $\{\pm\ell, \pm(\ell+1)\}$ for some positive integer ℓ. Partition the empty cells of A into $a(b+c-3)/4$ 2×2 sub-arrays and fill each 2×2 sub-array with a 4-subset in such a way that the column sums are zero and the row sum for the first row is 1 and for the second row is -1 in each sub-array. Now the row sum for row $i = 2k-1$ is $\frac{b+c-3}{2}+k+1$ and for row $i = 2k$ is $-(\frac{b+c-3}{2}+k+1)$. Hence, A is the required array. Example 12 displays a 4×13 array A when $a = 4$, $b = 10$ and $c = 3$.

Part 2. We first note that if $ab+ac-bc-2a+2 < 0$, then $ab+bc-ac-2b+2 \geq 0$. Hence, by Part (1), the required $b \times (a+c)$ array exists. □

Example 12. The construction given in Part (1) of Lemma 11 with $a = 4$, $b = 10$ and $c = 3$ provides the following 4×13 array with column sums zero and row sums $\{\pm 7, \pm 8\}$. This array is employed to extend a SEGL of $K_{10,3}$ to a SEGL of $K_{4,10,3}$ (see Lemma 13).

16	18	−34	−20	21	−24	25	−28	29	−32	33	−38	39
−16	−18	34	20	−21	24	−25	28	−29	32	−33	38	−39
17	19	−35	−22	23	−26	27	−30	31	−36	37	−40	41
−17	−19	35	22	−23	26	−27	30	−31	36	−37	40	−41

Lemma 13. Let a, b and c be positive integers, a and b even, $a \not\equiv b \pmod 4$, and c odd. If

1. $ab + ac - bc - 2a + 2 \geq 0$, $a < bc$ and $(b, c) \neq (2, 3)$, or

2. $ab + ac - bc - 2a + 2 < 0$, $b < ac$ and $(a, c) \neq (2, 3)$,

then there exists a SEGL of $K_{a,b,c}$.

Proof. We only present a proof for Part (1). The proof for Part (2) is similar.
By Part (1) of Lemma 11, there is an $a \times (b + c)$ array $A = [a_{i,j}]$ whose entries are precisely $\{\pm(\frac{bc}{2} + i) \mid 1 \leq i \leq \frac{ab+ac}{2}\}$ and whose column sums are all zero and row sums are $\{\pm(\frac{b+c-1}{2} + i) \mid 1 \leq i \leq \frac{a}{2}\}$. Consider the graph $K_{a,b,c}$ with partite sets $U = \{u_1, u_2, \ldots, u_a\}$, $W_1 = \{w_1, w_2, \ldots, w_b\}$ and $W_2 = \{w_{b+1}, w_{b+2}, \ldots, w_{b+c}\}$. By Theorem 2, there is a SEGL of $K_{b,c}$. Use this labeling to label the edges between W_1 and W_2. In addition, label the edge $u_i w_j$ with $a_{i,j}$ for $1 \leq i \leq a$ and $1 \leq j \leq b + c$. The resulting labeling is a SEGL of $K_{a,b,c}$. □

Corollary 14. Let a and b be positive even integers and $a \not\equiv b \pmod 4$. Then there exists a SEGL of $K_{a,b,1}$.

Proof. Without loss of generality we may assume $a \leq b$. Now apply Lemma 13 with $c = 1$. □

Since $K_{2,3}$ is not SEG by Theorem 2, we use the following result as a detour.

Lemma 15. Let $b \equiv 0 \pmod 4$ be a positive integer. Then there exists a SEGL of $K_{2,b,3}$.

Proof. Let $W = \{w_1, w_2, \ldots, w_{b+3}\}$ be the vertices of $K_{b,3}$. By Theorem 2, there is a SEGL of $K_{b,3}$. Join two new vertices u and v to every vertex in W to obtain a $K_{2,b,3}$. The labels we need to assign to the new edges are $\{\pm(\frac{3b}{2} + 1), \pm(\frac{3b}{2} + 2), \ldots, \pm(\frac{3b}{2} + (b + 3))\}$.

Label the edge uw_i with $(-1)^i(\frac{3b}{2}+i)$ and the edge vw_i with $(-1)^{i+1}(\frac{3b}{2}+i)$ for $i \in \{1,2,3,\ldots,b+3\}$. Note that with this labeling the vertex labels of W do not change and $(\ell(u),\ell(v)) = (-(2b+2), 2b+2)$. To obtain a SEGL of $K_{2,b,3}$, we need to make $\{\ell(u),\ell(v)\} = \{\pm(\frac{b}{2}+2)\}$.

If $b/4$ is odd, we swap the edge labels $3b/2+2$ and $-(3b/2+3b/4+2)$ at u with the edge labels $-(3b/2+2)$ and $(3b/2+3b/4+2)$ at v. Then $\ell(u) = -(b/2+2)$ and $\ell(v) = b/2+2$, as required.

If $b/4$ is even, we swap the edge labels $3b/2+2$ and $-(3b/2+3b/4+3)$ at u with the edge labels $-(3b/2+2)$ and $(3b/2+3b/4+3)$ at v. Then $\ell(u) = -b/2$ and $\ell(v) = b/2$. We also swap the edge labels $3b/2+4$ and $-(3b/2+3)$ with $-(3b/2+4)$ and $3b/2+3$. Then $\ell(u) = -(b/2+2)$ and $\ell(v) = b/2+2$, as required. □

Lemma 16. Let c be a positive odd integer. Then $K_{2,4,c}$, $K_{2,8,c}$ and $K_{4,6,c}$ are super edge-graceful.

Proof. For $c=1$ apply Corollary 14. By Lemma 15, $K_{2,4,3}$ and $K_{2,8,3}$ are SEG. For the other values apply Lemma 13. □

Theorem 17. Let a, b and c be positive integers, a and b even, $a \not\equiv b \pmod 4$, and c odd. Then there exists a SEGL of $K_{a,b,c}$.

