Some remarks on a paper of Chetwynd and Hilton on critical star multigraphs

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Abstract

In [A.G. Chetwynd and A.J.W. Hilton, *Critical star multigraphs*, Graphs and Combinatorics 2 (1986), 209-221] Chetwynd and Hilton started the investigations of the edge-chromatic properties of a particular class of multigraphs, which they called *star multigraphs*. A star multigraph is a multigraph such that there exists a vertex v^* that is incident with each multiple edge. Star multigraphs turn out to be useful tools in the study of the chromatic index of simple graphs. The main goal of this paper is to provide shorter and simpler proofs of all the main theorems contained in the above mentioned paper. Most simplifications are achieved by means of a formula for the chromatic index recently obtained by the author and by a careful use of arguments involving fans.

Keywords: star multigraph, chromatic index, edge-colouring, fans

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1 Introduction

All graphs considered in this paper are loopless, undirected and finite, but may contain multiple edges. The terms "multigraph" and "graph" will have, in this paper, the same meaning. Let G be a graph. The vertex set and edge set of G shall be denoted by $V(G)$ and $E(G)$, respectively. The *degree* of a vertex v in G, denoted by $d_G(v)$, is the number of edges incident with v in G. $\Delta(G)$ denotes the *maximum degree* of G. If u, v are vertices of G, we denote by uv the set of edges joining u and v. The cardinality of uv is denoted by $\mu(uv)$ and called the *multiplicity of* uv. If $\mu(uv) = 1$, we say that uv is a *simple edge* and, if $\mu(uv) > 1$, we say that uv is a *multiple edge.* A graph is *simple* if all its edges are simple. If an edge e joins the vertices u and v, we denote this fact by $e \in uv$, or $e = uv$ when uv is a simple edge. Two edges are *adjacent* if they are distinct and have at least one common endpoint, and *parallel* if they are distinct and have two common endpoints. If S is a set of vertices or edges of G, we denote by $G - S$ the graph obtained from G by deleting all the elements of S, together with the edges incident to any vertex in S if any.

An *edge-colouring* of G is a map $\varphi : E(G) \to C$, where C is a set, called the *colour-set*, whose elements are called *colours*, and φ assigns distinct colours to every pair of adjacent edges. If C is chosen so that $|C|$ is minimum, then φ is called an *optimal colouring* and the integer $|C|$ is called the *chromatic index* of G, denoted by $\chi'(G)$. G is said to be *k-edge-colourable* if $k \geq \chi'(G)$.

Clearly $\chi'(G) \geq \Delta(G)$, since all the edges incident with a vertex of maximum degree must receive a distinct colour. If $\chi'(G) = \Delta(G)$, we say that G is *Class 1* and, otherwise, we say that G is *Class 2.* An edge e of G is called *critical* if $\chi'(G - e) < \chi'(G)$. G itself is called *critical* if it is Class 2, has no isolated vertices, and all its edges are critical. It is well known and easy to see that every Class 2 graph G contains a critical subgraph with the same chromatic index. For an introduction to edge-colouring, and for graph-theoretic notation and terminology, not explicitly introduced here, we refer the reader to Fiorini and Wilson [8].

A celebrated theorem of Vizing [12] is equivalent to the statement that every Class 2 simple graph G satisfies $\chi'(G) = \Delta(G) + 1$. An extension of this theorem (Theorem 2 below) was first proved by Chetwynd and Hilton in [5], where the study of a new class of graphs, known as *star multigraphs,* was begun. A *star multigraph* is a graph G such that there exists a vertex v^* (called a *star centre*) to which all multiple edges of G are incident. Equivalently, G is a star multigraph if there exists a vertex v^* such that $G - v^*$ is a simple graph. Thus a star centre may not be unique, but is unique unless G is a simple graph or there exists only one multiple edge in G. Star multigraphs, as stated in [5], are "vital tools in the investigations of the chromatic index of certain kinds of simple graphs". The underlying philosophy is that, if the chromatic index of a certain simple graph H is unknown, it may be helpful to embed H into a star multigraph G (typically by the addition of a vertex v^* and suitably chosen multiple edges joining v^* to the vertices of H) in the attempt to draw information about H from the knowledge of the edge-colouring properties of G. It was this line of investigation that led Chetwynd and Hilton in [6] to formulate the *Overfull Conjecture*, which is now considered one of the most interesting and difficult conjectures in edge-colouring. Thus, it appears that star multigraphs offer a very fruitful line of investigation.

