Ramsey set numbers in balanced complete multipartite graphs.

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Abstract

One natural extension of classical Ramsey numbers to multipartite graphs is to consider 2-colorings of the complete multipartite graph consisting of n parts, each of size k, denoted $K_{n \times k}$. We may then ask for the minimum integer n such that $K_{n \times k} \to \{G, H\}$ for two given graphs G and H. We study this number for the the cases when G and H are paths or cycles and show some general bounds and relations to classical Ramsey theory.

Keywords: multipartite graph, Ramsey theory

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1 Introduction

Classical Ramsey theory is a search for order in disorder. A graph G is said to *arrow* a set of graphs $\{H_1, H_2\}$, denoted $G \to \{H_1, H_2\}$, if any 2-coloring of the edges of G contains a copy of H_i in color i for some i. Denoted $R(G, H)$, the Ramsey number for graphs G and H is the smallest integer n such that $K_n \to \{G, H\}$.

Define $K_{n \times k}$ to be the complete multipartite graph consisting of n parts each of order k. There are two natural extensions of Ramsey theory to multipartite graphs. One involves fixing the number of partite sets n and finding the minimum number of vertices k in each set which will force the desired subgraph.

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Definition. Given a positive integer n and graphs G and H, the *size multipartite Ramsey number* $m_n(G, H)$ is the minimum integer k_0 such that $K_{n\times k} \to \{G, H\}$ for all $k \geq k_0$.

See [2, 4, 6, 9] for some of the work done in this area. The other natural extension to multipartite graphs, and the focus of this paper, is the following as defined in [3].

Definition. Given a positive integer k and graphs G and H, the *multipartite Ramsey number* $M_k(G, H)$ is the minimum integer n_0 such that $K_{n\times k} \to \{G, H\}$ for all $n \geq n_0$.

Remark 1. *$M_k(G, H) \leq M_1(G, H) = R(G, H)$ for all positive integers k.*

This implies that, for any k, the multipartite Ramsey number is bounded by the classical Ramsey number. In particular, $M_k(G, H)$ is well-defined for all graphs G, H.

2 Preliminaries

In this section, we present some easy observations and related corollaries. Some of these observations will be used in later proofs.

Claim. If $M_k(G, H) = 2$ then $M_{k'}(G, H) = 2$ for all $k' > k$.

Proof. $M_k(G, H) = 1$ only if there are no edges in G or H. For any nontrivial graphs, $M_k(G, H) \geq 2$. Otherwise note $K_{k\times n} \subset K_{k'\times n}$. □

Proposition 1. *$M_k(K_n, K_m) = R(K_n, K_m)$ for all positive integers k, n and m.*

Proof. By Remark 1, we know that $M_k(K_n, K_m) \leq R(K_n, K_m)$ so we need only show $M_k(K_n, K_m) \geq R(K_n, K_m)$. Let $r = R(K_n, K_m)$ and consider a 2-coloring of K_{r-1} with no red K_n and no blue K_m. Blow up this colored graph by making $k-1$ independent copies of each vertex copying all edges (with colors) to other vertices and their copies. This provides a 2-coloring of $K_{(r-1)\times k}$ with no red K_n or blue K_m. □

Proposition 2. *For graphs G and H and for integers $k \geq 1$ and $k' > k$, let $r = M_k(G, H) - 1$. Then $M_{k'}(G, H) \geq \max\{r' : K_{r'\times k'} \subseteq K_{r\times k}\} + 1$.*

Proof. Let $r = M_k(G,H) - 1$ and let K be a 2-coloring of $K_{r\times k}$ which contains no red copy of G or blue copy of H. Let r' be any integer such that $K_{r'\times k'} \subseteq K_{r\times k}$. Then the subgraph $K' \subseteq K$ corresponding to a copy of $K_{r'\times k'}$ contains no red copy of G and no blue copy of H. Thus, $M_{k'}(G,H) \geq r' + 1$. □

In particular, this implies the following corollary.

Corollary 1. $M_k(G,H) \geq \left\lfloor \frac{M_1(G,H)-1}{k} \right\rfloor + 1 = \left\lfloor \frac{R(G,H)-1}{k} \right\rfloor + 1.$

3 General Results

In this section, we provide more general results and relationships to classical Ramsey theory. Our first result uses the chromatic number to provide a general lower bound on the multipartite Ramsey number.