Proof. By Lemma 16, the theorem is true if $a+b \le 10$. Now let $a+b \ge 12$. First consider the case $c < 4(a+b)-6$. By Corollary 14, there is a SEGL of $K_{a,b,1}$. Now by induction on c, Lemma 3 and Remark 4, one can obtain a SEGL of $K_{a,b,c}$. Second let $c > 4(a+b)-6$. By Theorem 2, there is a SEGL of $K_{b,c}$. By induction on a and Lemma 6, we extend this labeling to a SEGL of $K_{a,b,c}$. This completes the proof.

□

3.3 $a \equiv b \equiv 2 \pmod 4$ and c odd

For this case we employ a technique similar to that explained in Subsection 3.2 to find a SEGL of $K_{a,b,c}$. The proof of the following lemma is similar to the proof of Lemma 15.

Lemma 18. Let $b \equiv 2 \pmod 4$. Then there exists a SEGL of $K_{2,b,1}$.

Proof. For a SEGL of $K_{2,2,1}$ see the Appendix. Now assume $b \ge 6$. By Theorem 2, there is a SEGL of $K_{b,1}$. Let $W = \{w_1, w_2, \ldots, w_{b+1}\}$ be the vertices of $K_{b,1}$. In addition, suppose that the induced labels for vertices w_1 and w_2 are ℓ and $\ell+1$ for some positive integer ℓ, respectively. Join two new vertices u and v to every vertex in W to obtain a $K_{2,b,1}$. The labels we need to assign to the new edges are $\{\pm(\frac{b}{2}+1), \pm(\frac{b}{2}+2), \ldots, \pm(\frac{b}{2}+(b+1))\}$.

Label the edge uw_i with $(-1)^i(\frac{b}{2}+i)$ and the edge vw_i with $(-1)^{i+1}(\frac{b}{2}+i)$ for $i \in \{1,2,3,\ldots,b+1\}$. Note that with this labeling the vertex labels of W do not change and $(\ell(u),\ell(v)) = (-(b+1), b+1)$. To obtain a SEGL of $K_{2,b,1}$, we need to make $\{\ell(u),\ell(v)\} = \{\pm(\frac{b}{2}+1)\}$. Swap the edge label $\frac{b}{2}+2$ at u with the edge label $\frac{b}{2}+1$ at v. Now $\ell(w_1) = \ell+1$, $\ell(w_2) = \ell$, $\ell(u) = -(b+2)$ and $\ell(v) = b+2$.

If $(b+2)/4$ is odd, we swap the edge labels $b/2+4$ and $-(b/2+4+(b+2)/4)$ at u with the edge labels $-(b/2+4)$ and $(b/2+4+(b+2)/4)$ at v. Then $\ell(u) = -(b/2+1)$ and $\ell(v) = b/2+1$, as required.

If $(b+2)/4$ is even and $b \geq 10$, we swap the edge labels $b/2+4$ and $-(b/2+4+(b+6)/4)$ at u with the edge labels $-(b/2+4)$ and $(b/2+4+(b+6)/4)$ at v. Then $\ell(u) = -(b/2-1)$ and $\ell(v) = b/2-1$. We also swap the edge labels $-(b/2+b-1)$ and $(b/2+b)$ with $(b/2+b-1)$ and $-(b/2+b)$. Then $\ell(u) = -(b/2+1)$ and $\ell(v) = b/2+1$, as required.

Finally, if $b = 6$, we swap the edge labels 7, -8, 9 and -10 at u with their opposites. Then $\ell(u) = -4$ and $\ell(v) = 4$, as desired. □

Lemma 19. Let $a \equiv 0 \pmod 4$, $b \equiv 2 \pmod 4$ and $a < b$. Then there exists an $a \times (b+1)$ array whose entries are precisely $\{\pm(\frac{3b+2}{2}+i) \mid 1 \leq i \leq \frac{(a-2)(b+1)}{2}\}$ and whose column sums are all zero and row sums are $\{\pm(\frac{b+2}{2}+i) \mid 1 \leq i \leq \frac{a}{2}\}$.

Proof. **Part 1.** Define an $a \times (b+1)$ array $A = [a_{i,j}]$ as follows.

$$a_{i,j} = \begin{cases} \frac{3b+2}{2}+k & \text{if } i = 2k-1, j = 1 \\ -(\frac{3b+2}{2}+k) & \text{if } i = 2k, j = 1 \\ \frac{3b+a+2}{2}+k & \text{if } i = 2k-1, j = 2 \\ -(\frac{3b+a+2}{2}+k) & \text{if } i = 2k, j = 2 \\ -(3b+\frac{a}{2}+k) & \text{if } i = 2k-1, j = 3 \\ 3b+\frac{a}{2}+k & \text{if } i = 2k, j = 3, \end{cases}$$

for $1 \leq i \leq a$. So far, we have filled the first three columns of A. Since $4 \leq a < b$, no entry is repeated in the first three columns and the entries are all in $\{\pm(\frac{3b+2}{2}+i) \mid 1 \leq i \leq \frac{(a-2)(b+1)}{2}\}$. Note that the column sums for the first three columns are zero and the row sums are $k+2$ if $i = 2k-1$ and $-(k+2)$ if $i = 2k$. The remaining entries are

$$\begin{aligned} L_1 &= \{\pm(\tfrac{3b}{2}+a+1+i) \mid 1 \leq i \leq \tfrac{3b-a-2}{2}\} \\ L_2 &= \{\pm(3b+a+i) \mid 1 \leq i \leq \tfrac{ab-a-3b+2}{2}\}. \end{aligned}$$

Since $a \equiv 0 \pmod 4$ and $b \equiv 2 \pmod 4$, it follows that $L_1 \cup L_2$ can be partitioned into $a(b-2)/4$ 4-subsets of the form $\{\pm\ell, \pm(\ell+1)\}$ for some positive integer ℓ. Partition the empty cells of A into $a(b-2)/4$ 2×2 sub-arrays and fill each 2×2 sub-array with a 4-subset in such a way that the column sums are zero and the row sum for the first row is 1 and for the second row is -1 in each sub-array. The resulting array A is the required array. Example 20 displays an 8×11 array A when $a = 8$ and $b = 10$. □

Example 20. The construction given in Lemma 19 with $a = 8$ and $b = 10$ provides the following 8×11 array with column sums zero and and row sums $\{\pm 7, \pm 8, \pm 9, \pm 10\}$. This array is employed to extend a SEGL of $K_{2,10,1}$ to a SEGL of $K_{10,10,1}$ (see Lemma 21).

