Combinatorial Identities on q-Harmonic Numbers

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Abstract

By means of the q-finite differences and the derivative operator, we derive, from an alternating q-binomial sum identity with a free variable x, several interesting identities concerning the generalized q-harmonic numbers.

1 Introduction and Notation

Let $\mathbb{N}$ and $\mathbb{N}_0$ be the sets of natural numbers and nonnegative integers respectively. Define the generalized q-harmonic numbers by

$$H_0^{\langle m\rangle}(x) = 0 \text{ and } H_n^{\langle m\rangle}(x) = \sum_{k=1}^{n} \frac{q^{k(m-1)}}{(1-xq^k)^m} \text{ for } m,\ n \in \mathbb{N}; \tag{1.1a}$$

$$\mathcal{H}_0^{\langle m\rangle}(x) = 0 \text{ and } \mathcal{H}_n^{\langle m\rangle}(x) = \sum_{1\le k_1\le k_2\le\cdots\le k_m\le n} \prod_{\ell=1}^{m} \frac{q^{k_\ell}}{1-xq^{k_\ell}} \text{ for } m,\ n \in \mathbb{N}. \tag{1.1b}$$

When $x = 1$, we shall omit the variables and denote, respectively, the corresponding higher q-harmonic numbers by $H_n^{\langle m\rangle}$ and $\mathcal{H}_n^{\langle m\rangle}$. In particular for $m = 1$, we shall write $H_n(x) := H_n^{\langle 1\rangle}(x)$ and $\mathcal{H}_n(x) := \mathcal{H}_n^{\langle 1\rangle}(x)$ as well $H_n := H_n^{\langle 1\rangle}$ and $\mathcal{H}_n := \mathcal{H}_n^{\langle 1\rangle}$, for simplicity.

In 1982, Van Hamme [9] (cf. [1, Eq 1.3]) found the following interesting identity:

$$\sum_{k=1}^{n}(-1)^{k-1}\begin{bmatrix} n \\ k \end{bmatrix}\frac{q^{\binom{k+1}{2}}}{1-q^k} = \sum_{k=1}^{n}\frac{q^k}{1-q^k} \tag{1.2}$$

where the Gaussian binomial coefficient is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(q^{1+n-k};q)_k}{(q;q)_k} = \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}$$

with the q-shifted factorial given by

$$(x;q)_0 = 1 \quad \text{and} \quad (x;q)_n = \prod_{k=0}^{n-1}(1-xq^k) \quad \text{for} \quad n \in \mathbb{N}.$$

For $m \in \mathbb{C}$ but $-m \notin \mathbb{N}_0$, Uchimura [14, 1987] generalized (1.2) as follows:

$$\sum_{k=0}^{n}(-1)^k\begin{bmatrix} n \\ k \end{bmatrix}\frac{q^{\binom{k+1}{2}}}{1-q^{m+k}} = \frac{(q;q)_n}{(q^m;q)_{n+1}}. \tag{1.3}$$

Then in 1995 Dilcher [6] established a more general expression:

$$\sum_{k=1}^{n}(-1)^{k-1}\begin{bmatrix} n \\ k \end{bmatrix}\frac{q^{\binom{k}{2}+mk}}{(1-q^k)^m} = \mathcal{H}_n^{\langle m\rangle}. \tag{1.4}$$

Prodinger [11] discovered its dual relation

$$\sum_{k=1}^{n}(-1)^{k-1}\begin{bmatrix} n \\ k \end{bmatrix}\frac{q^{\binom{n-k}{2}}}{1-q^k}\mathcal{H}_k^{\langle m\rangle} = q^{\binom{n}{2}}H_n^{\langle m+1\rangle}. \tag{1.5}$$

Several generalizations of (1.3) and (1.4) are recently published by Bradley [2], Fu-Lascoux [7] and Zeng [15].

In this paper, we will further investigate combinatorial identities concerning the generalized q-harmonic numbers just mentioned. By means of the q-finite differences and the derivative operator, we derive, from an alternating q-binomial sum identity with a free variable x, several interesting identities concerning q-harmonic numbers. Some of them generalize the identities displayed in (1.3), (1.4) and (1.5).