Theorem 1. *If $\chi(G) = x_G$ and $\chi(H) = x_H$, then $M_k(G,H) \geq (x_G - 1)(x_H - 1) + 1$.*

Proof. Let $n = (x_G - 1)(x_H - 1)$ and consider the following coloring of K_n. Color $x_G - 1$ copies of $K_{x_H - 1}$ with color 2 and the rest of the edges with color 1. In order to produce a coloring of $K_{n\times k}$, we simply blow-up this coloring of K_n in the classical sense: Make $k-1$ copies of each vertex (for a total of k of each) and copy not only the neighborhood but also the coloring at each vertex. It is now easy to see that the resulting graph induced on color 1 has chromatic number at most $x_G - 1$, meaning that there can be no subgraph isomorphic to G in color 1. Similarly, there can be no subgraph isomorphic to H in color 2. □

Conversely, we get the following from [4].

Theorem 2 ([4])**.** *$m_k(K_{n\times \ell}, K_{s\times t})$ exists for any $n, s \geq 2$ and $\ell, t \geq 1$ if and only if $k \geq R(K_n, K_s)$.*

This result implies the following corollary which is sharp since every k-chromatic graph is a subgraph of a complete k-partite graph.

Corollary 2. *For all graphs G and H, there exists an integer k_0 such that $M_k(G,H) \leq R(K_{\chi(G)}, K_{\chi(H)})$ for all $k \geq k_0$.*

By using three colors to color a complete graph, we can actually get an exact relationship to classical Ramsey theory.

Theorem 3. *For all graphs G and H, $M_2(G,H) = \left\lceil \frac{R(G,H,P_3)}{2} \right\rceil$.*

Proof. First we show that $M_2(G,H) \geq \left\lceil \frac{R(G,H,P_3)}{2} \right\rceil$. Let $n = R(G,H,P_3) - 1$ and consider a 3-coloring of K_n which contains no red copy of G, no blue copy of H and no green copy of P_3. In order to avoid a green P_3, no two green edges share a vertex so the green edges form at most a matching. If n is odd, remove a vertex which is not incident to a green edge. Let $m = 2\left\lfloor \frac{n}{2} \right\rfloor$ be the order of the remaining graph. Now remove the edges of a perfect matching by first removing all the green edges and then removing arbitrary independent edges from the remaining edges. The remaining graph is a 2-coloring of $K_{(m/2)\times 2}$ which contains no red G and no blue H. This means that

$$M_2(G,H) \geq \frac{m}{2} + 1 = \left\lfloor \frac{n}{2} \right\rfloor + 1 = \left\lceil \frac{R(G,H,P_3)}{2} \right\rceil,$$

as desired.

Next, we need to show that $M_2(G,H) \leq \left\lceil \frac{R(G,H,P_3)}{2} \right\rceil$. Choose $n = M_2(G,H) - 1$ and consider a 2-coloring of $K_{n\times 2}$ which contains no red copy of G and no blue copy of H. For every non-edge, we add a green edge. This forms a 3-colored complete graph of order $2n$ which contains no red G, no blue H and no green P_3. Hence,

$$R(G,H,P_3) \geq 2n + 1 = 2M_2(G,H) - 1.$$

This implies $M_2(G,H) \leq \left\lceil \frac{R(G,H,P_3)}{2} \right\rceil$ as desired, completing the proof of Theorem 3. □

4 Paths

This section concerns multipartite Ramsey numbers for paths. We begin with specific numbers for some short paths and finally observe general results.

Proposition 3. *$M_k(P_3,P_3) = 3$ for $k \leq 2$ and $M_k(P_3,P_3) = 2$ for $k \geq 3$.*

Proof. If G contains a triangle or a vertex of degree at least 3, then any 2-coloring of G contains a monochromatic P_3.