17	21	−35	−25	26	−33	34	−45	46	−53	54
−17	−21	35	25	−26	33	−34	45	−46	53	−54
18	22	−36	−27	28	−39	40	−47	48	−55	56
−18	−22	36	27	−28	39	−40	47	−48	55	−56
19	23	−37	−29	30	−41	42	−49	50	−57	58
−19	−23	37	29	−30	41	−42	49	−50	57	−58
20	24	−38	−31	32	−43	44	−51	52	−59	60
−20	−24	38	31	−32	43	−44	51	−52	59	−60

Lemma 21. Let $a \equiv b \equiv 2 \pmod 4$. Then there exists a SEGL of $K_{a,b,1}$.

Proof. Without loss of generality, we may assume $a \leq b$. By Lemma 18, we may also assume $a \geq 6$. Let the partite sets of $K_{a,b,1}$ be $U = \{u_1, u_2, \ldots, u_a\}$, $W = \{w_1, w_2, \ldots, w_b\}$ and $\{w_{b+1}\}$. Consider the subgraph $K_{2,b,1}$ of $K_{a,b,1}$ with vertices $\{u_{a-1}, u_a\} \cup W \cup \{w_{b+1}\}$. By Lemma 18, there is a SEGL for this subgraph. Let $A = [a_{i,j}]$ be an $(a-2) \times (b+1)$ array given in Lemma 19. Label the edge $u_i w_j$ with $a_{i,j}$ for $i \in \{1, 2, \ldots, a-2\}$ and $j \in \{1, 2, \ldots, b+1\}$. The resulting labeling is a SEGL of $K_{a,b,1}$. □

Lemma 22. Let c be an odd integer and $b \in \{2, 6\}$. Then there exists a SEGL of $K_{2,b,c}$.

Proof. For $c = 1$ we apply Lemma 18. For a SEGL of $K_{2,2,3}$ see the Appendix. Now let $c \geq 3$ and $(b, c) \neq (2, 3)$. By Theorem 2, there is a SEGL of $K_{b,c}$. Apply a technique similar to that described in the proof of Lemma 18 to extend this labeling to a SEGL of $K_{2,b,c}$. □

Theorem 23. Let a, b and c be positive integers, $a \equiv b \equiv 2 \pmod 4$ and $c \equiv 1 \pmod 2$. Then there exists a SEGL of $K_{a,b,c}$.

Proof. Without loss of generality we may assume $a \leq b$. By Lemma 22, we may also assume $b \neq 2$ and $a + b \geq 12$. First let $c < 4(a + b) - 6$. By Lemma 21, there is a SEGL of $K_{a,b,1}$. Apply Lemma 3 and Remark 4 to obtain a SEGL of $K_{a,b,c}$. Next let $c > 4(a+b) - 6$. By Theorem 2, if $a = 2$, and Theorem 17, if $a \neq 2$, there is a SEGL of $K_{a-2,b,c}$. Now apply Lemma 6 to extend this labeling to a SEGL of $K_{a,b,c}$. □

By Theorems 17, 10 and 23, we can state the main result of this section.

Theorem 24. Let a, b and c be positive integers, a and b even and c odd. Then there exists a SEGL of $K_{a,b,c}$.

4 A SEGL of $K_{a,b,c}$, where a is even and b and c are odd

In this section we prove that for every positive even integer a and positive odd integers b and c, $(a, b, c) \neq (2, 1, 1)$, the complete tripartite graph $K_{a,b,c}$ is SEG. It is easy to see that $K_{2,1,1}$ is not SEG. By Theorem 2, $K_{b,1}$ is not SEG. Hence, we cannot apply Lemma 3 to obtain a SEGL of $K_{2,b,1}$. Our first result in this section shows that $K_{2,b,1}$ is SEG.

Lemma 25. Let $b \neq 1$ be a positive odd integer. Then there exists a SEGL of $K_{2,b,1}$.

Proof. We split this proof into two cases.
Case 1: $b \equiv 1 \pmod 4$. First consider the graph $K_{2,1,1}$ with partite sets $\{v_1, v_2\}$, $\{u\}$ and $\{w_0\}$. Label uw_0, uv_1, uv_2, v_1w_0 and v_2w_0 with $0, 2, -1, -2$ and 1, respectively. Then $\ell(u) = 1$, $\ell(w_0) = -1$ and $\ell(v_1) = \ell(v_2) = 0$. Now define a $(b-1) \times 3$ array $A = [a_{i,j}]$ as follows.

$$a_{i,j} = \begin{cases} k+2 & \text{if } \ i = 2k-1, j = 1 \\ -(k+2) & \text{if } \ i = 2k, j = 1 \\ \frac{b-1}{2} + i + 2 & \text{if } \ j = 2 \\ -(\frac{b-1}{2} + i + 2) & \text{if } \ j = 3, \end{cases}$$

for $1 \leq i \leq b-1$. Note the row sums of A are $\{\pm 3, \pm 4, \ldots, \pm \frac{b+3}{2}\}$. Swap $a_{2,2}$ with $a_{2,3}$ and $a_{4,2}$ with $a_{4,3}$. In addition, swap $a_{4k+2,2}$ with $a_{4k+2,3}$ and $a_{4k+3,2}$ with $a_{4k+3,3}$ for $1 \leq k \leq \frac{b-5}{4}$. It is easy to see that the resulting $(b-1) \times 3$ array, say $B = [b_{i,j}]$, has the same row sums as array A and the column sums of B are $0, -2$ and 2.