In [5] Chetwynd and Hilton completely classified star multigraphs with at most two vertices of maximum degree. In [6] they classified certain star multigraphs with three vertices of maximum degree. (To the best of our knowledge the problem of the classification of <u>all</u> the star multigraphs with three vertices of maximum degree remains open to this date.) The proofs of the results contained in [6] are considerably more involved than those in [5], and we shall not be concerned with them in this paper. Instead, we shall provide simplifications to the proofs of all the main results of [5], and sometimes our simplifications will be substantial. We remark that all the proofs contained in [5] and [6] are (in the style of the authors) very precise and, at times, ingenious. However the reading of [5] and [6] can prove to be difficult for those readers not accustomed with edge colouring, in particular for those parts of the proofs where *fans* or the so-called *fan argument* is used. One of our goals will be to provide some conceptual simplifications to the Chetwynd-Hilton proofs. Indeed, having put on a formal basis the theory of fans in [2], we shall use some of the results proved there to express, very succinctly, facts concerning fans in a rigorous way, in the hope to give to the reader a better understanding and to increase the clarity of the material presented. In particular, we shall frequently use an expression for the chromatic index of a Class 2 multigraph which we recently obtained [3, 4] (Lemma 1 below). We call this expression the *Fan Formula*. A similar formula has been discovered independently by Favrholdt et al. [7]. Quite unexpectedly, the Fan Formula gives an exact expression for the chromatic index of a Class 2 graph under very general conditions. Thus the paper may also be viewed as an attempt to show the power and wide applicability of this formula. Before we state it, however, we need to give some technical definitions.

An *e-tense colouring* ϕ of a graph G is a partial edge-colouring of G which assigns no colour to e and whose restriction to $E(G-e)$ is an optimal colouring of $G-e$. The colour set of ϕ is defined to be the colour set of its restriction to $G-e$. The edge e is called the *uncoloured edge*.

Given an e-tense colouring ϕ of G with colour set C, and a vertex $w \in V(G)$, we say that a colour $\alpha \in C$ is *missing* at w (or that w is missing the colour α) if there is no edge, having w as an endpoint, which is assigned the colour α by ϕ.

Let e be an edge of G and let u be an endpoint of e. Let ϕ be an e-tense colouring of G. A *fan* at u with respect to ϕ is a sequence of edges of the form

$$F = [e_0, e_1, e_2, \ldots, e_{k-1}, e_k],$$

where $e_0 = e$, $e_i \in uv_i$, and where the vertex v_i is missing the colour of the edge e_{i+1}, for every $i = 0, 1, \ldots, k-1$. The vertex u is called the *pivot* of the fan. The fan F is said to *terminate* at the edge e_k. A fan is *maximal* if it cannot be extended to a larger fan. An edge f is called a *fan edge* at u if it appears in at least one fan at u. A vertex w is called a *fan vertex* at u if it is joined to u by at least one fan edge. The set[1] of fan vertices is denoted by $V(\mathcal{F})$. If w is a fan vertex at u, we denote by $\mu^*(uw)$ the number of *fan edges* joining u and w, and call $\mu^*(uw)$ the *fan multiplicity of the edge uw*.

We are now ready to state the *Fan Formula* [4, 3].

Lemma 1 *Let G be a Class 2 multigraph and let $e \in uv$ be a critical edge. Let ϕ be a tense colouring with respect to the edge e, and let $V(\mathcal{F})$ be the set of fan vertices at u with respect to ϕ. Then*

$$\chi'(G) = \frac{1}{|V(\mathcal{F})|} \cdot \sum_{w \in V(\mathcal{F})} (\deg_G(w) + \mu^*(uw)) + \frac{|V(\mathcal{F})|-2}{|V(\mathcal{F})|}$$

$$= \left\lceil \frac{1}{|V(\mathcal{F})|} \cdot \sum_{w \in V(\mathcal{F})} (\deg_G(w) + \mu^*(uw)) \right\rceil$$

We shall say that the Fan Formula is written *at u*, to indicate that the pivot of the fans is the vertex u. Notice that $|V(\mathcal{F})| \geq 2$ holds always under the hypotheses of Lemma 1, from which the second equality above follows easily. The Fan Formula is a direct consequence of [2, Theorem 3], which was named "Fan Theorem".

