Star graphs: threaded distance trees and E-sets

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Abstract

The distribution of distances in the star graph ST_n, $(1 < n \in \mathbf{Z})$, is established, and subsequently a threaded binary tree is obtained that realizes an orientation of ST_n whose levels are given by the distances to the identity permutation, via a pruning algorithm followed by a threading algorithm. In the process, the distributions of distances of the efficient dominating sets of ST_n are determined.

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1 Introduction

The *star graph* ST_n, $(1 < n \in \mathbf{Z})$, is the Cayley graph of the symmetric group S_n with set of generators $\Theta_n = \{(1\ i),\ i = 2, \dots, n\}$, ([1, 2]). The *weight* of a vertex u of ST_n is its distance to the identity-permutation vertex $12\dots n$. In this work, based on DIMACS Technical Report 2001-05, the weight distributions of certain subsets C of ST_n are determined, including that of ST_n itself. Theorems 8 and 6 below attain these objectives. (A variation of Theorem 6 was obtained in a different fashion in [6]).

An independent set C of vertices in a graph is an efficient dominating set [4], or E-set [3], or 1-perfect codes [5], if each vertex not in C is adjacent to exactly one vertex of C. In Section 5, we determine the weight distributions of these E-sets; see Theorem 8 and subsequent remark. In obtaining this, we use a binary directed tree $\Lambda_n = \Lambda(ST_n)$ whose arcs are of two types: (1) horizontal, left-to-right, arcs; (2) vertical, top-to-bottom, arcs, (as in the subsequent figures). In Section 6, we extend Λ_n to an orientation Γ_n of ST_n, (that is: an oriented graph Γ_n). Moreover, the graphs Γ_n form a nested sequence that converges to a universal graph Γ_∞ associated to the infinite star graph ST_∞.

2 Definition and examples of Λ_n

Let $n > 1$ and let $\Sigma \in S_n$. We write $\Sigma = \sigma_1\sigma_2\dots\sigma_n$, where $\Sigma(i) = \sigma_i$, for $i = 1, 2, \dots, n$. A cycle $(\sigma_{i_1}\sigma_{i_2}\dots\sigma_{i_r})$ of the permutation Σ is given by $\Sigma(\sigma_{i_j}) = \sigma_{i_{j+1}}$, for $j = 1\dots r$, where $j+1$ is taken as 1 if $j = r$. Then, Σ has *length* r. Now, Σ is said to be *proper* if $r > 1$. The *cycle structure* $\Pi(\Sigma)$ of $\Sigma = \sigma_1\sigma_2\ldots\sigma_n$ is defined as the set of proper cycles of by Σ.

Two vertices Σ^1 and Σ^2 of ST_n, with 1 in cycles τ^1 of Σ^1 and τ^2 of Σ^2 of the same length, have a common 1-*invariant cycle structure* if there is $\Phi \in S_n$ with $\Phi(\Sigma^1) = \Sigma^2$ inducing a 1-1 correspondence $\Phi^* : \Pi(\Sigma^1) \to \Pi(\Sigma^2)$ sending τ^1 onto τ^2 and with each $\tau \in \Pi(\Sigma^1)$ and $\Phi(\tau) \in \Pi(\Sigma^2)$ having the same length. We say that Σ^2 has the 1-*invariant cycle structure*, (or 1-ics), of Σ^1. Each vertex u of Λ_n is written

$$\boxed{\begin{array}{c} w(u), c(u) \\ \Sigma(u) \end{array}}$$

where (a) $\Sigma(u) = \sigma_1 \ldots \sigma_{i-1}$ is shorthand for a permutation $\sigma_1\sigma_2\ldots\sigma_n$ of $12\ldots n$ having i as the smallest index in $\{2,\ldots,n\}$ satisfying $\sigma_j = j$, for $i \le j \le n$, and $\sigma_j \neq j$ for $1 < j < i$; (b) $w(u)$ is the weight of $\Sigma(u)$; (c) $c(u)$ is the cardinality of the set $S(u)$ of permutations having the 1-ics $\Pi(u)$ of $\Sigma(u)$.

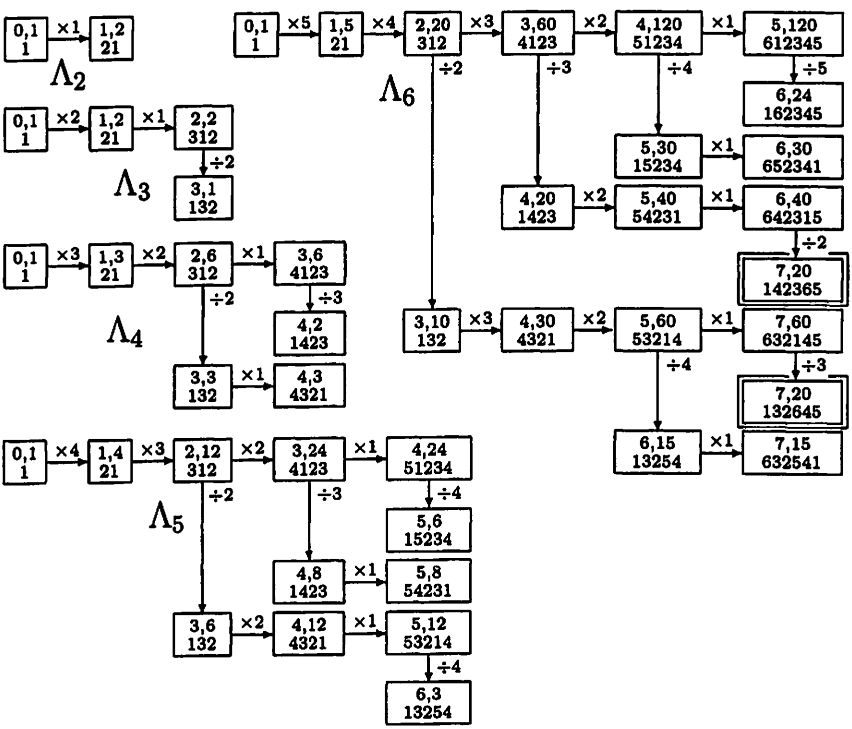

Figure 1: Representations of Λ_n, for $n = 2,3,4,5,6$.