Add $b-1$ new vertices $\{w_1, w_2, \ldots, w_{b-1}\}$ to the partite set $\{w_0\}$ of $K_{2,1,1}$ and join these vertices to u, v_1 and v_2 to obtain a $K_{2,b,1}$. Label uw_i, v_1w_i and v_2w_i with $b_{i,1}$, $b_{i,2}$ and $b_{i,3}$, respectively, for $1 \leq i \leq b-1$. Then $\ell(u) = 1$, $\ell(w_0) = -1$, $\ell(v_1) = -2$ and $\ell(v_2) = 2$. Hence, the resulting labeling is a SEGL of $K_{2,b,1}$.

Case 2: $b \equiv 3 \pmod 4$. First note that there is a SEGL of $K_{2,3,1}$ (see the Appendix). We construct a $(b-3) \times 3$ array $A = [a_{i,j}]$ whose entries are precisely $\{\pm 6, \pm 7, \ldots, \pm \frac{3b+1}{2}\}$ and whose column sums are all zero and row sums are $\{\pm 4, \pm 5, \ldots, \pm \frac{b+3}{2}\}$. Set $a_{i,1} = 5 + k$ if $i = 2k - 1$ and $a_{i,1} = -(5 + k)$ if $i = 2k$ for $1 \leq i \leq b - 3$. The remaining entries are $\{\frac{b-3}{2} + i + 5 \mid 1 \leq i \leq b - 3\}$. By assumption, we can partition these entries into $(b-3)/4$ 4-subsets of the form $\{\pm \ell, \pm(\ell + 2)\}$ for some positive integer ℓ. Partition the empty cells of A into $(b - 3)/4$, 2×2 sub-arrays and fill each 2×2 sub-array with a 4-subset in such a way that the column sums are zero and the row sum for the first row is -2 and for the second row is 2 in each sub-array. Then A is the required array. As in Case 1, we can use this array to extend a SEGL of $K_{2,3,1}$ to a SEGL of $K_{2,b,1}$. □

Theorem 26. Let a, b and c be positive integers, a even, b and c odd and $a < 4(b + c) - 18$. Then there is a SEGL of $K_{a,b,c}$.

Proof. Without loss of generality, we may assume $b \geq c$. If $c = 1$, then by Lemma 25, there is a SEGL of $K_{2,b,1}$. If $c \neq 1$, then by Theorem 2, there is a SEGL of $K_{b,c}$. Now since $b + c$ is even, by induction on a and Lemma 3, the result follows. □

Lemma 27. Let a and b be positive integers, a even, $b \equiv 1 \pmod 4$, $b \geq 5$ and $a \geq (b + 3)/2$. Then there exists a $(b - 1) \times (a + 1)$ array whose entries are precisely $\{\pm(a + i) \mid 1 \leq i \leq \frac{ab-a+b-1}{2}\}$ and whose column sums are all zero and row sums are $\{\pm(\frac{a}{2} + 1 + i) \mid 1 \leq i \leq \frac{b-1}{2}\}$.

Proof. Define a $(b - 1) \times (a + 1)$ array $A = [a_{i,j}]$ as follows.

$$a_{i,j} = \begin{cases} a + k & \text{if } i = 2k - 1, j = 1 \\ -(a + k) & \text{if } i = 2k, j = 1 \\ \frac{2a+b-1}{2} + k & \text{if } i = 2k - 1, j = 2 \\ -(\frac{2a+b-1}{2} + k) & \text{if } i = 2k, j = 2 \\ -(\frac{4a+b-5}{2} + k) & \text{if } i = 2k - 1, j = 3 \\ \frac{4a+b-5}{2} + k & \text{if } i = 2k, j = 3, \end{cases}$$

for $1 \leq i \leq b - 1$. So far, we have filled the first three columns of A. Note that the column sums for the first three columns are zero and the row sums are $k + 2$ if $i = 2k - 1$ and $-(k + 2)$ if $i = 2k$. The remaining entries are

$$\begin{aligned} L_1 &= \{\pm(a + b + i - 1) \mid 1 \leq i \leq \tfrac{2a-b-3}{2}\} \\ L_2 &= \{\pm(2a + b + i - 3) \mid 1 \leq i \leq \tfrac{ab-3a-b+5}{2}\}. \end{aligned}$$

By assumptions, $L_1 \cup L_2$ can be partitioned into $(b - 1)(a - 2)/4$ 4-subsets of the form $\{\pm \ell, \pm(\ell + 1)\}$ for some positive integer ℓ. Partition the empty cells of A into $(b - 1)(a - 2)/4$ 2×2 sub-arrays and fill each 2×2 sub-array with a 4-subset in such a way that the column sums are zero and the row sum for the first row is 1 and for the second row is -1 in each sub-array.

The resulting array A is the required array. Example 28 displays an 8×11 array A when $a = 10$ and $b = 9$. $\square$

Example 28. The construction given in Lemma 27 with $a = 10$ and $b = 9$ provides the following 8×11 array with column sums zero and and row sums $\{\pm 7, \pm 8, \pm 9, \pm 10\}$. This array can be employed to extend a SEGL of $K_{10,1,1}$ to a SEGL of $K_{10,9,1}$.