For $k = 1$, the result follows from the easy fact that $R(P_3,P_3) = 3$. For $k = 2$, we observe that $K_{2,2} = C_4$ so we color the edges of the C_4 with alternating colors. This coloring has no monochromatic P_3 so $M_2(P_3,P_3) \geq 3$. The graph $K_{2,2,2}$ contains a triangle so necessarily arrows P_3.

For $k \geq 3$, the graph $K_{k,k}$ contains a vertex of degree at least 3, meaning it also arrows P_3. □

Proposition 4. *$M_1(P_3, 2K_2) = 4$ and $M_k(P_3, 2K_2) = 2$ for $k \geq 2$.*

Proof. First note that any 2-coloring of a C_4 contains either a red P_3 or two disjoint blue edges. Since an all blue triangle contains no red P_3 and no pair of blue independent edges, we get $M_1(P_3, 2K_2) \geq 4$. The graph $K_{1\times 4} = K_4$ contains C_4 so we get equality.

For $k \geq 2$, we have $M_k(P_3, 2K_2) = 2$ for $k \geq 2$ because $G_{n\times k}$ contains a copy of C_4 for all $k, n \geq 2$. □

Proposition 5. *$M_1(2K_2, 2K_2) = 5$, $M_2(2K_2, 2K_2) = 3$ and, for $k \geq 3$, $M_k(2K_2, 2K_2) = 2$.*

Proof. For $k = 1$, it is known [5] that $R(2K_2, 2K_2) = M_1(2K_2, 2K_2) = 5$. For $k = 2$, $K_{2,2} = C_4 = v_1v_2v_3v_4v_1$ can be colored so v_1v_2 and v_2v_3 are red while v_3v_4 and v_4v_1 are blue. Notice this coloring does not contain a monochromatic $2P_2$. Hence, $M_2(2K_2, 2K_2) \geq 3$.

If there exists a set of three independent edges, then two must have the same color, producing a monochromatic $2K_2$. Since $K_{2,2,2}$ and $K_{3,3}$ each contain 3 independent edges, we know that $M_2(2K_2, 2K_2) = 3$ and $M_k(2K_2, 2K_2) = 2$ for $k \geq 3$. □

Proposition 6. *$M_1(P_4, P_4) = 5$, $M_2(P_4, P_4) = 3$ and $M_k(P_4, P_4) = 2$ for $k \geq 3$.*

Proof. Clearly we have $M_1(P_4, P_4) = 5$ as $R(P_n, P_m) = n + \lfloor \frac{m}{2} \rfloor - 1$ in [8]. When $k = 2$, we can color $K_{2,2} = C_4 = v_1v_2v_3v_4v_1$ as follows. As in the previous proof, we color v_1v_2 and v_2v_3 with red while v_3v_4 and v_4v_1 are blue. This coloring contains no monochromatic P_4 so $M_2(P_4, P_4) \geq 3$.

Let $K_{3,3}^-$ be the complete bipartite graph minus a single edge. For the remainder of this proof, it suffices to show that $K_{3,3}^- \to \{P_4, P_4\}$ since $K_{3,3}^- \subseteq K_{2,2,2}$ and $K_{3,3}^- \subseteq K_{k,k}$ for $k \geq 3$.

Let $G = K_{3,3}$ and label the vertices of one partite set with $A = \{a_1, a_2, a_3\}$ and the other with $B = \{b_1, b_2, b_3\}$ and suppose a_3b_3 is the missing edge. From a_1 to B, there must be two edges of the same color. Without loss of generality, suppose a_1b_1 and a_1b_2 are both red. If any edge between $\{b_1, b_2\}$ and $\{a_2, a_3\}$ is red, we have produced a red P_4 so all these edges must be blue. This contains a blue P_4. Hence, $K_{3,3}^- \to \{P_4, P_4\}$, completing the proof. □

We now cite two results for long paths from classical Ramsey theory.

Theorem 4 ([8]). *$R(P_n, P_m) = m + \lfloor n/2 \rfloor - 1$.*

Theorem 5 ([7]). *$R(P_n, P_m, P_3) = m + \lfloor n/2 \rfloor - 1$ for $m \geq 6(n+3)^2$.*

Using Theorem 3, these results imply the following corollary for the case when one path is much longer than the other and k is small.