The *string length* $\lambda(\Sigma(u))$ of u is defined as the number of entries ($\le n$) of $\Sigma(u)$. Given an arc e of Λ_n, let u_e and u^e be the tail and the head of e, respectively. The two types of arcs in Λ_n are selected as follows: **(1)** arcs e with $\lambda(\Sigma(u^e)) = 1 + \lambda(\Sigma(u_e))$, as shown in Figures 1–3, indicated with a multiplicative operator $\times m_e$, where $c(u^e) = c(u_e) \times m_e$, noticing that $\sigma_1(u^e) \neq 1$; **(2)** the remaining arcs f, indicated with a divisive operator $\div d_f$ determined by $c(u^f) = c(u_f) \div d_f$, noticing that $\sigma_1(u^f) = 1$ and that there is not an arc e of type (1) with $u_e = u^f$ and $u^e = u_f$.

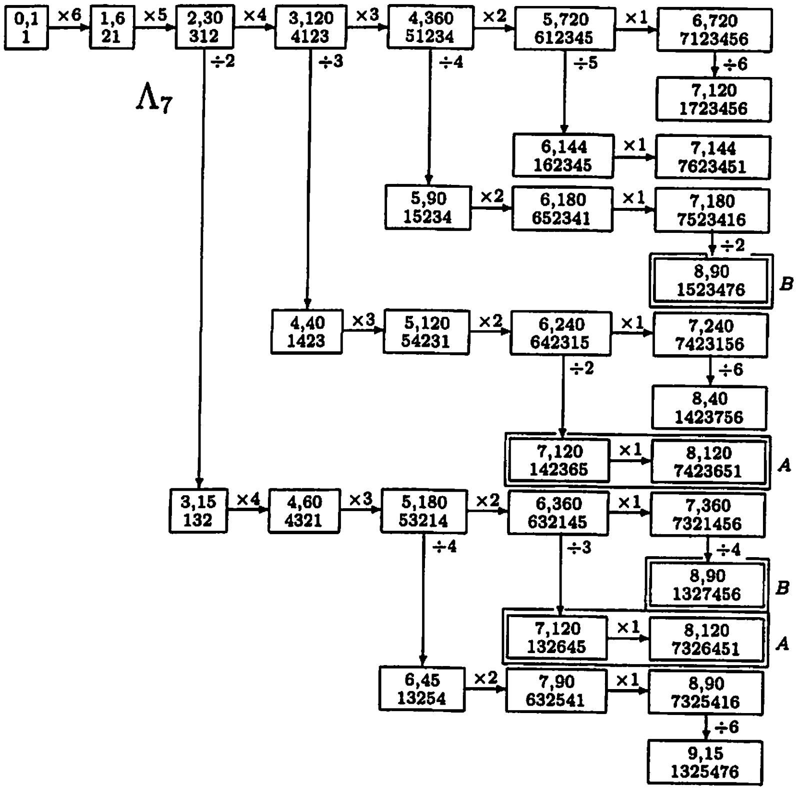

Figure 2: Representation of Λ_7.

An additional requirement in the definition of Λ_n is that it is a rooted tree; its root is denoted $u_0 = u_0^n$, with $w(u_0) = 0$, $c(u_0) = 1$ and $\Sigma(u_0) = 1$.

Given a maximal horizontal directed path, (or mhdp), P of Λ_n, the *depth* of P is the number of vertical arcs of Λ_n preceding P from u_0.

Examples. Figures 1–3 contain the representations of Λ_n for $n = 2, \ldots, 8$ (with the root of Λ_8 in Figure 3 squeezed on the bottom left), where pairs of encased mhdp's U_I, V_I, either improper, (i.e. consisting of one vertex), or proper, and indicated with a common capital letter $I = A, B, \ldots$ on their right, have corresponding vertex sets $\{u_j^I\}, \{v_j^I\}$ representing each a complete set of permutations with a common 1-ics, and thus having a common cardinality.

In fact, to determine the weight distribution of ST_n, the Pruning Algorithm of section 3 below will leave only one of these encased mpdh's with a common