11	15	−23	−19	20	−31	32	−39	40	−47	48
−11	−15	23	19	−20	31	−32	39	−40	47	−48
12	16	−24	−21	22	−33	34	−41	42	−49	50
−12	−16	24	21	−22	33	−34	41	−42	49	−50
13	17	−25	−27	28	−35	36	−43	44	−51	52
−13	−17	25	27	−28	35	−36	43	−44	51	−52
14	18	−26	−29	30	−37	38	−45	46	−53	54
−14	−18	26	29	−30	37	−38	45	−46	53	−54

Lemma 29. Let a and b be positive integers, a even and $b \equiv 3 \pmod 4$, $b \geq 7$ and $a \geq (b+3)/4$. Then there exists a $(b-3) \times (a+1)$ array whose entries are precisely $\{\pm(2a+i+1) \mid 1 \leq i \leq \frac{ab-3a+b-3}{2}\}$ and whose column sums are all zero and row sums are $\{\pm(\frac{a}{2}+2+i) \mid 1 \leq i \leq \frac{b-3}{2}\}$.

Proof. Define a $(b-3) \times (a+1)$ array $A = [a_{i,j}]$ as follows.

$$a_{i,j} = \begin{cases} 2a+k+1 & \text{if } \ i = 2k-1, j = 1 \\ -(2a+k+1) & \text{if } \ i = 2k, j = 1 \\ \frac{4a+b-1}{2}+k & \text{if } \ i = 2k-1, j = 2 \\ -(\frac{4a+b-1}{2}+k) & \text{if } \ i = 2k, j = 2 \\ -(\frac{8a+b-5}{2}+k) & \text{if } \ i = 2k-1, j = 3 \\ \frac{8a+b-5}{2}+k & \text{if } \ i = 2k, j = 3, \end{cases}$$

for $1 \leq i \leq b-3$. So far, we have filled the first three columns of A. Note that the column sums for the first three columns are zero and the row sums are $k+3$ if $i = 2k-1$ and $-(k+3)$ if $i = 2k$. The remaining entries are

$$\begin{aligned} L_1 &= \{\pm(2a+b+i-2) \mid 1 \leq i \leq \tfrac{4a-b-1}{2}\} \\ L_2 &= \{\pm(4a+b+i-4) \mid 1 \leq i \leq \tfrac{ab-7a-b+7}{2}\}. \end{aligned}$$

By assumptions, $L_1 \cup L_2$ can be partitioned into $(b-3)(a-2)/4$ 4-subsets of the form $\{\pm\ell, \pm(\ell+1)\}$ for some positive integer ℓ. Partition the empty cells of A into $(b-3)(a-2)/4$ 2×2 sub-arrays and fill each 2×2 sub-array with a 4-subset in such a way that the column sums are zero and the row sum for the first row is 1 and for the second row is -1 in each sub-array. The resulting array A is the required array. Example 30 displays a 8×11 array A when $a = 10$ and $b = 11$. $\square$

Example 30. The construction given in Lemma 27 with $a = 10$ and $b = 11$ provides the following 8×11 array with column sums zero and and row sums $\{\pm 8, \pm 9, \pm 10, \pm 11\}$. This array can be employed to extend a SEGL of $K_{10,3,1}$ to a SEGL of $K_{10,11,1}$.

22	26	−44	−30	31	−38	39	−50	51	−58	59
−22	−26	44	30	−31	38	−39	50	−51	58	−59
23	27	−45	−32	33	−40	41	−52	53	−60	61
−23	−27	45	32	−33	40	−41	52	−53	60	−61
24	27	−46	−34	35	−42	43	−54	55	−62	63
−24	−28	46	34	−35	42	−43	54	−55	62	−63
25	29	−47	−36	37	−48	49	−56	57	−64	65
−25	−29	47	36	−37	48	−49	56	−57	64	−65

Lemma 31. There exists a SEGL of $K_{1,1,c}$ for every positive integer $c \neq 2$.

Proof. There is a SEGL of $K_{1,1,1} = K_3$ by Theorem 1. A SEGL of $K_{1,1,3}$ is given in the Appendix. Now let $c \geq 4$. By Theorem 2, there is a SEGL of $K_{2,c}$. Join the two vertices in the partite set of size two with an edge and assign label zero to this edge. The resulting labeling is a SEGL of $K_{1,1,c}$. □

Lemma 32. Let a be a positive even integer, b a positive odd integer and $(a, b) \neq (2, 1)$. Then $K_{a,b,1}$ is SEG.

Proof. By Lemma 25, there exists a SEGL of $K_{2,b,1}$ if $b \neq 1$. Now let $a \geq 4$. First consider the case $a < b$. Hence, we may assume $b \geq 5$. Now since $b + 1$ is even, the result follows by induction on a and Lemma 3.

Second let $a > b$. By Lemma 31, there is a SEGL of $K_{a,1,1}$. If $b \equiv 1 \pmod 4$, using a $(b-1) \times (a+1)$ array given in Lemma 27 we can extend a SEGL of $K_{a,1,1}$ to a SEGL of $K_{a,b,1}$.

Now assume $b \equiv 3 \pmod 4$. First we construct a SEGL of $K_{a,3,1}$. For $a \geq 12$ apply Lemma 6 to extend a SEGL of $K_{a,1,1}$, to a SEGL of $K_{a,3,1}$. The Appendix displays a SEGL of $K_{2,3,1}$. By adding two new vertices in the partite set of size two, one can extend this labeling to a SEGL of $K_{4,3,1}$ (see Lemma 3). Now let $a \in \{6, 8, 10\}$. Add two new vertices to the partite set of size one of a $K_{a,1,1}$. Apply a method similar to that described in Lemma 6 to extend a SEGL of $K_{a,1,1}$ to a SEGL of $K_{a,3,1}$. Finally, use a $(b-3) \times (a+1)$ array given in Lemma 29 to extend a SEGL of $K_{a,3,1}$ to a SEGL of $K_{a,b,1}$. □

Lemma 33. Let a and b be positive integers, a even and $b \equiv 1 \pmod 4$, $b \geq 9$ and $a \geq \frac{b-1}{6}$. Then there exists a $(b-3) \times (a+3)$ array whose entries are precisely $\{\pm(3a+i+4) \mid 1 \leq i \leq \frac{ab-3a+3b-9}{2}\}$ and whose column sums are all zero and row sums are $\{\pm(\frac{a}{2}+3+i) \mid 1 \leq i \leq \frac{b-3}{2}\}$.