2 q-Finite Differences

As our starting point, we recall the following finite sum identity.

Lemma 1 (Zeng [15]). *With n being a natural number and x an indeterminate, there is a useful algebraic identity:*

$$F_n(x) := \frac{(q;q)_n}{(x;q)_{n+1}} = \sum_{k=0}^{n}(-1)^k \begin{bmatrix} n \\ k \end{bmatrix} \frac{q^{\binom{k+1}{2}}}{1-xq^k}.$$

Different from Zeng [15], this can be verified by means of the partial fraction decomposition method. Writing $F_n(x)$ in terms of simple fractions

$$F_n(x) = \sum_{k=0}^{n} \frac{A_k}{1-xq^k}$$

and then determining the coefficients by the following limiting relation:

$$A_k = \lim_{x\to q^{-k}} (1-xq^k)\, F_n(x) = (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{k+1}{2}}$$

we derive the identity stated in Lemma 1.

We will use the well-known q-analogue of the finite difference operator that can be found explicitly in Carlitz [3]. Related operators were used earlier by Heine, Rogers and Jackson (cf. [8, Ex 1.12]) and their applications can be found, for example, in Charalambides [4] and Johnson [10].

For any given function $f(x)$ and the fixed indeterminate q, we define

$$\Delta_q f(x) := \frac{f(x)-f(xq)}{x} \quad \text{and} \quad \Delta_q^n f(x) := \Delta_q(\Delta_q^{n-1})f(x) \quad \text{for } n = 2, 3, \cdots$$

with the convention that $\Delta_q^0 f(x) = f(x)$ for the identity operator. By means of induction principle, we can prove the following explicit formula

$$\Delta_q^n f(x) = x^{-n} q^{-\binom{n}{2}} \sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{n-k}{2}} f(xq^k) \tag{2.1a}$$

$$= x^{-n} \sum_{k=0}^{n} q^k \frac{(q^{-n};q)_k}{(q;q)_k} f(xq^k). \tag{2.1b}$$

Now we are ready to generalize (1.3) to the following identity.

Theorem 2. *For $m, n \in \mathbb{N}_0$ and an indeterminate x, there hold two identities:*

$$\sum_{k=0}^{n}(-1)^k \begin{bmatrix} n \\ k \end{bmatrix} \frac{q^{\binom{k+1}{2}+mk}}{(xq^k;q)_{m+1}} = \frac{(q;q)_{m+n}}{(q;q)_m(x;q)_{m+n+1}}, \tag{2.2a}$$

$$\sum_{k=0}^{n}(-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{n-k}{2}} \frac{(q;q)_{m+k}}{(x;q)_{m+k+1}} = \frac{(q;q)_m}{(xq^n;q)_{m+1}} q^{\binom{n+1}{2}+mn}. \tag{2.2b}$$

Proof. Combining (2.1b) with the q-Chu-Vandermonde formula (cf. [8, II-6]), we first compute the q-finite differences of a rational function:

$$\begin{aligned}
\Delta_q^m \frac{1}{(xq^k;q)_{n+1}} &= x^{-m} \sum_{i=0}^{m} \frac{(q^{-m};q)_i}{(q;q)_i} \frac{q^i}{(xq^{k+i};q)_{n+1}} \\
&= \frac{x^{-m}}{(xq^k;q)_{n+1}} \sum_{i=0}^{m} \frac{(q^{-m};q)_i (xq^k;q)_i}{(q;q)_i (xq^{n+k+1};q)_i} q^i \\
&= \frac{q^{km}}{(xq^k;q)_{n+1}} \frac{(q^{n+1};q)_m}{(xq^{n+k+1};q)_m}
\end{aligned}$$

which leads us to

$$\Delta_q^m \frac{1}{(xq^k;q)_{n+1}} = \frac{q^{km}}{(q;q)_n} \frac{(q;q)_{m+n}}{(xq^k;q)_{m+n+1}}. \tag{2.3}$$

When $k = 0$ and $n = 0$, we have two particular expressions:

$$\begin{aligned}
\Delta_q^m \frac{1}{(x;q)_{n+1}} &= \frac{(q;q)_{m+n}}{(q;q)_n (x;q)_{m+n+1}}, \\
\Delta_q^m \frac{1}{1-xq^k} &= q^{km} \frac{(q;q)_m}{(xq^k;q)_{m+1}}.
\end{aligned}$$

Then (2.2a) follows from applying Δ_q^m to both sides of Lemma 1.