We shall often use the following property, discovered independently by Andersen [1] and Goldberg [9, 10] and implicit in the work of Vizing (see [2, Lemma 2]).

Lemma 2 *Let G be a Class 2 graph, let e be a critical edge and let ϕ be an e-tense colouring of G. Let u be an endpoint of e. Let $V(\mathcal{F})$ be the set of fan vertices at u. Then, for any colour α, there is at most one vertex $x \in V(\mathcal{F}) \cup \{u\}$ which is missing colour α.*

The exposition and the organization of the results of this paper does not necessarily follow the same order of [5]. Instead the paper is organized as follows. Section 2 is dedicated to star multigraphs with one vertex of maximum degree and to the proof that every Class 2 star multigraph G satisfies $\chi'(G) = \Delta(G) + 1$. In Section 3 we completely classify star multigraphs with two vertices of maximum degree. There are some additional results in [5] which we will not prove. Their proof is, in our view, sufficiently simple as it is in the original source, and most of them are corollaries of the results

[1] As the notation suggests, the set of fan vertices is the vertex set of a graph, which is called the Fan Digraph and was introduced and studied in [2]. However, this concept will not be necessary in the present context. We refer the reader to [2] for further details.

proved by us, or can be proved by an easy adaptation of the arguments adopted by us.

2 Star multigraphs with one vertex of maximum degree

The first of our theorems will be stated in terms of list-colouring and is a partial extension of [5, Theorem 1]. We first give some relevant definitions. If G is a graph, an *edge-list-assignment* $\mathcal{L}$ is a function which assigns to each edge e of G a list $L(e)$ of colours (i.e. a set). We say that G is *$\mathcal{L}$-choosable* if it is possible to select a colour from each list and assign it to the corresponding edge in such a way that the resulting colouring is a proper edge-colouring. If all the lists have cardinality k, we call $\mathcal{L}$ a *k-list-assignment.* G is said to be *k-edge-choosable* if, for every k-list-assignment $\mathcal{L}$, G is $\mathcal{L}$-choosable. The minimum integer k for which G is k-edge-choosable is called the *list-chromatic index* of G and denoted by $\chi'_l(G)$. Clearly $\chi'_l(G) \geq \chi'(G)$, as may be seen by taking all the lists to be coincident with a fixed colour-set C, and the well known List-Colouring Conjecture asserts that $\chi'_l(G) = \chi'(G)$ for any graph G.

We are ready to prove our first theorem. We shall use the simple fact that the List-Colouring Conjecture holds for graphs G with $\Delta(G) \leq 2$ and a result of Harris [11] to the effect that, if a simple graph satisfies $\Delta(G) \geq 3$, then $\chi'_l(G) \leq 2\Delta(G) - 2$.

Theorem 1 *Let G be a star multigraph such that every vertex is incident with a multiple edge. Then*

$$\chi'_l(G) = \chi'(G) = \begin{cases} \Delta(G)+1 & \text{if } |V(G)| = 3 \text{ and } |E(G)| = \Delta(G)+1; \\ \Delta(G) & \text{otherwise.} \end{cases}$$

Proof. If $|V(G)| \leq 3$ the statement of the theorem is immediate, so we can assume $|V(G)| \geq 4$. Let v^* be the star centre (which is necessarily unique in this case) and let $V(G) = \{v^*, v_1, v_2, \ldots, v_s\}$, where $s \geq 3$. Let $\mu(v^* v_i) = k_i$ for all $i = 1, 2, \ldots, s$. By assumption, $k_i \geq 2$, so v^* has degree $\Delta(G)$. Let $\mathcal{L} = \{L(e) : e \in E(G)\}$ be a $\Delta(G)$-list-assignment. Choose a distinct colour from each of the lists assigned to the edges incident with v^*, thus obtaining a partial colouring ψ. We now aim to extend ψ to the edges of the simple graph $H = G - v^*$. Consider the list-assignment $\mathcal{L}_1 = \{L_1(e) : e \in E(H)\}$, where, if $e = v_i v_j$, $L_1(e)$ is obtained from $L(e)$ by suppressing the colours assigned by ψ to the edges of the form $v^* v_i$ and