Corollary 3. *For $m \geq 6(n+3)^2$ and $k \in \{1, 2\}$, we have $M_k(P_n, P_m) = \left\lceil \frac{m + \lfloor n/2 \rfloor - 1}{k} \right\rceil$.*

As k gets larger, this problem becomes more difficult and it is not easily implied by classical Ramsey results. Hence, we pose the following problem.

Problem 1. *Provide non-trivial general bounds on $M_k(P_n, P_m)$.*

5 Cycles

For a triangle versus a path, we get the following.

Theorem 6. *For $n \geq 2k$, we have $\left\lfloor \frac{2(n-1)}{k} \right\rfloor < M_k(K_3, P_n) \leq \left\lceil \frac{2(n-1)}{k} \right\rceil + 1$.*

In particular, this means that $M_k(K_3, P_n) = \frac{2(n-1)}{k} + 1$ for $1 \leq k \leq 2$ and for $n \geq 2k$.

Proof. The lower bound comes from the classical construction of a 2-colored $K_{2(n-1)}$ with no red triangle or blue P_n. This is two disjoint blue cliques of order $n - 1$ with all other edges in red. This graph contains no red triangle and no blue P_n and the removal of a matching from this graph certainly preserves this property.

For the upper bound, we let G be a 2-colored complete multipartite graph with $\left\lceil \frac{2(n-1)}{k} \right\rceil + 1$ parts each of order k. Notice that the degree of each vertex is at least $(2n - 2 + k) - k = 2n - 2$. If every vertex has blue degree at least $n - 1$, then there is a blue copy of P_n in G so there exists a vertex v with degree at most $n - 2$ in blue. This means v has at least n edges in red. In order to avoid a red triangle, no edge among these vertices can be red. Hence, there is a subgraph of G of size n with only blue edges. This subgraph is complete except for the removal of the edges of disjoint cliques of order at most k. Since $n \geq 2k$, this subgraph contains a copy of P_n. Hence, $K_{\lceil \frac{2(n-1)}{k} \rceil \times k} \to \{K_3, P_n\}$ for $n \geq 2k$. □

For short cycles, we get the following table of values for $M_k(C_n, C_m)$. Here "..." means $M_k(C_n, C_m)$ remains the same for larger values of k. Also, the inequalities come from Corollary 1 and recall that, by Remark 1, these sequences are monotone decreasing with k.

We prove one case of this (see Proposition 7). All other proofs are similar or use a computer search.

$k \setminus n, m$	3, 3	3, 4	3, 5	3, 6	4, 4	4, 5	4, 6	5, 5	5, 6	6, 6
1	6	7	9	11	6	7	7	9	11	8
2	⋮	4	5	6	4	4	4	5	6	5
3		3	⋮	≥ 4	3	3	3	⋮	≥ 4	3
4		⋮		?	3	⋮	≥ 2		?	≥ 2
5					2		?			?
6					⋮					

Proposition 7. *$M_1(C_4, C_4) = 6$, $M_2(C_4, C_4) = 4$, $M_3(C_4, C_4) = 3$, $M_4(C_4, C_4) = 3$ and $M_k(C_4, C_4) = 2$ for all $k \geq 5$.*

Proof. Clearly $M_1(C_4, C_4) = R(C_4, C_4) = 6$. From [1], it is also known that $R(C_4, C_4, P_3) = 8$ so, by Theorem 3, we get

$$M_2(C_4, C_4) = \left\lceil \frac{8}{2} \right\rceil = 4.$$

For the next case, we need a more substantial proof.

Claim 1. $M_3(C_4, C_4) = 3$.

By Corollary 1, we know that $M_3(C_4, C_4) \geq \frac{M_1(C_4, C_4) - 1}{3} + 1 = 3$ so we need only show that $K_{3,3,3} \to (C_4, C_4)$. Consider any 2-coloring of $K_{3,3,3}$ and let A, B, C be the three independent sets. If a pair of vertices in A shares 2 blue (or red) neighbors, there would be a blue (respectively red) C_4 so each pair of vertices in A (similarly in B and C) shares at most two monochromatic neighbors (at most one in each color). Suppose a pair shares only one monochromatic neighbor, say a_1 and a_2 share only a blue neighbor $b_1 \in B$. Then a_1 and a_2 disagree on colors to all 5 vertices in $C \cup B \setminus b_1$. If we let a_3 be the remaining vertex of A, this means a_3 must share at least 3 monochromatic neighbors with either a_1 or a_2, a contradiction. Hence, every pair of vertices must share exactly two monochromatic neighbors, one in each color.