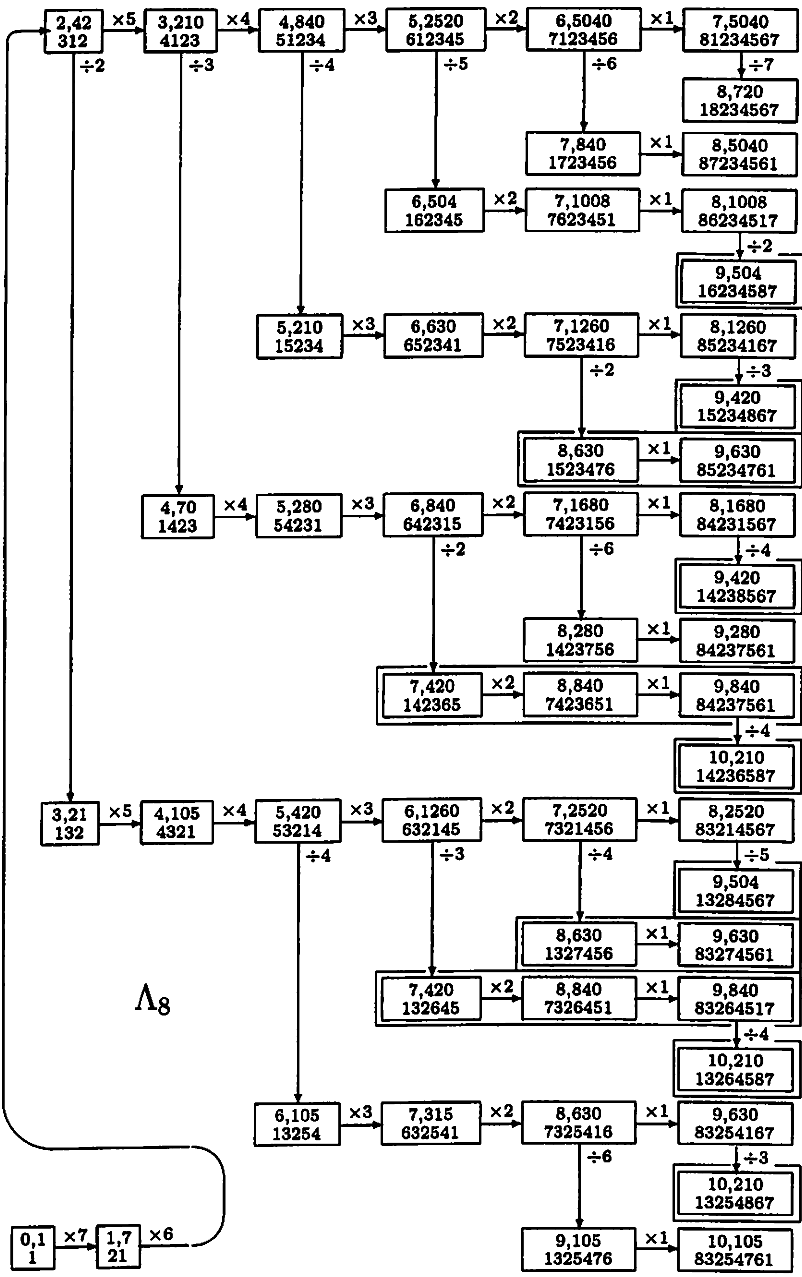

Figure 3: Representation of Λ_8.

capital letter I, provided a denomination $u_{i_0 i_1 \ldots i_{(j-2)} i_{(j-1)}}$ for each vertex of Λ_{n+1} is given via the following inductive definition of Axiom (j), for $j = 0, 1 \ldots \lfloor n/2 \rfloor$, and exemplified in Figure 4, showing the strings $i_0 i_1 \ldots i_{(j-2)} i_{(j-1)}$ in those denominations, for the vertices of Λ_8 in their positions in Figure 3. **Axiom (0):** there is an mhdp $u_0 u_1 \ldots u_n$ of depth 0 in Λ_{n+1}. **Axiom (j):** for each $u_{i_0 i_1 \ldots i_{(j-2)} i_{(j-1)}}$ as in property $(j-1)$ with $i_{j-2} + 1 < i_{j-1}$, there is a vertical arc $u_{i_0 i_1 \ldots i_{(j-1)}} u_{i_0 i_1 \ldots i_{(j-1)} i_{(j-1)}}$ and an mhdp from $u_{i_0 i_1 \ldots i_{(j-1)} i_{(j-1)}}$ to $u_{i_0 i_1 \ldots i_{j-1} n}$ whose depth is j in Λ_{n+1}.

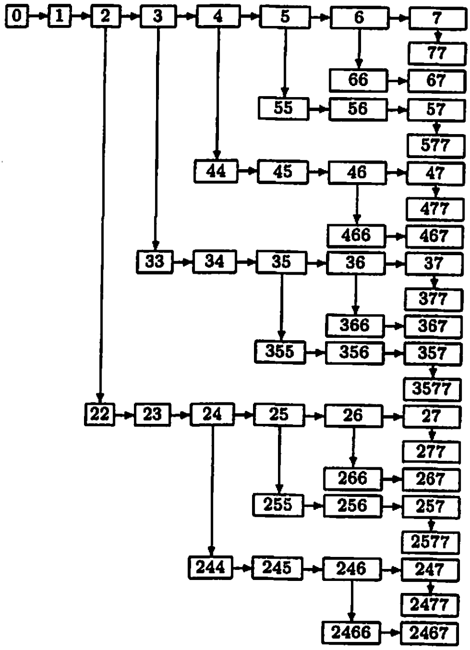

Figure 4: Index string representation of Λ_8.

3 Redefinition and pruning of Λ_n

If a vertex u of Λ_n is the tail of a horizontal (vertical) arc e indicated $\times m_e$ $(\div d_e)$ then we write $m_u = m_e$ $(d_u = d_e)$. If u is not the tail of an arc, then we write $m_u = 0$, $(d_u = 0)$. The following redefinition allows to consider Λ_n as a subdigraph of Λ_{n+1} for any n, so the limit Λ_∞ of the nested sequence $\{\Lambda_n; n > 0\}$ of binary directed trees makes sense: replace the indication $\times m_u = \times m_e$ of the horizontal arc e of Λ_n departing from a tail u of an arc in Λ_n by $\bullet \ell_u$, where $\ell_u = n - m_u$ and $\bullet$ is the operation given by $c(u) \bullet \ell_u = c(u) \times (n - m_u)$; let $\Lambda_n^\bullet$ be the resulting indicated digraph; redefine $\Lambda_n = \Lambda_n^\bullet$. The new indications for the horizontal arcs allow now the containment of indicated subdigraphs.

Incidentally, the cardinalities of the set of vertices of the resulting Λ_∞ that are tails of arcs indicated $\bullet_i$ form a Fibonacci sequence according to the increasing values of $i = 0, 1, \ldots$. This is apparent from the number of vertices in the successive columns from left to right, in the representations of the Λ_n's as in Figures 1–3, for increasing values of $n = 2, 3, \ldots$.

Pruning Algorithm. The vertices $u_{i_0 i_1 \ldots i_j}$ of Λ_n are treated first in the increasing order of their string lengths $j+1$ and then, for each fixed string length $j+1$, in the lexicographical order of their subindex strings $i_0 i_1 \ldots i_j$, namely:

$$u_0, u_1, \ldots, u_{n-1}, u_{2,2}, u_{2,3}, \ldots, u_{2,n-1}, u_{3,3}, \ldots,$$
$$u_{n-1,n-1}, u_{2,4,4}, \ldots, u_{2,4,6,6}, \ldots$$

Each such a vertex $u = u_{i_0 i_1 \ldots i_j}$ has the following fields associated with it: (1) the notation $u = u_i^w = u_{i_0 i_1 \ldots u_j}^{w(u)}$, where $w(u) = w(\Sigma(u))$; (2) the notation $\Sigma(u)$ of the corresponding permutation of $\{1, \ldots n\}$ associated to u; (3) the 1-ics $\Pi(u)$ of $\Sigma(u)$; (4) the number $\ell_u = n - m_u$; (5-7) either a blank in each of the three cases (5), (6) and (7), if u is the first or second vertex of an mhdp, or: (5) the notation $\Sigma[u]$ of the permutation obtained from $\Sigma(u)$ by permuting σ_1 and $\sigma_k = 1$, $(k \neq 1)$; (6) the 1-ics $\Pi[u]$ of $\Sigma[u]$; (7) a tuple $C(u) = s_1, \ldots, s_h$ composed by the orders s_j of the cycles composing $\Pi[u]$; (8) the number d_u expressed as a product $b_u a_u$, where (8a) $a_u = i_j - i_{j-1}$ under the convention $i_{-1} = 0$ and (8b) $b_u \neq 0$ if the value of item (7) above is not a blank and the resulting tuple $C(u)$ was not present in previously treated vertices of Λ_n; $b_u = 0$, otherwise.