The second identity (2.2b) results from the dual relation of (2.2a) in view of the q-binomial inverse series relation (cf. [5, Eqs 02d-02e]):

$$F(n) = \sum_{k=0}^{n}(-1)^k \begin{bmatrix} n \\ k \end{bmatrix} G(k), \tag{2.4a}$$

$$G(n) = \sum_{k=0}^{n}(-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{n-k}{2}} F(k). \tag{2.4b}$$

This completes the proof of Theorem 2. □

Corollary 3 (Two q-harmonic number identities).

$$\sum_{k=1}^{n}(-1)^{k-1}\begin{bmatrix}n\\k\end{bmatrix}\frac{q^{\binom{k+1}{2}+mk}}{(q^k;q)_{m+1}}=\frac{\mathcal{H}_{m+n}-\mathcal{H}_m}{(q;q)_m}, \tag{2.5a}$$

$$\sum_{k=0}^{n}(-1)^{k-1}\begin{bmatrix}n\\k\end{bmatrix}q^{\binom{n-k}{2}}\mathcal{H}_{m+k}=\frac{(q;q)_m}{(q^n;q)_{m+1}}q^{\binom{n+1}{2}+mn}. \tag{2.5b}$$

Proof. Reformulating (2.2a) and then applying the L'Hôspital rule, we have

$$\begin{aligned}\sum_{k=1}^{n}(-1)^{k-1}\begin{bmatrix}n\\k\end{bmatrix}\frac{q^{\binom{k+1}{2}+mk}}{(q^k;q)_{m+1}}&=\lim_{x\to1}\left\{\frac{1}{(x;q)_{m+1}}-\frac{(q;q)_{m+n}}{(q;q)_m(x;q)_{m+n+1}}\right\}\\&=\frac{1}{(q;q)_m}\lim_{x\to1}\left\{\frac{(q;q)_m}{(xq;q)_m}-\frac{(q;q)_{m+n}}{(xq;q)_{m+n}}\right\}\Big/(1-x)\\&=\frac{1}{(q;q)_m}\lim_{x\to1}\left\{\frac{(q;q)_{m+n}}{(xq;q)_{m+n}}\mathcal{H}_{m+n}(xq)-\frac{(q;q)_m}{(xq;q)_m}\mathcal{H}_m(xq)\right\}\end{aligned}$$

which is equal to the right member displayed in (2.5a).

By invoking the following almost trivial binomial sum:

$$\sum_{k=0}^{n}(-1)^k\begin{bmatrix}n\\k\end{bmatrix}q^{\binom{n-k}{2}}=0 \quad \text{for} \quad n>0, \tag{2.6}$$

we see that the second identity (2.5b) is the dual relation of (2.5a) in view of the q-binomial inversions (2.4a-2.4b). □

3 Identities on Higher q-Harmonic Numbers

In this section we are going to compute derivatives of Lemma 1 with respect to x. This will allow us to establish another interesting identity (3.1a), which is similar to (2.2a) but with shifted factorial $(xq^k;q)_m$ in the summand being replaced by the binomial factor $(1-xq^k)^m$. It can be considered as the common generalization of (1.3) and (1.4).

Theorem 4. *For $m,n\in\mathbb{N}_0$ and an indeterminate x, there hold two iden-*

tities:

$$\sum_{k=0}^{n}(-1)^k \begin{bmatrix} n \\ k \end{bmatrix} \frac{q^{\binom{k+1}{2}+m(k+1)}}{(1-xq^k)^{m+1}} = F_n(x)\mathcal{H}_{n+1}^{\langle m\rangle}(x/q), \tag{3.1a}$$

$$\sum_{k=0}^{n}(-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{n-k}{2}} F_k(x)\, \mathcal{H}_{k+1}^{\langle m\rangle}(x/q) = \frac{q^{\binom{n+1}{2}+m(n+1)}}{(1-xq^n)^{m+1}}. \tag{3.1b}$$

Proof. Obviously the first identity (3.1a) is equivalent to the following formula on higher derivatives of $F_n(x)$ with respect to x:

$$\mathcal{D}_x^m F_n(x) = m! q^{-m} F_n(x)\mathcal{H}_{n+1}^{\langle m\rangle}(x/q) \tag{3.2}$$

which can be verified by induction principle.