v^*v_j. Clearly G is $\mathcal{L}$-choosable if H is $\mathcal{L}_1$-choosable. By construction,

$$|L_1(e)| \geq |L(e)| - k_i - k_j = \Delta(G) - k_i - k_j = \sum_{\ell \neq i,j} k_\ell \geq 2(s-2) \geq 2\Delta(H) - 2, \tag{1}$$

where we have used the fact that $d_G(v^*) = \sum_{\ell=1}^{s} k_\ell = \Delta(G)$ and $s = |V(H)| \geq \Delta(H) + 1$. By Harris' theorem and (1), H is $\mathcal{L}_1$-choosable as long as $\Delta(H) \geq 3$. We may then assume $\Delta(H) \leq 2$. If $\Delta(H) = 0$, there is clearly nothing to prove. If $\Delta(H) = 1$, then H is $\mathcal{L}_1$-choosable unless there is an edge $e \in E(H)$ such that $L_1(e) = \emptyset$. But, in view of (1), this may occur only if $s = 2$, which is contrary to assumption. If $\Delta(H) = 2$ and H contains no odd cycle, then $\chi'_l(H) = 2$, and hence, from (1), H is $\mathcal{L}_1$-choosable. If $\Delta(H) = 2$ and H contains an odd cycle, then $\chi'_l(H) = 3$; in this case it suffices to show that at least one of the inequalities in (1) is strict. If $s \geq 4$, the last inequality is strict. Hence we may assume that $s = 3$. It is easy to see that, by a different initial choice for the partial list-colouring ψ, the first inequality in (1) may be assumed to be strict for all the edges of H. Hence, in any case, H may be assumed to be $\mathcal{L}_1$-choosable and thus G is $\mathcal{L}$-choosable, concluding the proof. □

Using Theorem 1, we can now prove [5, Theorem 1], i.e. the assertion that Vizing's theorem for simple graphs extends to star multigraphs. Our proof is very short.

Theorem 2 *Let G be a star multigraph. Then $\chi'(G) \leq \Delta(G) + 1$.*

Proof. Without loss of generality, we may assume that G is Class 2. Furthermore, replacing G with a critical subgraph with the same chromatic index, we may assume that G is critical. If every vertex is incident with a multiple edge, then, by Theorem 1, G is $(\Delta(G) + 1)$-edge-choosable, and hence $(\Delta(G) + 1)$-edge-colourable. Therefore we may assume the existence of a vertex u incident only with simple edges. Choosing an arbitrary edge e incident with u and any e-tense colouring ϕ, and writing the Fan Formula at u, we have

$$\chi'(G) = \lceil \tfrac{1}{|V(\mathcal{F})|} \cdot \textstyle\sum_{w \in V(\mathcal{F})} (\deg_G(w) + \mu^*(uw)) \rceil$$

$$\leq \lceil \tfrac{1}{|V(\mathcal{F})|} \cdot \textstyle\sum_{w \in V(\mathcal{F})} (\deg_G(w) + 1) \rceil \leq \Delta(G) + 1,$$

where we have used the fact that $\mu^*(uw) = \mu(uw) = 1$ for any $w \in V(\mathcal{F})$. This proves the theorem. □

By Theorem 2, any Class 2 star multigraph with maximum degree Δ has necessarily chromatic index $\Delta + 1$. We proceed to consider star multigraphs with only one vertex of maximum degree.

Theorem 3 *Let G be a star multigraph with only one vertex of maximum degree. Then G is Class 2 if and only if G contains a subgraph on 3 vertices with $\Delta(G)+1 \geq 3$ edges.*

Proof. If G contains a subgraph on 3 vertices as stated by the theorem then G is clearly Class 2. Assume now that G is Class 2. Replacing G with any of its critical subgraphs with the same chromatic index, we may assume that G is critical. If every vertex of G is incident with a multiple edge, then G, in view of Theorem 1, has necessarily the form prescribed by the theorem. Hence we may assume that G has a vertex u incident with simple edges only. Writing the Fan Formula at u (with respect to any edge e incident with u and any e-tense colouring ϕ), we have