The Pruning Algorithm consists in determining these fields for the vertices of Λ_n, in the prescribed order. This allows a partial reconstruction of Λ_n in the form of the maximal subdigraph Λ'_n, which accepts and copies all the vertices and arcs of Λ_n into Λ'_n except for one case: If $b_u = 0$ and there is a vertical arc e whose tail is u, then e is not copied from Λ_n into Λ'_n; this is interpreted as the pruning of e and descendant vertices and arcs (performed to avoid repetitions of mhdp's, as in the encased mpdh's having a common capital-letter indication at their right in Figures 1–3). □

Let ρ_n be the relation defined on the vertex set of ST_n by $u \rho_n v$ if and only if u and v represent permutations with a common 1-ics. The following second redefinition of Λ_n allows to have its vertex set in bijective correspondence with the family of equivalence classes of ST_n under ρ_n, which in turn allows to use Λ_n in computing the weight distribution of ST_n: perform the Pruning Algorithm of Λ_n, whose output is a maximal subdigraph Λ'_n of Λ_n in which there are not pairs of mhdp's $v_0 v_1 \ldots v_s$ and $v'_0 v'_1 \ldots v'_s$ of the same string length s with corresponding vertices v_i and v'_i having common 1-ics $\Pi(v_i) = \Pi(v'_i)$, for $i = 0, 1, \ldots, s$; redefine $\Lambda_n = \Lambda'_n$. We still have Λ_n as a subdigraph of Λ_{n+1} for every n, so a Λ_∞ persists.

Example. The algorithm yields the list $\mathcal{P}_9$ for $n = 9$, (commas are deleted in subindices i of $u_i^w = u(i, w)$ in item 1 and in the tuples $C(u)$ in item 7; the $\Pi(u)$ and $\Pi[u]$ are shown to the right of their corresponding $\Sigma(u)$ and $\Sigma[u]$):

$u(i,w)$	$\Sigma(u)\Pi(u)$	ℓ_u	$\Sigma[u]\Pi[u]$	$C(u)$	$b_u a_u$
u(0,0)	1				00
u(1,1)	21(12)	2		1	01
u(2,2)	312(132)	3	132(32)	2	12
u(3,3)	4123(1432)	4	1423(432)	3	13
u(4,4)	51234(15432)	5	15234(5432)	4	14
u(5,5)	612345(165432)	6	162345(65432)	5	15
u(6,6)	7123456(1765432)	7	1723456(765432)	6	16
u(7,7)	81234567(18765432)	8	18234567(8765432)	7	17
u(8,8)	912345678(198765432)	9	192345678(98765432)	8	18
u(22,3)	132(23)	3			00
u(23,4)	4321(14.32)	4			01
u(24,5)	53214(154.32)	5	13254(54.32)	22	22
u(25,6)	632145(1654.32)	6	132645(654.32)	23	13
u(26,7)	7321456(17654.32)	7	1327456(7654.32)	24	14
u(27,8)	83214567(187654.32)	8	13284567(87654.32)	25	15
u(28,9)	932145678(1987654.32)	9	132945678(987654.32)	26	16
u(33,4)	1423(432)	4			00
u(34,5)	54231(15.432)	5			01
u(35,6)	642315(165.432)	6	142365(65.432)	32	02
u(36,7)	7423156(1765.432)	7	1423756(765.432)	33	23
u(37,8)	84231567(18765.432)	8	14238567(8765.432)	34	14
u(38,9)	942315678(198765.432)	9	142395678(98765.432)	35	15
u(44,5)	15234(5432)	5			00
u(45,6)	652341(16.5432)	6			01
u(46,7)	7523416(176.5432)	7	1523476(76.5432)	42	02
u(47,8)	85234167(1876.5432)	8	15234867(876.5432)	43	03
u(48,9)	952341678(19876.5432)	9	152349678(9876.5432)	44	24
u(55,6)	162345(65432)	6			00
u(56,7)	7623451(17.65432)	7			01
u(57,8)	86234517(187.65432)	8	16234587(87.65432)	52	02
u(58,9)	962345178(1987.65432)	9	162345978(987.65432)	53	03
u(66,7)	1723456(765432)	7			00
u(67,8)	87234561(18.765432)	8			01
u(68,9)	972345618(198.765432)	9	172345698(98.765432)	62	02
u(77,8)	18234567(8765432)	8			00
u(78,9)	982345671(19.8765432)	9			01
u(88,9)	192345678(98765432)	9			00
u(244,6)	13254(54.32)	5			00
u(245,7)	632541(16.54.32)	6			01
u(246,8)	7325416(176.54.32)	7	1325476(23.45.67)	222	32
u(247,9)	83254167(1876.54.32)	8	13254867(23.45.687)	223	13
u(248,10)	932541678(19876.54.32)	9	132549678(23.45.6987)	224	14
u(255,7)	132645(654.32)	6			00
u(256,8)	7326451(17.654.32)	7			01
u(257,9)	83264517(187.654.32)	8	13264587(23.465.78)	232	02
u(258,10)	932645178(1987.654.32)	9	132645978(23.465.798)	233	13
u(266,8)	1327456(7654.32)	7			00
u(267,9)	83274561(18.7654.32)	8			01
u(268,10)	932745618(198.7654.32)	9	132745698(23.4765.89)	242	02
u(277,9)	13284567(87654.32)	8			00
u(278,10)	932845671(19.87654.32)	9			01
u(288,10)	132945678(987654.32)	9			00
u(366,8)	1423756(765.432)	7			00
u(367,9)	84237561(18.765.432)	8			01
u(368,10)	942375618(198.765.432)	9	142375698(243.576.89)	332	02
u(377,9)	14238567(8765.432)	8			00
u(378,10)	942385671(19.8765.432)	9			01
u(388,10)	142395678(98765.432)	9			00
u(488,10)	152349678(9876.5432)	9			00
u(2466,9)	1325476(76.54.32)	7			00
u(2467,10)	83254761(18.76.54.32)	8			01
u(2468,11)	932547618(198.76.54.32)	9	132547698(23.45.67.89)	2222	42
u(2477,10)	13254867(876.54.32)	8			00
u(2478,11)	932548671(19.876.54.32)	9			01
u(2488,11)	132549678(9876.54.32)	9			00
u(2588,11)	132645978(987.654.32)	9			00
u(24688,12)	u(132547698(98.76.54.32)	9			00

This list $\mathcal{P}_n$ generalizes to the patterns expressed in the following theorem. For $u = u_{i_0 i_1 \ldots i_j}$ in Λ_n, let $\ell_u = \ell_{i_0 i_1 \ldots i_j}$, etc.