Proof. Define a $(b-3) \times (a+3)$ array $A = [a_{i,j}]$ as follows.

$$a_{i,j} = \begin{cases} 3a+k+4 & \text{if } i = 2k-1, j = 1 \\ -(3a+k+4) & \text{if } i = 2k, j = 1 \\ \frac{6a+b+5}{2} + k & \text{if } i = 2k-1, j = 2 \\ -(\frac{6a+b+5}{2} + k) & \text{if } i = 2k, j = 2 \\ -(\frac{12a+b+7}{2} + k) & \text{if } i = 2k-1, j = 3 \\ \frac{12a+b+7}{2} + k & \text{if } i = 2k, j = 3, \end{cases}$$

for $1 \leq i \leq b-3$. So far, we have filled the first three columns of A. Note that the column sums for the first three columns are zero and the row sums are $k+3$ if $i = 2k-1$ and $-(k+3)$ if $i = 2k$. The remaining entries are

$$\begin{aligned} L_1 &= \{\pm(3a+b+i+1) \mid 1 \leq i \leq \tfrac{6a-b+5}{2}\} \\ L_2 &= \{\pm(6a+b+i+2) \mid 1 \leq i \leq \tfrac{ab-9a+b-5}{2}\}. \end{aligned}$$

By assumptions, $L_1 \cup L_2$ can be partitioned into $a(b-3)/4$ 4-subsets of the form $\{\pm\ell, \pm(\ell+1)\}$ for some positive integer ℓ. Partition the empty cells of A into $a(b-3)/4$ 2×2 sub-arrays and fill each 2×2 sub-array with a 4-subset in such a way that the column sums are zero and the row sum for the first row is 1 and for the second row is -1 in each sub-array. The resulting array A is the required array. Example 34 displays a 8×11 array A when $a = 8$ and $b = 9$. □

Example 34. The construction given in Lemma 33 with $a = 8$ and $b = 9$ provides the following 6×11 array with column sums zero and and row sums $\{\pm 8, \pm 9, \pm 10\}$. This array can be employed to extend a SEGL of $K_{8,3,3}$ to a SEGL of $K_{8,9,3}$.

29	32	−57	−35	36	−41	42	−47	48	−53	54
−29	−32	57	35	−36	41	−42	47	−48	53	−54
30	33	−58	−37	38	−43	44	−49	50	−55	56
−30	−33	58	37	−38	43	−44	49	−50	55	−56
31	34	−59	−39	40	−45	46	−51	52	−60	61
−31	−34	−59	39	−40	45	−46	51	−52	60	−61

Lemma 35. Let a and b be positive integers, a even and $b \equiv 3 \pmod 4$, $b \geq 7$ and $a \geq \frac{b+3}{4}$. Then there exists a $(b-1)\times(a+3)$ array whose entries are precisely $\{\pm(2a+i+1) \mid 1 \leq i \leq \frac{ab-a+3b-3}{2}\}$ and whose column sums are all zero and row sums are $\{\pm(\frac{a}{2}+2+i) \mid 1 \leq i \leq \frac{b-1}{2}\}$.

Proof. Define a $(b-1)\times(a+3)$ array $A = [a_{i,j}]$ as follows.

$$a_{i,j} = \begin{cases} 2a+k+1 & \text{if } \ i = 2k-1, j = 1 \\ -(2a+k+1) & \text{if } \ i = 2k, j = 1 \\ \frac{4a+b+1}{2}+k & \text{if } \ i = 2k-1, j = 2 \\ -(\frac{4a+b+1}{2}+k) & \text{if } \ i = 2k, j = 2 \\ -(\frac{8a+b-1}{2}+k) & \text{if } \ i = 2k-1, j = 3 \\ \frac{8a+b-1}{2}+k & \text{if } \ i = 2k, j = 3, \end{cases}$$

for $1 \leq i \leq b-1$. So far, we have filled the first three columns of A. Note that the column sums for the first three columns are zero and the row sums are $k+2$ if $i = 2k-1$ and $-(k+2)$ if $i = 2k$. The remaining entries are

$$\begin{aligned} L_1 &= \{\pm(2a+b+i) \mid 1 \leq i \leq \tfrac{4a-b-1}{2}\} \\ L_2 &= \{\pm(4a+b+i-1) \mid 1 \leq i \leq \tfrac{ab-5a+b+1}{2}\}. \end{aligned}$$

By assumptions, $L_1 \cup L_2$ can be partitioned into $a(b-1)/4$ 4-subsets of the form $\{\pm\ell, \pm(\ell+1)\}$ for some positive integer ℓ. Partition the empty cells of A into $a(b-1)/4$ 2×2 sub-arrays and fill each 2×2 sub-array with a 4-subset in such a way that the column sums are zero and the row sum for the first row is 1 and for the second row is -1 in each sub-array. The resulting array A is the required array. Example 36 displays a 10×11 array A when $a = 8$ and $b = 11$. □

Example 36. The construction given in Lemma 35 with $a = 8$ and $b = 11$ provides the following 10×11 array with column sums zero and and row sums $\{\pm7, \pm8, \pm9, \pm10, \pm11\}$. This array can be employed to extend a SEGL of $K_{8,1,3}$ to a SEGL of $K_{8,11,3}$.