With slight difference from $H_n^{\langle m\rangle}(x)$, we define, for convenience, the power sum symmetric functions by

$$p_m(x) = \sum_{k=0}^{n}\left\{\frac{q^k}{1-xq^k}\right\}^m.$$

For $m=0$, it is trivial to see that (3.2) is true. According to the definition of $F_n(x)$, the first derivative of $F_n(x)$ reads as

$$\mathcal{D}_x F_n(x) = F_n(x)p_1(x) \tag{3.3}$$

which is consistent with (3.2).

Suppose that (3.2) is true for $m \le M$. We proceed to show its validity when the order of differentiation is equal to $M+1$. Applying the Leibniz rule to (3.3), we have

$$\mathcal{D}_x^{M+1} F_n(x) = \mathcal{D}_x^M\{F_n(x)p_1(x)\} = \sum_{m=0}^{M}\binom{M}{m}\mathcal{D}_x^m p_1(x)\mathcal{D}_x^{M-m}F_n(x).$$

According to the induction hypothesis and

$$\mathcal{D}_x^m p_1(x) = m! p_{m+1}(x)$$

we get the following expression:

$$\mathcal{D}_x^{M+1} F_n(x) = M! F_n(x)\sum_{m=0}^{M} p_{m+1}(x) q^{m-M}\mathcal{H}_{n+1}^{\langle M-m\rangle}(x/q).$$

Then the formula (3.2) for $M+1$ follows if we can show that the last sum admits the following closed form:

$$\sum_{m=0}^{M} p_{m+1}(x)q^{m-M}\mathcal{H}_{n+1}^{(M-m)}(x/q) = (M+1)q^{-(M+1)}\mathcal{H}_{n+1}^{(M+1)}(x/q). \quad (3.4)$$

This can be justified by means of generating function method.

For the exponential generating function defined by

$$F(\tau) = \exp\Big\{\sum_{\ell\geq 1}\frac{\tau^\ell}{\ell}p_\ell(x)\Big\}$$

it is not hard to check that this is the ordinary generating function of the sequence $\{q^{-m}\mathcal{H}_{n+1}^{(m)}(x/q)\}_{m\geq 0}$. The derivative of $F(\tau)$ with respect to τ reads as

$$F'(\tau) = F(\tau)\sum_{\ell\geq 1}\tau^{\ell-1}p_\ell(x).$$

Extracting the coefficient of τ^M from both sides of the last equation, we derive the recurrence relation (3.4). This proves (3.1a).

Recalling the inversion pair (2.4a-2.4b), we infer that the second identity (3.1b) is just the dual formula of (3.1a). This completes the proof of Theorem 4. □

Specifying in (3.1a) and (3.1b) by $x \to q$, $n \to n-1$ and then making some routine simplifications, we recover Dilcher's formula (1.4) and its dual form:

$$\sum_{k=1}^{n}(-1)^{k-1}\begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{n-k}{2}}\mathcal{H}_k^{(m)} = \frac{q^{\binom{n}{2}+mn}}{(1-q^n)^m}. \quad (3.5)$$

In addition, the limiting case $n \to \infty$ of (3.1a) reads as the following identity:

$$\sum_{k=0}^{\infty}\frac{(-1)^k q^{\binom{k+1}{2}+m(k+1)}}{(q;q)_k(1-xq^k)^{m+1}} = \frac{(q;q)_\infty}{(x;q)_\infty}\sum_{n=0}^{\infty}\frac{q^{n+1}}{1-xq^n}\mathcal{H}_{n+1}^{(m-1)}(x/q) \quad (3.6)$$

which contains an identity due to Uchimura [14, Theorem 3.1] as the very particular case $x = q^M$ with $M \in \mathbb{N}_0$.

Now we use (3.1a) to prove another interesting q-series identity.