Theorem 1 *Let $i_{-1} = 0$ and let $t_k = i_k - i_{k-1}$, for $k = 0, 1, \ldots, j-1$. Then:* **(1)** *the 1-ics $C(u)$ in the penultimate field of the line associated to a vertex $u = u_{i_0 i_1 \ldots i_j}$ in $\mathcal{P}_n$ is of the form $t_0, t_1, \ldots, t_j$, where the order of the integers t_k is irrelevant;* **(2)** *the vertices $u_{i_0 i_1 \ldots i_j}$ of Λ_n, (remaining after applying the Pruning Algorithm), have subindex strings $i_0 i_1 \ldots i_j$ completely determined by the following conditions:*

(a) $0 \leq i_0 \leq n-1$; (b) *if* $j > 0$, *then* $2 \leq i_0$;
(c) $t_k \leq t_{k+1}$, *for* $k = 0, \ldots, j-2$; (d) $i_{j-1} \leq i_j$;

(3) *the weight $w(u)$ of a vertex $u = u_{i_0 i_1 \ldots i_j}$ of Λ_n is $w(u) = w(u_{i_0 i_1 \ldots i_j}) = i_j + j$;* **(4)** *the number ℓ_u associated to a vertex $u_{i_0 i_1 \ldots i_j}$ of Λ_n is $\ell_u = \ell_{i_0 i_1 \ldots i_j} = i_j + 1$; thus, the corresponding multiplicative factor m_u is $m_u = m_{i_0 i_1 \ldots i_j} = n - i_j - 1$;* **(5)** *the divisive-operator number $d_u = b_u . a_u$ has $a_u = t_j$; moreover, $b_u > 0$ if and only if either $j = 0$ and $i_0 > 1$ or $j > 0$ and $2 \leq i_0 \leq t_1 \leq t_2 \leq \ldots \leq t_j$; furthermore, if $b_u > 0$, then $b_u = 1$, unless $i_0 = t_1 = t_2 = \ldots = t_j$, in which case $b_u = j + 1$.*

4 The weight distribution of ST_n

To compute the weight distribution of ST_n, a table $\mathcal{T}_n$ constructed from the resulting pruned version of Λ_n and satisfying the following additional conditions will be used: **(a)** the subindex strings $i_0 i_1 \ldots i_j$ of the vertices $u_{i_0 i_1 \ldots i_j}$ of Λ_n are distributed on columns according to their weights; **(b)** each row is to contain the subindex strings of the vertices of an mhdp P of Λ_n, given from left to right according to the orientation of P; **(c)** each mhdp is presented in lexicographical order in its containing row; **(d)** the rows of each complete set of common-depth mhdp's are presented contiguously and in the decreasing order of their path lengths, thus forming upper triangular matrices, because of item (a), above; **(e)** these upper triangular matrices are given from top to bottom in the increasing order of their depths.

Example. $\mathcal{T}_{11}$ is as follows, where $a = 10$ and $b = 11$, vertices with $j = 2$ and $i_0 = 3$ previous to 366 do not appear since they were pruned, and one additional row should be added for the 15-th column, containing solely the string $2468aa$, (which, for insufficient margin, remained excluded):

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
0	1	2	3	4	5	6	7	8	9	a				
			22	23	24	25	26	27	28	29	$2a$			
				33	34	35	36	37	38	39	$3a$			
					44	45	46	47	48	49	$4a$			
						55	56	57	58	59	$5a$			
							66	67	68	69	$6a$			
								77	78	79	$7a$			
									88	89	$8a$			
										99	$9a$			
											aa			

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
						244	245	246	247	248	249	24a		
							255	256	257	258	259	25a		
								266	267	268	269	26a		
									277	278	279	27a		
										288	289	28a		
											299	29a		
												2aa		
								366	367	368	369	36a		
									377	378	379	37a		
										388	389	38a		
											399	39a		
												3aa		
										488	489	48a		
											499	49a		
												4aa		
												5aa		
									2466	2467	2468	2469	246a	
										2477	2478	2479	247a	
											2488	2489	248a	
												2499	249a	
													24aa	
											2588	2589	258a	
												2599	259a	
													25aa	
													26aa	
												3699	369a	
													36aa	
												24688	24689	2468a
													24699	2469a
														246aa
														247aa

Each vertex $u = u_{i_0 i_1 \ldots i_j} = i_0 i_1 \ldots i_j$ of Λ_n, reachable from $u_0 = 0$ by a path P, has associated cardinality $c(u) = M/(A.B)$, where: **(a)** M, (respectively A), is the product of the numbers $m_{i_0 i_1 \ldots i_j} = n - i_j - 1$, (resp. $a_{i_0 i_1 \ldots i_j} = t_j = i_i - i_{j-1}$), of all tails $i_0 i_1 \ldots i_j$ of horizontal, (resp. vertical), arcs in P; **(b)** B is the product of all the numbers $b_{i_0 i_1 \ldots i_j}$ of tails $i_0 i_1 \ldots i_j$ of vertical arcs in P with $i_0 = t_1 = \ldots = t_j$.

A procedure to compute the path from u_0 to any given vertex u of Λ_n, performed by going backwards from $i_0 i_1 \ldots i_j$ to 0 by means of table $\mathcal{T}_n$, consists of the following steps: **(1)** set $u = i_0 i_1 \ldots i_j$; **(2)** if u is not the first vertex of an mhdp, then go backwards through the vertices of the mhdp containing $i_0 i_1 \ldots i_j$; **(3)** once arrived to the first vertex v of an mhdp, or in the case that $u = v$ is such a first vertex, consider its vertical predecessor, that is the tail z of the vertical arc in Λ_n with head v, (which is in the column previous to that containing v); **(4)** set $u = z$ and repeat item (2); **(5)** continue until vertex 0 is reached.