18	23	−38	−28	29	−43	44	−53	54	−63	64
−18	−23	38	28	−29	43	−44	53	−54	63	−64
19	24	−39	−30	31	−45	46	−55	56	−65	66
−19	−24	39	30	−31	45	−46	55	−56	65	−66
20	25	−40	−32	33	−47	48	−57	58	−67	68
−20	−25	40	32	−33	47	−48	57	−58	67	−68
21	26	−41	−34	35	−49	50	−59	60	−69	70
−21	−26	41	34	−35	49	−50	59	−60	69	−70
22	27	−42	−36	37	−51	52	−61	62	−71	72
−22	−27	42	36	−37	51	−52	61	−62	71	−72

Lemma 37. Let a be a positive even integer. Then there exist SEGLs of $K_{a,3,3}$ and $K_{a,3,5}$.

Proof. First we construct a SEGL of $K_{a,3,3}$. By Lemma 32, there is a SEGL of $K_{a,1,3}$. Add two new vertices to the partite set of size one of $K_{a,1,3}$. Let $W = \{w_1, w_2, \ldots, w_{a+3}\}$ be the vertices in partite sets of sizes a and 3. In addition, suppose that the induced labels for vertices w_1 and w_2 are ℓ and $\ell + 1$ for some positive integer ℓ, respectively. Add two new vertices u and v to the partite set of size one and join them to every vertex in W to obtain a $K_{a,3,3}$. The labels we need to assign to the new edges are $\{\pm(2a+1) + i \mid 1 \le i \le a+3\}$.

Label the edge uw_i with $(-1)^i(2a+1+i)$ and the edge vw_i with $(-1)^{i+1}(2a+1+i)$ for $i \in \{1, 2, 3, \ldots, a+3\}$. Note that with this labeling the vertex labels of $K_{a,1,3}$ do not change and $(\ell(u), \ell(v)) = (-2a-\frac{a}{2}-3, 2a+\frac{a}{2}+3)$. To obtain a SEGL of $K_{a,3,3}$, we need to make $\{\ell(u), \ell(v)\} = \{\pm(\frac{a}{2}+3)\}$. Swap the edge labels $2a+1+2$ and $-(2a+1+a+3)$ at u with their opposites, then $(\ell(u), \ell(v)) = (-\frac{a}{2}-1, \frac{a}{2}+1)$. We also swap the edge labels $-(2a+1+3)$ and $2a+1+4$ at u with their opposites, then $(\ell(u), \ell(v)) = (-\frac{a}{2}-3, \frac{a}{2}+3)$, as required.

Now consider the graph $K_{a,3,5}$. By Theorem 2, there is a SEGL of $K_{3,5}$. Apply Lemma 3 to extend this labeling to a SEGL of $K_{2,3,5}$ and then to a SEGL of $K_{4,3,5}$. Now let $a \ge 6$. Extend a SEGL of $K_{a,3,3}$ to a SEGL of $K_{a,3,5}$ in a similar fashion explained above. This completes the proof. □

Lemma 38. Let a be a positive even integer and b a positive odd integer. Then $K_{a,b,3}$ is SEG.

Proof. By Lemma 37, we may assume $b \ge 7$. First let $a < b$. By Theorem 2, there is a SEGL of $K_{b,3}$. Apply Lemma 3 to extend this labeling to a SEGL of $K_{a,b,3}$. Now let $a > b$. If $b \equiv 3 \pmod 4$ we proceed as follows. By Lemma 32, there is a SEGL of $K_{a,1,3}$. Use a $(b-1) \times (a+3)$ array given in Lemma 35 to extend this labeling to a SEGL of $K_{a,b,3}$. Finally, if $b \equiv 1 \pmod 4$, we start with a SEGL of $K_{a,3,3}$, which exists by Lemma 37. Then we employ a $(b-3) \times (a+3)$ array given in Lemma 33 to extend this labeling to a SEGL of $K_{a,b,3}$. □

Theorem 39. Let a, b and c be positive integers, a even, b and c odd, $(a, b, c) \ne (2, 1, 1)$ and $a \ge 4(b+c) - 18$. Then there exists a SEGL of $K_{a,b,c}$.

Proof. By Lemmas 32 and 38 the theorem is true for $c = 1, 3$. Without loss of generality, we may assume $b \ge c \ge 5$. Now since $a \ge 4(b+c) - 18$, by induction on c and Lemma 6, we can extend a SEGL of $K_{a,b,c-2}$ to a SEGL of $K_{a,b,c}$. □

By Theorems 26 and 39, we can state the main theorem of this section.

Theorem 40. The graph $K_{a,b,c}$ is SEG for all even positive integer a and all positive odd integers b and c except for $(a,b,c) = (2,1,1)$.

5 A SEGL of $K_{a,b,c}$, where a, b and c are all odd

In this section we prove that for all positive odd integers a, b and c the complete tripartite graph $K_{a,b,c}$ is SEG.

Lemma 41. There exists a SEGL of $K_{1,b,c}$ for every positive odd integers b and c.

Proof. Without loss of generality, we may assume $b \le c$. By Lemma 31 and the Appendix, the theorem is true for $(b,c) \in \{(1,1),(1,3),(3,3),(1,5),(1,7)\}$. Using the notation in the proof of Lemma 3, the following table displays the labels for the new edges uw_i and vw_i. Now one can easily extend a SEGL of $K_{1,b-2,c}$ to a SEGL of $K_{1,b,c}$, where $(b,c) \in \{(3,5),(5,5),(3,7),(5,7),(7,7)\}$.