$$\chi'(G) = \tfrac{1}{|V(\mathcal{F})|} \cdot \textstyle\sum_{w \in V(\mathcal{F})} (\deg_G(w) + \mu^*(uw)) + \tfrac{|V(\mathcal{F})|-2}{|V(\mathcal{F})|}$$

$$= \tfrac{1}{|V(\mathcal{F})|} \cdot \textstyle\sum_{w \in V(\mathcal{F})} (\deg_G(w) + 1) + \tfrac{|V(\mathcal{F})|-2}{|V(\mathcal{F})|}$$

$$\leq \Delta(G) + \tfrac{1}{|V(\mathcal{F})|} + \tfrac{|V(\mathcal{F})|-2}{|V(\mathcal{F})|} = \Delta(G) + \tfrac{|V(\mathcal{F})|-1}{|V(\mathcal{F})|} < \Delta(G) + 1,$$

where we have used the fact that at most one of the neighbours of u has degree $\Delta(G)$. Hence G is Class 1, contradicting the assumption. This contradiction proves the theorem. □

3 Star multigraphs with two vertices of maximum degree

We now prove a simple lemma which will be helpful in the sequel. For positive integers n, r, we denote[2] by K_n^{+r} the complete graph K_n with one edge replaced by an edge of multiplicity r. It is obvious that K_3^{+r} is Class 2 for every r, and easy to see that K_{2n}^{+r} is Class 1 for every r and n. Indeed

$$\chi'(K_{2n}^{+r}) \leq \chi'(K_{2n}) + r - 1 = 2n + r - 2 = \Delta(K_{2n}^{+r}).$$

The following lemma deals with the remaining cases. We shall use the fact that K_{2n+1} is Class 2 for every n and, in each optimal colouring of K_{2n+1}, every vertex is missing a distinct colour.

Lemma 3 *Let r, n be integers, where $n \geq 2, r \geq 2$. Then the following are equivalent properties:*

[2] The corresponding notation for the graph K_n^{+2} in the Chetwynd-Hilton paper is K_n^{+}.

(a) $r = 2$;

(b) K_{2n+1}^{+r} *is Class 2;*

(c) K_{2n+1}^{+r} *is critical.*

Proof. Assume $r = 2$. We prove that K_{2n+1}^{+2} is Class 2. This is immediate since, if K_{2n+1}^{+2} was Class 1, it would have a $(2n+1)$ edge-colouring; but then we would have a $(2n+1)$-edge-colouring of K_{2n+1} with two vertices missing the same colour, which is impossible. Hence K_{2n+1}^{+2} is Class 2. We now prove that it is critical. Let xy be the multiple edge of K_{2n+1}^{+2}. Removing one of the two parallel edges yields the graph K_{2n+1} and, since K_{2n+1}^{+2} is Class 2, we have $\chi'(K_{2n+1}^{+2}) > 2n+1 = \chi'(K_{2n+1})$. Remove now any other edge of K_{2n+1}^{+2}, say e, and consider the graph $K_{2n+1} - e$. It is easy to construct a $(2n+1)$-edge-colouring of $K_{2n+1} - e$ with the two vertices x, y missing the same colour α, and hence it is possible to colour with colour α an additional edge joining x and y, proving the inequality $\chi'(K_{2n+1}^{+2} - e) = 2n+1 < \chi'(K_{2n+1}^{+2})$. Hence K_{2n+1}^{+2} is critical. Assume now $r = 3$. Let xy be the multiple edge of K_{2n+1}^{+3} and let $e \in xy$. Let $f \neq e$ be an edge of K_{2n+1}^{+3} not incident with e (which certainly exists since $n \geq 2$). Since $K_{2n+1}^{+2} \cong K_{2n+1}^{+3} - e$ is critical, we can assume the existence of a colouring ϕ of $K_{2n+1}^{+3} - e$ with colours $1, 2, \ldots, 2n+2$, such that $\phi^{-1}(2n+2) = \{f\}$. Clearly such a colouring can be extended to a colouring of K_{2n+1}^{+3} by colouring the edge e with colour $2n+2$. Thus

$$\chi'(K_{2n+1}^{+3}) = \chi'(K_{2n+1}^{+2}) = 2n+2 = \Delta(K_{2n+1}^{+3}),$$

and hence K_{2n+1}^{+3} is Class 1. It follows by an easy induction argument that, for every $r \geq 4$, K_{2n+1}^{+r} is Class 1, since

$$\chi'(K_{2n+1}^{+r}) \leq \chi'(K_{2n+1}^{+(r-1)}) + 1 = 2n + r = \Delta(K_{2n+1}^{+r}).$$

Thus the proof is completed. □

We are now ready to prove the following theorem.