Example. Let $i_0 i_1 \ldots i_j = 2468aa$ be the vertex of Λ_{11} whose weight is 15, (the one left out of the encased table above). This is the first (and only) vertex of its (improper) mhdp. Its vertical predecessor, in column 14, is $2468a$. This is preceded horizontally by 24689 and this by 24688, in respective columns 13 and 12. The vertical predecessor of 24688 is 2468, in column 11, preceded horizontally by 2467 and this by 2466, in respective columns 10 and 9. The vertical predecessor of 2466 is 246, in column 8, preceded horizontally by 245 and this by 244, in respective columns 7 and 6. The vertical predecessor of 244 is 24, in column 5, preceded horizontally by 23 and this by 22, in respective columns 4 and 3. The vertical predecessor of 22 is 2, in column 2, preceded horizontally by 1 and this by $0 = u_0$, in respective columns 1 and 0. Thus we get the following path, with commas replaced by superindices m_u, for horizontal-arc tails u, and subindices d_u, for vertical-arc tails u, respectively:

$$0^{10}1^{9}2_{2}22^{8}23^{7}24_{4}244^{6}245^{5}246_{6}2466^{4}2467^{3}2468_{8}24688^{2}24689^{1}2468a_{10}2468aa.$$

We arrive at $c(2468aa) = 9 \times 7 \times 5 \times 3$.

This generalizes to the following statement.

Theorem 2 *If $n = 2k+1$, then the paths realizing the diameter $D(ST_n)$ of ST_n and starting at $12\ldots n$ end up at exactly $(n-2)(n-4)\ldots 3$ vertices u of the form $\Sigma(u) = \sigma_1\sigma_2\ldots\sigma_n$, with $\sigma_1 = 1$ and $\Pi(u)$ expressible as a product of $k+1$ independent transpositions.*

A string $i_0i_1\ldots i_j$ is said to be admissible if $u_{i_0i_1\ldots i_j}$ is a vertex of Λ_∞. Given a positive integer $\omega \leq D(ST_n)$, we want first to find an expression for the cardinality of the set V_ω of vertices of Λ_∞ having ω as their weight in $ST_{\omega+1}$. Toward this end, we start exemplifying some sequences of admissible strings for lower values of ω, where subindex strings $i_0i_1\ldots u_j$ of vertices $u_{i_0i_1\ldots u_j}$ are expressed in a suitable order without commas and employing the following shorthand dot-notation rule for certain subsequences: let $i_0i_1\ldots i_{k-1}i_k.i_{k+1}\ldots i_{j-1}i_j$ stand for the subsequence composed by all the admissible strings $i_0i_1\ldots i_{k-1}\iota_k.\iota_{k+1}\ldots\iota_{j-1}i_j$ in Λ_n with $\iota_\ell \geq i_\ell$, for $k \leq \ell < j$.

Examples. Some subsequences of admissible strings in Λ_∞ are:

2.2	=	$\{22\}$	$2.i_1$	=	$\{2i_1,3i_1,\ldots,i_1i_1\}$	$i_1>2$
24.4	=	$\{244\}$	$24.i_2$	=	$\{24i_2,25i_2,\ldots,2i_2i_2\}$	$i_2>4$
36.6	=	$\{366\}$	$36.i_2$	=	$\{36i_2,37i_2,\ldots,3i_2i_2\}$	$i_2>6$
246.6	=	$\{2466\}$	$246.i_3$	=	$\{246i_3,247i_3,\ldots,24i_3i_3\}$	$i_3>6$
369.9	=	$\{3699\}$	$369.i_3$	=	$\{369i_3,36ai_3,\ldots,36i_3i_3\}$,	$i_3>9$

For $\omega = 0,1,\ldots,15 = f$ we can express V_ω as follows, where hexadecimal notation is used:

V_0={0}						V_3={3,	2.2}
V_1={1}						V_4={4,	2.3}
V_2={2}						V_5={5,	2.4}
V_6={6,	2.5,	24.4}					
V_7={7,	2.6,	24.5}					
V_8={8,	2.7,	24.6,	36.6}				
V_9={9,	2.8,	24.7,	36.7,				
		246.6}					
V_a={a,	2.9,	24.8,	36.8,	48.8,			
		246.7,	257.7}				
V_b={b,	2.a,	24.9,	36.9,	48.9,			
		246.8,	257.8,	268.8}			
V_c={c,	2.b,	24.a,	36.a,	48.a,	5a.a,		
		246.9,	257.9,	268.9,	279.9,		
		369.9					
		2468.8}					
V_d={d,	2.c,	24.b,	36.b,	48.b,	5a.b,		
		246.a,	257.a,	268.a,	279.a,	28a.a,	
		369.a,	37a.a,				
		2468.9,	2579.9}				
V_e={e,	2.d,	24.c,	36.c,	48.c,	5a.c,	6c.c,	
		246.b,	257.b,	268.b,	279.b,	28a.b, 29b.b,	
		369.b,	37a.b,	38b.b,			
		2468.a,	2579.a,	268a.a}			
V_f={f,	2.e,	24.d,	36.d,	48.d,	5a.d,	6c.d,	
		246.c,	257.c,	268.c,	279.c,	28a.c,29b.c, 2ac.c,	
		369.c,	37a.c,	38b.c,	39c.c		
		2468.b,	2579.b,	268a.b,	279b.b,		
		2468a.a}					

The ten last V_ω here are expressible as:

$V_6=\{6,$	$2.5,$	$2.44\}$	$V_9=\{9,$	$2.8,$	$2.47,$	$2.466\}$
$V_7=\{7,$	$2.6,$	$2.45\}$	$V_a=\{a,$	$2.9,$	$2.48,$	$2.467\}$
$V_8=\{8,$	$2.7,$	$2.46\}$	$V_b=\{b,$	$2.a,$	$2.49,$	$2.468\}$

$V_c=\{c,$	$2.b,$	$2.4a,$	$2.469,$	$2.4688\}$	
$V_d=\{d,$	$2.c,$	$2.4b,$	$2.46a,$	$2.4689\}$	
$V_e=\{e,$	$2.d,$	$2.4c,$	$2.46b,$	$2.468a\}$	
$V_f=\{f,$	$2.e,$	$2.4d,$	$2.46c,$	$2.468b,$	$2.468aa\}$