$K_{1,1,5}$	Label of uw_i : −6 6 −8 −9 10 11	$\ell(u) = 4$
to $K_{1,3,5}$	Label of vw_i : −7 7 8 9 −10 −11	$\ell(v) = -4$
$K_{1,3,5}$	Label of uw_i : −12 −13 14 15 −16 17	$\ell(u) = 5$
to $K_{1,5,5}$	Label of vw_i : 12 13 −14 −15 16 −17	$\ell(v) = -5$
$K_{1,1,7}$	Label of uw_i : −8 8 −10 −11 12 13 −14 15	$\ell(u) = 5$
to $K_{1,3,7}$	Label of vw_i : −9 9 10 11 −12 −13 14 −15	$\ell(v) = -5$
$K_{1,3,7}$	Label of uw_i : 16 −17 −18 −19 20 −21 22 23	$\ell(u) = 6$
to $K_{1,5,7}$	Label of vw_i : −16 17 18 19 −20 21 −22 −23	$\ell(v) = 6$
$K_{1,5,7}$	Label of uw_i : −24 24 −26 −27 28 −29 30 31	$\ell(u) = 7$
to $K_{1,7,7}$	Label of uw_i : −25 25 26 27 −28 29 −30 −31	$\ell(v) = -7$

Finally, assume $c \ge 9$. By Lemma 31, there is a SEGL of $K_{1,1,c}$. Now by induction on b and Lemma 3 the result follows. □

Theorem 42. Let a, b and c be positive odd integers. Then there exists a SEGL of $K_{a,b,c}$.

Proof. By Lemma 41, there is a SEGL of $K_{1,b,c}$. Now, without loss of generality, we may assume $3 \le a \le b \le c$. First we find a SEGL of $K_{3,3,3}$. By Lemma 41, there is a SEGL of $K_{1,3,3}$. Let the partite sets of $K_{1,3,3}$ be $\{x\}$, $\{w_1, w_2, w_3\}$ and $\{w_4, w_5, w_6\}$. Add two new vertices u and v to the partite set $\{x\}$ and join them to w_i for $1 \le i \le 6$ to obtain a $K_{3,3,3}$. Label the new edges uw_i, $1 \le i \le 6$, with −8, 8, −10, −11, 12 and 13 and

vw_i with -9, 9, 10, 11, -12 and -13. The resulting labeling is a SEGL of $K_{3,3,3}$. Now let $b \geq 3$ and $c \geq 5$. By Lemma 41, there is a SEGL of $K_{1,b,c}$. By induction on a and Lemma 3, there exists a SEGL of $K_{a,b,c}$. □

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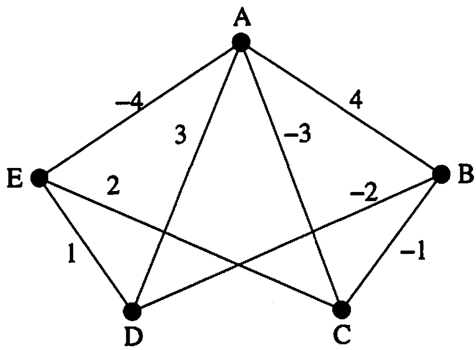
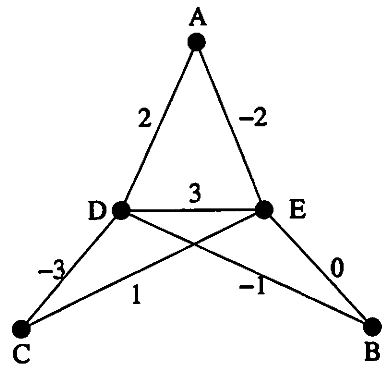
Appendix

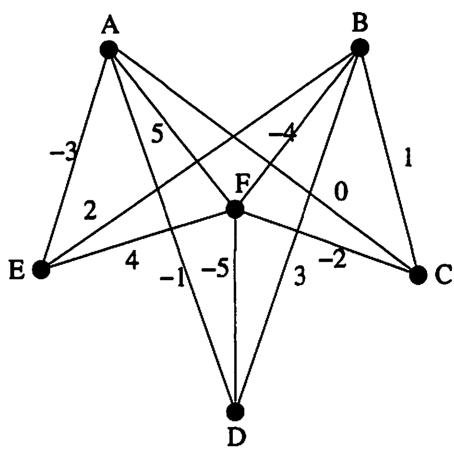
In this appendix we present SEGLs of $K_{1,2,2}$, $K_{1,1,3}$, $K_{1,2,3}$, $K_{1,3,3}$, $K_{2,2,2}$, and $K_{2,2,3}$. These labelings cannot be obtained by the constructions given in the paper.



A SEGL for $K_{1,2,2}$



A SEGL for $K_{1,1,3}$



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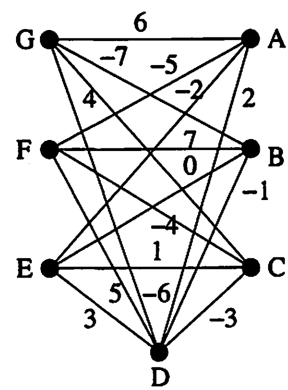
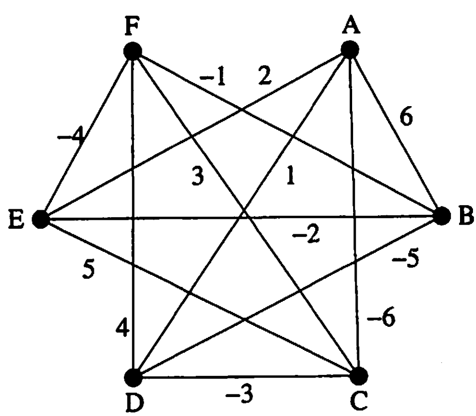
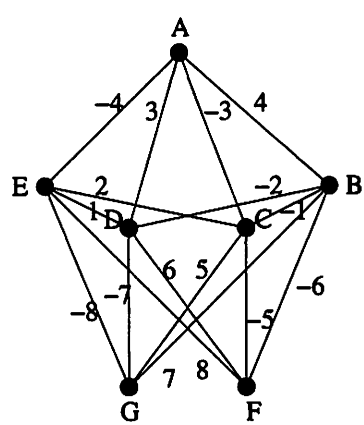


A SEGL for $K_{1,3,3}$



A SEGL for $K_{2,2,2}$



A SEGL for $K_{2,2,3}$