Theorem 5.

$$\sum_{k=m}^{\infty}\binom{k}{m}x^k(q^{k+1};q)_n=(x/q)^m\frac{(q;q)_n}{(x;q)_{n+1}}\mathcal{H}_{n+1}^{(m)}(x/q).$$

Proof. Recalling two binomial formulae

$$(x;q)_n=\sum_{k=0}^{n}(-1)^k\begin{bmatrix}n\\k\end{bmatrix}q^{\binom{k}{2}}x^k \quad\text{and}\quad \sum_{k=m}^{\infty}\binom{k}{m}x^k=\frac{x^m}{(1-x)^{m+1}}$$

we can manipulate the following infinite series:

$$\begin{aligned}\sum_{k=m}^{\infty}\binom{k}{m}x^k(q^{k+1};q)_n &= \sum_{k=m}^{\infty}\binom{k}{m}x^k\sum_{i=0}^{n}(-1)^i\begin{bmatrix}n\\i\end{bmatrix}q^{\binom{i+1}{2}+ki}\\ &= \sum_{i=0}^{n}(-1)^i\begin{bmatrix}n\\i\end{bmatrix}q^{\binom{i+1}{2}}\sum_{k=m}^{\infty}\binom{k}{m}(xq^i)^k\\ &= x^m\sum_{i=0}^{n}(-1)^i\begin{bmatrix}n\\i\end{bmatrix}\frac{q^{\binom{i+1}{2}+mi}}{(1-xq^i)^{m+1}}.\end{aligned}$$

Evaluating the last sum by means of (3.1a), we get the identity stated in the theorem. □

Specifying Theorem 5 by $x=q$, we find the identity below.

Corollary 6 (Extension of Dilcher [6, Corollary 2]).

$$\sum_{k=m}^{\infty}\binom{k}{m}q^k(q^{k+1};q)_n = \frac{\mathcal{H}_{n+1}^{(m)}}{1-q^{n+1}}.$$

The limiting case $n\to\infty$ of the last identity has appeared in Dilcher [6, Corollary 2], which further includes Uchilmura [12, Theorem 2] as the case $m=1$.

Finally we show an interesting identity similar to Theorem 5, but with the binomial coefficient being replaced by its q-counterpart.

Proposition 7.

$$\sum_{k=m}^{\infty}\begin{bmatrix}k\\m\end{bmatrix}x^k(q^{k+1};q)_n = \frac{(q^{m+1};q)_n}{(x;q)_{m+n+1}}x^m.$$

It is interesting to observe that this identity has a simpler right member than that stated in Theorem 5.

Proof. Similar to the proof of Theorem 5, applying the q-binomial expansion

$$\sum_{k=m}^{\infty} \begin{bmatrix} k \\ m \end{bmatrix} x^k = \frac{x^m}{(x;q)_{m+1}}$$

we can reduce the following infinite series:

$$\begin{aligned}
\sum_{k=m}^{\infty} \begin{bmatrix} k \\ m \end{bmatrix} x^k (q^{k+1};q)_n &= \sum_{k=m}^{\infty} \begin{bmatrix} k \\ m \end{bmatrix} x^k \sum_{i=0}^{n} (-1)^i \begin{bmatrix} n \\ i \end{bmatrix} q^{\binom{i+1}{2}+ki} \\
&= \sum_{i=0}^{n} (-1)^i \begin{bmatrix} n \\ i \end{bmatrix} q^{\binom{i+1}{2}} \sum_{k=m}^{\infty} \begin{bmatrix} k \\ m \end{bmatrix} (xq^i)^k \\
&= x^m \sum_{i=0}^{n} (-1)^i \begin{bmatrix} n \\ i \end{bmatrix} \frac{q^{\binom{i+1}{2}+mi}}{(xq^i;q)_{m+1}}
\end{aligned}$$

which leads us to the identity stated in the proposition thanks to (2.2a). □

Similarly, one can write down the limiting case $n \to \infty$ and other formulae corresponding to particular values of x. We are not going to reproduce them here.

4 Further Identities on Higher q-Harmonic Numbers

In this section, we shall reformulate Lemma 1 as a q-binomial sum involving generalized q-harmonic numbers. Then applying the derivative operator and q-binomial inversions, we further derive several identities concerning higher q-harmonic numbers.