First we will show that no single vertex is the red (or similarly blue) shared neighbor of all the vertices in A (or similarly any other set). Suppose b_1 has all red edges to A. Each vertex in $C \cup B \setminus b_1$ has at most 1 red edge to A meaning that each vertex has at least 2 blue edges to A. This clearly creates a blue C_4 so we know that no vertex is monochromatic to another set.

Now suppose all red shared neighbors of pairs in A are in B. This means that all blue shared neighbors of pairs in A are in C so the coloring of edges between A and the rest of the graph is already fixed to be that of Figure 1 (here red edges are represented by solid lines while blue edges are represented by dotted lines). Each pair in B also needs a blue common neighbor so this neighbor must be in C. Also each pair in C needs a red common neighbor, which must be in B. This is clearly a contradiction.

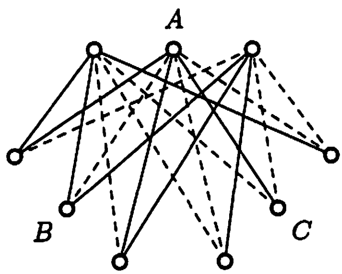

Figure 1: Partial coloring of $K_{3,3,3}$.

Finally, suppose two red shared neighbors of pairs in A are in B while the third is in C. This also implies two blue shared neighbors are in C while one is in B. Up to symmetry, we get the coloring in Figure 2.

The pairs c_1, c_3 and c_2, c_3 must share a red (solid) neighbor in B. This means that c_3 must have 2 red edges to B. Since all other pairs in B share a red neighbor already, these red edges must be c_3b_1 and c_3b_2. Also, c_3 and c_2 must share a blue (dashed) neighbor in B, so this shared neighbor must be b_3. Since we have eliminated the case where one vertex has all one color to a set, this implies that c_1b_3 must be red. Now recall that c_1 and c_3 must share a red neighbor in B so c_1 has a red edge to either b_1 or b_2, either case producing a red C_4. $\square_{Claim\ 1}$

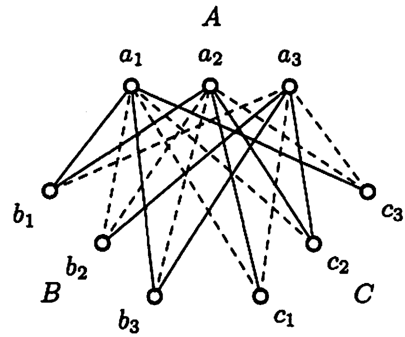

Figure 2: Partial coloring of $K_{3,3,3}$.

The coloring in Figure 3 shows that $M_4(C_4, C_4) > 2$ so, by Remark 1, we know that $M_4(C_4, C_4) = 3$.

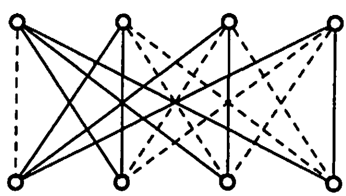

Figure 3: Coloring of $K_{4,4}$.

Finally, by Theorem 1 and Remark 1, it suffices to show that $K_{5,5} \rightarrow (C_4, C_4)$ to complete the proof of Proposition 7. Consider any 2-coloring of $K_{5,5} = A \cup B$. Let $a \in A$ and notice that a has at least 3 edges of a single color (suppose red) to B. Let $B' \subseteq B$ be a set of 3 vertices with red edges to a. Then each vertex in $A \setminus a$ must have at most one red edge to B'. Hence, each vertex of $A \setminus a$ has two blue edges to B'. There are 3 distinct pairs of vertices in B' and 4 vertices in $A \setminus a$ so, by the pigeon hole principle, there must be two vertices in $A \setminus a$, each with red edges to the same pair of vertices in B', thereby creating a red C_4. □

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