Theorem 4 *Let G be a star multigraph with two vertices of maximum degree, one of which is a star centre. Then G is Class 2 if and only if G contains a subgraph on 3 vertices and $\Delta(G)+1$ edges or $G = K_{2n+1}^{+2}$, for some $n \geq 2$.*

Proof. If G has the form prescribed by the theorem then clearly G is Class 2. Assume now that G is Class 2. Let H be a critical subgraph of G with the same chromatic index (and hence with the same maximum degree). Arguing by contradiction, we assume that H has more than 3 vertices. By Theorem 3 and the assumption of criticality, H must have two vertices of maximum degree, and hence H and G have the same vertices of maximum degree. Let v^* be a star centre. If there is a vertex u of H not adjacent to v^*, then, using the Fan Formula at u and the fact that u is adjacent to at most one vertex of maximum degree of H, we have

$$\chi'(G) = \tfrac{1}{|V(\mathcal{F})|} \cdot \textstyle\sum_{w \in V(\mathcal{F})} (\deg_G(w) + \mu^*(uw)) + \tfrac{|V(\mathcal{F})|-2}{|V(\mathcal{F})|}$$

$$= \tfrac{1}{|V(\mathcal{F})|} \cdot \textstyle\sum_{w \in V(\mathcal{F})} (\deg_G(w) + 1) + \tfrac{|V(\mathcal{F})|-2}{|V(\mathcal{F})|}$$

$$\leq \Delta(G) + \tfrac{1}{|V(\mathcal{F})|} + \tfrac{|V(\mathcal{F})|-2}{|V(\mathcal{F})|} = \Delta(G) + \tfrac{|V(\mathcal{F})|-1}{|V(\mathcal{F})|} < \Delta(G) + 1,$$

whence a contradiction with the fact that H is Class 2. It follows that every vertex is either adjacent or coincident with v^*. Let $V(H) = \{v^*, v_1, v_2, \ldots, v_s\}$, where $d_H(v^*) = d_H(u) = \Delta(H)$, and let $k_i = \mu(v^* v_i)$ for every $i = 1, 2, \ldots, s$. Then

$$\Delta(H) = \sum_{i=1}^{s} k_i = k_1 + \sum_{i=2}^{s} k_i \tag{2}$$

and

$$d_H(v_1) = k_1 + t = \Delta(H), \tag{3}$$

where t is the number of neighbours of v_1 in $J = H - v^*$. By comparison of (2) and (3), we see that

$$t = s - 1 \tag{4}$$

and

$$k_i = 1 \text{ for every } i = 2, 3, \ldots, s. \tag{5}$$

It follows that

$$H \subset K_n^{+r} \tag{6}$$

and

$$\Delta(H) = \Delta(K_n^{+r}) \tag{7}$$

for some $n \geq 4$ and $r \geq 2$. Since H is Class 2, it follows from (6) and (7) that K_n^{+r} is Class 2 and, by Lemma3, that n is odd, $n \geq 5$ and $r = 2$. That is to say, we have $H \subset K_{2n+1}^{+2}$ for some $n \geq 2$ and $\Delta(H) = \Delta(K_{2n+1}^{+2}) = 2n + 1$. Since K_{2n+1}^{+2} is critical and H is Class 2 and $\Delta(H) = \Delta(K_{2n+1}^{+2})$, we conclude that $H = K_{2n+1}^{+2}$. Hence every vertex of H has degree $2n$, except v^* and v_1, which have degree $2n+1$. The addition of any edge incident with any vertex of H would therefore either increase the maximum degree of H or increase the number of vertices of maximum degree in H. Since H has the same maximum degree and the same number of vertices of maximum degree as G, it follows that $G = H$, and the proof is completed. □

To complete the classification of star multigraphs with two vertices of maximum degree, we now prove the following.