Lemma 8 (Equivalent form of Lemma 1).

$$F_{n-1}(xq) = \frac{(q;q)_{n-1}}{(xq;q)_n} = \sum_{k=1}^{n} (-1)^{k-1} \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{k}{2}} H_k(x).$$

Proof. Recalling the following simple binomial identity

$$(-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{k+1}{2}} = \sum_{j=k}^{n} (-1)^j \begin{bmatrix} n+1 \\ j+1 \end{bmatrix} q^{\binom{j+1}{2}},$$

and substituting it into Lemma 1, we can then manipulate the double series as follows:

$$\begin{aligned} F_n(x) = \frac{(q;q)_n}{(x;q)_{n+1}} &= \sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} \frac{q^{\binom{k+1}{2}}}{1-xq^k} \\ &= \sum_{k=0}^{n} \frac{1}{1-xq^k} \sum_{j=k}^{n} (-1)^j \begin{bmatrix} n+1 \\ j+1 \end{bmatrix} q^{\binom{j+1}{2}} \\ &= \sum_{j=0}^{n} (-1)^j \begin{bmatrix} n+1 \\ j+1 \end{bmatrix} q^{\binom{j+1}{2}} H_{j+1}(x/q). \end{aligned}$$

Under parameter replacements $x \to xq$, $n \to n-1$ and $j \to k-1$, we get the identity stated in the lemma. □

Replacing x by xq^λ in Lemma 8 and then applying the equation (2.6) and

$$H_{\lambda+k}(x) = H_\lambda(x) + H_k(xq^\lambda) \quad \text{for} \quad \lambda \in \mathbb{N}_0,$$

we get

$$F_{n-1}(xq^{\lambda+1}) = \frac{(q;q)_{n-1}}{(xq^{\lambda+1};q)_n} = \sum_{k=0}^{n} (-1)^{k-1} \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{k}{2}} H_{\lambda+k}(x). \tag{4.1}$$

Theorem 9. *For $m, n, \lambda \in \mathbb{N}$ and an indeterminate x, there hold the identities:*

$$\sum_{k=0}^{n} (-1)^{k-1} \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{k}{2}} H_{\lambda+k}^{\langle m+1 \rangle}(x) = F_{n-1}(xq^{\lambda+1}) \mathcal{H}_n^{\langle m \rangle}(xq^\lambda) q^{m\lambda}, \tag{4.2a}$$

$$\sum_{k=1}^{n} (-1)^{k-1} \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{n-k}{2}} F_{k-1}(xq^{\lambda+1}) \mathcal{H}_n^{\langle m \rangle}(xq^\lambda) = H_{\lambda+n}^{\langle m+1 \rangle}(x) q^{\binom{n}{2} - m\lambda}. \tag{4.2b}$$

Proof. On account of the m-time differentiation

$$\mathcal{D}_x^m F_{n-1}(xq^{\lambda+1}) = m! F_{n-1}(xq^{\lambda+1}) \mathcal{H}_n^{\langle m \rangle}(xq^\lambda) q^{m\lambda}, \tag{4.3}$$

we get immediately (4.2a) from (4.1). The second identity (4.2b) is the dual formula of (4.2a) in view of q-binomial inversions (2.4a-2.4b). □

Combining (4.2a) with the case $n \to n-1$ and $x = xq^{\lambda+1}$ of Theorem 5, we obtain the following rather strange transformation formula:

$$\sum_{k=m}^{\infty} \binom{k}{m} (xq^{\lambda+1})^k (q^{k+1};q)_{n-1} = x^m \sum_{\ell=0}^{n} (-1)^{\ell-1} \begin{bmatrix} n \\ \ell \end{bmatrix} q^{\binom{\ell}{2}} H_{\lambda+\ell}(x). \quad (4.4)$$

When $x = 1$, two identities (4.2a) and (4.2b) become the following ones.