Let $V_\omega^{i_0}$ be the subset of strings of V_ω starting at i_0. We draw the following conclusions, where the dot-notation rule is used: **(1)** $\lambda = 1$ happens in V_ω just for each subsequence $\omega \geq 0$, and only in V_ω^1; **(2)** $\lambda = 2$ happens in V_ω for the members of $2.(\omega - 1)$, where $\omega \geq 3$, and only in V_ω^2; **(3)** $\lambda = 3$ happens in V_ω: **(a)** for the members of $24.(\omega - 2)$, where $\omega \geq 6$, and only in V_ω^2; **(b)** for the members of $36.(\omega - 2)$, where $\omega \geq 8$, and only in V_ω^3; ... **(z)** for the members of $k(2k).(\omega - 2)$, where $\omega \geq 2(k+1)$, and only in V_ω^k, $(k \geq 2)$; **(4)** $\lambda = 4$ happens in V_ω: **(a)** for the members of $246.(\omega - 3)$ where $\omega \geq 9$, and only in V_ω^2; **(b)** for the members of $369.(\omega - 3)$ where $\omega \geq 12$, and only in V_ω^3; ... **(z)** for the members of $k(2k)(3k).(\omega - 3)$, where $\omega \geq 3(k+1)$, and only in V_ω^k, $(k \geq 2)$. The following result is obtained.

Theorem 3 **(a)** *$\lambda = 1$ happens in V_ω, and only for the strings of V_ω starting at i_0;* **(b)** *for each $k \geq 2$, any fixed $\lambda > 1$ happens in V_ω for the members of $k(2k)(3k)\ldots((\lambda-1)k).(\omega - \lambda + 1)$, where $\omega \geq (\lambda - 1)(k+1)$, and only for the subsets V_ω^k.*

Let $W_\omega^k \subseteq V_\omega$ consist of the strings of length $\lambda = k$ in the statement of Theorem 3. Then $|W_\omega^1| = 1$ and $|W_\omega^k| = 0$ whenever $\omega < 3k$, for $k \geq 2$. Moreover, if $S_j^0 = 1$ and $S_j^h = \sum_{k=1}^{j} S_k^{h-1}$, $(h > 0)$, for every $j \geq 1$, so $S_j^h - S_{j-1}^h + S_j^{h-1}$ for $h > 0$ and $j > 1$, then

$$S_j^h = \binom{j+h-1}{h}, \qquad |W_1^k| = S_1^k = k \quad \text{and in general} \tag{1}$$

$$|W_\omega^k| = \sum_{i=0}^{\lfloor\frac{\omega}{k+1}\rfloor} S^k_{\omega - i(k+1)} = \sum_{i=0}^{\lfloor\frac{\omega}{k+1}\rfloor} \binom{\omega - ik - i + k - 1}{k}, \tag{2}$$

for every weight ω valid in $ST_{\omega+1}$ and every string length k.

Theorem 4 *For $0 < \omega \in \mathbf{Z}$, the number of vertices of $ST_{\omega+1}$ having weight ω is given by the finite sum $|V_\omega| = |W_\omega^1| + |W_\omega^2| + \ldots + |W_\omega^k| + \ldots$.*

It is easy to establish the following expression for the diameter $D(n) = D(ST_n)$ of ST_n.

Proposition 5 *The diameter of ST_n is $D(n) = \lfloor\frac{n-1}{2}\rfloor + n - 1$.*

Let $V_\omega(n)$ be the set of vertices of Λ_n having weight ω. Let $W_\omega^k(n)$ be the subset of admissible strings corresponding to vertices of $V_\omega(n)$ whose length λ is equal to k. Then, from the tables $\mathcal{T}_n$ we get:

$$|W_\omega^k(n)| = |W_\omega^k|, \ (0 \leq k < n); \tag{3}$$

$$|W_\omega^k(n)| = |W_\omega^k| - \sum_{j=0}^{k-n} |W_\omega^j|, \ (n \leq k \leq D(n)). \tag{4}$$

The main result of the section follows.

Theorem 6 *The cardinality of the set of vertices of ST_n having weight ω is*

$$|V_\omega(n)| = |W_\omega^0(n)| + |W_\omega^1(n)| + \ldots + |W_\omega^{D(n)}(n)| = \sum_{i=0}^{D(n)} W_\omega^i,$$

where the terms of the displayed sum are obtained by means of equations (1), (2), (3) *and* (4) *presented above.*

Proof. The equations and the statement of the theorem arise naturally from the patterns in the tables $\mathcal{T}_n$ and the previous results. □

5 Weight distributions of E-sets in ST_n

It was proved in [3] that if $1 \leq i \leq n$, then, the vertex subset C_i of ST_n corresponding to the permutations $\sigma_1\sigma_2\ldots\sigma_n$ with a fixed $\sigma_1 = i$ forms an E-set. This is the only way of getting an E-set in ST_n. Furthermore, it can be seen that the E-sets of ST_n form a partition of the vertex set of ST_n.

Having established in Section 4 the distribution of weights of vertices of ST_n, we ask, How does such a distribution restricts to each C_i?

Proposition 7 *The vertices u of Λ_n with $\Sigma(u) = \sigma_1\sigma_2\ldots\sigma_n$ and $\sigma_1 = 1$ represent all the vertices of ST_n with $\sigma_1 = 1$. They have associated admissible strings $i_0i_1\ldots i_{j-1}i_j$ with $i_{j-1} = i_j$.*

Proof. This is clear from the developments above. □

Let $V_\omega^i(n)$ be the set of vertices of C_i having weight ω in ST_n, for $1 \leq i \leq n$.

Theorem 8 *The weight distribution of the subsets C_i of ST_{n+1}, for $2 \leq i \leq n+1$, is given by:*

$|V_0^i(n+1)| = 0;$
$|V_\omega^i(n+1)| = |V_{\omega-1}(n)|,$ *for* $\omega = 1, 2, \ldots, 2\lfloor\frac{D(n+1)}{2}\rfloor;$
$|V_{D(n+1)}(n+1)| = 0,$ *for n even, (only case not covered above)*

Proof. For each $i \in \{2,\ldots,n+1\}$, the permutations $\sigma_1\sigma_2\ldots\sigma_{n+1}$ with $\sigma_i = i$ induce a copy H_i of ST_n in ST_{n+1} containing the identity permutation $12\ldots(n+1)$. Each vertex h of H_i has a unique neighbor h^i in $ST_{n+1}\backslash H_i$. Then the collection of all h^i is C_i, for each $i \in \{2,\ldots n+1\}$ fixed. □

Remark. According to Theorem 8, the n vertex subsets C_i in ST_{n+1} with $1 < i \leq n+1$ have equivalent weight distributions. Thus, by multiplying the quantities obtained in the theorems by n and substracting the results correspondingly from those obtained for ST_{n+1}, the case for C_1 can be obtained, which uses that if n is odd then $|V_{D(n)}| = (n-2)(n-4)\ldots \times 5 \times 3$, by Theorem 2.