Theorem 5 *Let G be a star multigraph with two vertices of maximum degree, neither of which is a star centre. Then G is Class 1.*

Proof. We argue by contradiction, so let us assume that G is Class 2. Replacing G by a critical subgraph with the same maximum degree (which must necessarily satisfy the same conditions) we may assume that G is critical. Let v^* be a star centre and let v_1, v_2 be the vertices of maximum degree. By assumption, $v^* \neq v_1, v_2$. If every vertex of G was adjacent or coincident to v^*, then v^* would have maximum degree in G, contrary to assumption. Hence there exists a vertex u which is not adjacent to v^*. Writing the Fan Formula at u with respect to any edge e incident with u and any e-tense colouring, we see that

$$\chi'(G) = \tfrac{1}{|V(\mathcal{F})|} \cdot \textstyle\sum_{w \in V(\mathcal{F})}(\deg_G(w) + \mu^*(uw)) + \tfrac{|V(\mathcal{F})|-2}{|V(\mathcal{F})|}$$

$$= \tfrac{1}{|V(\mathcal{F})|} \cdot \textstyle\sum_{w \in V(\mathcal{F})}(\deg_G(w) + 1) + \tfrac{|V(\mathcal{F})|-2}{|V(\mathcal{F})|}$$

$$\leq \Delta(G) + \tfrac{2}{|V(\mathcal{F})|} + \tfrac{|V(\mathcal{F})|-2}{|V(\mathcal{F})|} = \Delta(G) + 1,$$

where we have used the fact that there are at most two neighbours of u of maximum degree. Since the sign of equality must hold in the last inequality above, both v_1 and v_2 are fan vertices at u, and in particular u is adjacent to both v_1 and v_2 and hence is distinct from v_1 and v_2 (which implies that both v_1 and v_2 are adjacent to v^*, otherwise, replacing u with one of them and repeating the argument, we reach a contradiction). Now, deleting[3] the edge uv_2 and adding an edge joining u to v^*, we obtain a star multigraph G' with star centre v^* and with $\Delta(G) = \Delta(G')$. Let $\Delta = \Delta(G) = \Delta(G')$. Notice that G' has at most two vertices of degree Δ, namely v_1 and (possibly) v^*. Suppose G' was Class 2. Then, by Theorem 3 and Theorem 4, either G' contains a subgraph on 3 vertices and $\Delta + 1$ edges (which is impossible,

[3]The following ingenious argument is the same used by Chetwynd and Hilton in [5, Lemma 15]. However the present proof is considerably shorter because we do not prove, as a preliminary step, that $\Delta(G) \leq |V(G)| - 1$, which is unnecessary for our purposes.

since this would force v^* to have only two neighbours in G', whereas has the three neighbours u, v_1, v_2), or $G' = K_{2n+1}^{+2}$ for some $n \geq 2$ (which is also impossible, since G' contains two non-adjacent vertices, namely u and v_2). We conclude that G is Class 1. Thus there exists a Δ-edge-colouring of G', and hence there exists a Δ-edge-colouring ϕ of $G' - uv^* = G - uv_2$ such that u and v^* are missing the same colour α. We can view ϕ as an e_0 tense colouring of G, where $e_0 = uv_2$ is the uncoloured edge. Then u is a fan vertex at v_2 missing colour α and, consequently, v^* cannot be a fan vertex at v_2, since it is also missing colour α. Since $d_G(u) < \Delta$, there are at least two missing colours at u, say α and β, and two maximal fans F_1, F_2, one of which of the form $F_1 = [e_0, e_1, \ldots]$, where e_1 is coloured α, and the other of the form $[e_0, f_1, \ldots]$, where f_1 is coloured β. Since no fan vertex at v_2 is joined to v_2 by a multiple edge, it follows that all the edges of F_1 (and, similarly, of F_2) are incident with distinct fan vertices, and the fan vertices of F_1 are distinct from those of F_2, except for u, which is an endpoint of e_0. Moreover both F_1 and F_2 must terminate with a fan edge incident with a vertex of maximum degree, which is necessarily v_1. Hence F_1 and F_2 both contain v_1 as a fan vertex, contradicting the fact that the only common fan vertex is u, and this contradiction establishes the theorem. □

Final Remarks: In this paper we have obtained simplifications of the proofs of the main results of [5] using some theoretical tool, viz. the Fan Formula. Such tool may be applied to the study of the classification problem of star multigraphs with more than two vertices of maximum degree, or to the classification of classes of graphs other than star multigraphs.

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