Corollary 10.

$$\sum_{k=0}^{n} (-1)^{k-1} \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{k}{2}} H_{\lambda+k}^{(m+1)} = \frac{(q;q)_{n-1}}{(q^{\lambda+1};q)_n} \mathcal{H}_n^{(m)}(q^\lambda) q^{m\lambda}, \quad (4.5a)$$

$$\sum_{k=1}^{n} (-1)^{k-1} \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{n-k}{2}} \frac{(q;q)_{k-1}}{(q^{\lambda+1};q)_k} \mathcal{H}_k^{(m)}(q^\lambda) = H_{\lambda+n}^{(m+1)} q^{\binom{n}{2}-m\lambda}. \quad (4.5b)$$

Taking further $\lambda = 0$ in the last identity, we recover Prodinger's identity (1.5). For $n \to \infty$, we record the limiting cases of (4.2a) and (4.5a) as follows:

$$\sum_{k=0}^{\infty} \frac{(-1)^{k-1} q^{\binom{k}{2}}}{(q;q)_k} H_{\lambda+k}^{(m+1)}(x) = \frac{(q;q)_\infty}{(xq^{\lambda+1};q)_\infty} \sum_{n=1}^{\infty} \frac{\mathcal{H}_n^{(m-1)}(xq^\lambda)}{1 - xq^{\lambda+n}} q^{m\lambda+n}, \quad (4.6a)$$

$$\sum_{k=0}^{\infty} \frac{(-1)^{k-1} q^{\binom{k}{2}}}{(q;q)_k} H_{\lambda+k}^{(m+1)} = \frac{(q;q)_\infty}{(q^{\lambda+1};q)_\infty} \sum_{n=1}^{\infty} \frac{\mathcal{H}_n^{(m-1)}(q^\lambda)}{1 - q^{\lambda+n}} q^{m\lambda+n}. \quad (4.6b)$$

Alternatively, we can also apply the derivative operator to Theorem 2 and get the following identities.

Theorem 11. *For $m, n, \lambda \in \mathbb{N}$ and an indeterminate x, there hold the identities:*

$$\sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{n-k}{2}} \frac{(q;q)_{\lambda+k-1}}{(xq;q)_{\lambda+k}} \mathcal{H}_{\lambda+k}^{(m)}(x) = \frac{(q;q)_{\lambda-1}}{(xq^{n+1};q)_\lambda} \mathcal{H}_\lambda^{(m)}(xq^n) q^{\binom{n}{2}+(\lambda+m)n}, \quad (4.7a)$$

$$\sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{k}{2}+(\lambda+m)k} \frac{(q;q)_{\lambda-1}}{(xq^{k+1};q)_\lambda} \mathcal{H}_\lambda^{(m)}(xq^k) = \frac{(q;q)_{\lambda+n-1}}{(xq;q)_{\lambda+n}} \mathcal{H}_{\lambda+n}^{(m)}(x). \quad (4.7b)$$

Proof. Making replacementS $x \to xq$ and $m \to \lambda - 1$ in (2.2b), we obtain

$$\sum_{k=0}^{n}(-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{n-k}{2}} F_{\lambda+k-1}(xq) = F_{\lambda-1}(xq^{n+1})q^{\binom{n}{2}+\lambda n}.$$

Applying (3.2), (4.3) and differentiating the last equation m-times with respect to x, we get the first identity. The second identity is in fact the dual formula in view of q-binomial inversions (2.4a-2.4b). □

The case $x = 1$ of (4.7a) and (4.7b) read as follows:

$$\sum_{k=0}^{n}(-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{n-k}{2}} \frac{\mathcal{H}^{(m)}_{\lambda+k}}{1-q^{\lambda+k}} = \frac{(q;q)_{\lambda-1}}{(q^{n+1};q)_\lambda} \mathcal{H}^{(m)}_\lambda(q^n) q^{\binom{n}{2}+(\lambda+m)n}, \quad (4.8a)$$

$$\sum_{k=0}^{n}(-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{k}{2}+(\lambda+m)k} \frac{(q;q)_{\lambda-1}}{(q^{k+1};q)_\lambda} \mathcal{H}^{(m)}_\lambda(q^k) = \frac{\mathcal{H}^{(m)}_{\lambda+n}}{1-q^{\lambda+n}}; \quad (4.8b)$$

where the first identity resembles Prodinger's identity (1.5).

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