6 Threading Λ_n into an orientation of ST_n

We now modify the Pruning Algorithm into a threading algorithm in order to produce an orientation Γ_n of ST_n whose vertices are those of Λ_n (remaining after applying the algorithm) and whose arc set contains the arc set of Λ_n.

The Threading Algorithm consists in running the Pruning Algorithm (on the previously defined Λ_n), checking whether the last field $b_u a_u$ of each line in the table $\mathcal{P}_n$ that is being generated has $b_u = 0$ and $a_u \geq 2$. If this is the case, then a *thread*, meaning a new arc, is added to Λ_n from u to a vertex $\psi(u)$ determined as follows. It happens that the penultimate field $C(u)$ was present in a previous line of $\mathcal{P}_n$ corresponding to the tail $\phi(u)$ of a vertical arc $e(u)$ of Λ_n having head $\psi(u)$. Then $\psi(u)$ is the head of $e(u)$.

Example. Working with $\mathcal{P}_9$, the threads appearing by means of the Threading Algorithm are departing from the vertices u with subindex strings 35, 46, 47, 57, 58, 68, 257, 268, 368, whose values $C(u)$ are respectively 32, 42, 43, 52, 53, 62, 232, 242, 332 and whose fields $b_u a_u = 0a_u$ have $a_u = 2,2,3,2,3,2,2,2,2$, respectively. But the vertices $\phi(u)$ with respective subindex strings 25, 26, 37, 27, 38, 28, 257, 268, 368, have the same corresponding values $C(u)$, presented in $\mathcal{P}_9$ in nondecreasing order: 23, 24, 34, 25, 35, 26, 223, 224, 233, so the corresponding 1-ics's are the same in both cases. We obtain the desired orientation of ST_9 by adding a thread from each one of the eight mentioned vertices respectively into the vertices $\psi(u)$ whose subindex strings are 255, 266, 377, 277, 388, 288, 2477, 2488, 2588, which are the heads of the respective arcs $e(u)$ (that departed from the vertices $\phi(u)$ mentioned above).

Theorem 9 *Any pair $(u,\phi(u))$ appearing during the running of the Threading Algorithm has the vertices u and $\phi(u)$ with $C(u) = C(\phi(u))$, where the order of the elements on each side of the equality is irrelevant. Thus, in the running of the Threading Algorithm, each consideration of a vertex u of Λ_n with $C(u)$ equal to the $C(v)$ of a previously considered vertex $v = \phi(u)$ determines a thread from u onto the corresponding $\psi(u)$.*

Proof. The statement follows from the previous discussion and Theorem 1, item 1. □

Remark. The Threading Algorithm insured by Theorem 9 produces an orientation Γ_n of ST_n whose vertices represent the 1-ics's of the permutations on n elements, that is each vertex of Γ_n represents all the permutations on n elements having a specific 1-ics, and there is a bijective correspondence between the vertices of Λ_n and the 1-ics's of permutations on n elements. Thus Γ_n may be referred to as the 1-ics orientation of ST_n. Each arc of ST_n projects into a specific arc of Γ_n. We still consider that the arcs of Γ_n are 'horizontal' and 'vertical', as in the case of Λ_n, where threads of Γ_n are 'vertical'. Moreover, the vertices and arcs of Γ_n may be considered as preserving the indications they inherit from Λ_n, including the threads, which preserve the indications of the arcs removed by the Pruning Algorithm. As said above, the indications of horizontal arcs are of the form $\bullet\ell_u$, so we still have that the orientations Γ_n form a nested sequence of indicated digraphs and that their limit indicated digraph Γ_∞ is well defined and constitutes a universal graph for this situation. This corresponds to the infinite star graph ST_∞ that can be defined as the Cayley graph of the symmetric group S_∞ with respect to the set of transpositions $\Theta_\infty = \{(1\ i),\ i = 2, \ldots n, \ldots\}$.

Theorem 10 *Γ_n can be interpreted as an orientation of ST_n via the map $\Phi_n : ST_n \to \Lambda_n$ given by $\Phi_n^{-1}(u) = \rho$-equivalence class of $\Sigma(u)$, for each vertex u of Λ_n. Then:* **(1)** *the value $c(u)$ of each vertex u of Γ_n is the cardinality of $\Phi_n^{-1}(u)$ and* **(2)** *the inverse image Φ_n^{-1} of an horizontal, (vertical), arc e of Λ_n is formed by $c(u^e)$, $(c(u_e))$, arcs subdivided into $c(u^e)/m_e$, $(c(u_e)/d_e)$, subsets of m_e, (d_e), arcs incident each to a common corresponding vertex in $\Phi_n^{-1}(u_e)$, $(\Phi_n^{-1}(u^e))$.*

References

[1] S.B. Akers and B. Krishanmurthy, *A group-theoretical model for symmetric interconnection networks*, IEEE Trans. on Comput. **38** (1989), 555–566.

[2] S. Arumugam and R. Kala, *Domination parameters of star graph*, Ars Combin. 44 (1996), 93–96.

[3] I. J. Dejter and O. Serra, *Efficient dominating sets in Cayley graphs*, Discrete Appl. Math. **129** (2003), 319–328.

[4] T. W. Haynes, S. T. Hedetniemi and P. J. Slater, Fundamentals of domination in graphs, M. Dekker Inc., 1998.

[5] J. Kratochvíl, *Perfect codes of graphs*, Jour. Combin. Theory, Ser. B, 40 (1986), 224–228.

[6] L. Wang, S. Subramanian, S. Latifi and P.K. Srimani, *Distance distribution of nodes in star graphs*, Applied Mathematics Letters, **18**(2006), 780–784.