A special class of convex polytopes with constant metric dimension*

M. Imran[1], A. Q. Baig[2]

[1] Abdus Salam School of Mathematical Sciences,
GC University,
68-B, New Muslim Town, Lahore, Pakistan
[2] National Textile University, Faisalabad, Pakistan
E–mail:{imrandhab, aqbaig1,}@gmail.com

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Abstract. A family $\mathcal{G}$ of connected graphs is a family with constant metric dimension if dim(G) is finite and does not depend upon the choice of G in $\mathcal{G}$.
The metric dimension of some classes of convex polytopes has been determined in [8–12] and an open problem was raised in [10]: *Let G be the graph of a convex polytope which is obtained by joining the graph of two different convex polytopes G_1 and G_2 (such that the outer cycle of G_1 is the inner cycle of G_2) both having constant metric dimension. Is it the case that G will always have the constant metric dimension?*
In this paper, we study the metric dimension of an infinite classes of convex polytopes which are obtained by the combinations of two different graph of convex polytopes. It is shown that this infinite class of convex polytoes has constant metric dimension and only three vertices chosen appropriately suffice to resolve all the vertices of these classes of convex polytopes.

Keywords: *Metric dimension, basis, resolving set, planar graph, prsism, antiprism, convex polytopes*

1 Notation and preliminary results

If G is a connected graph, the distance $d(u, v)$ between two vertices $u, v \in V(G)$ is the length of a shortest path between them. Let $W = \{w_1, w_2, \ldots, w_k\}$ be an ordered set of vertices of G and let v be a vertex

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of G. The representation $r(v|W)$ of v with respect to W is the k-tuple $(d(v,w_1), d(v,w_2), d(v,w_3), \ldots, d(v,w_k))$. W is called a resolving set [6] or locating set [20] if every vertex of G is uniquely identified by its distances from the vertices of W, or equivalently, if distinct vertices of G have distinct representations with respect to W. A resolving set of minimum cardinality is called a basis for G and this cardinality is the metric dimension of G, denoted by $dim(G)$ [3]. The concepts of resolving set and metric basis have previously appeared in the literature (see [3-6, 8-13, 16-23]).
For a given ordered set of vertices $W = \{w_1, w_2, \ldots, w_k\}$ of a graph G, the ith component of $r(v|W)$ is 0 if and only if $v = w_i$. Thus, to show that W is a resolving set it suffices to verify that $r(x|W) \neq r(y|W)$ for each pair of distinct vertices $x, y \in V(G) \backslash W$.
A useful property in finding $dim(G)$ is the following lemma:

Lemma 1. [23] *Let W be a resolving set for a connected graph G and $u, v \in V(G)$. If $d(u,w) = d(v,w)$ for all vertices $w \in V(G) \setminus \{u,v\}$, then $\{u,v\} \cap W \neq \emptyset$.*

Motivated by the problem of uniquely determining the location of an intruder in a network, the concept of metric dimension was introduced by Slater in [20, 21] and studied independently by Harary and Melter in [7]. Applications of this invariant to the navigation of robots in networks are discussed in [17] and applications to chemistry in [6] while applications to problem of pattern recognition and image processing, some of which involve the use of hierarchical data structures are given in [18].
By denoting $G + H$ the join of G and H a *wheel* W_n is defined as $W_n = K_1 + C_n$, for $n \geq 3$, a *fan* is $f_n = K_1 + P_n$ for $n \geq 1$ and *Jahangir graph* J_{2n}, $(n \geq 2)$ (also known as *gear graph*) is obtained from the *wheel* W_{2n} by alternately deleting n spokes. Buczkowski *et al.* [3] determined the dimension of *wheel* W_n, Caceres *et al.* [5] the dimension of *fan* f_n and Tomescu and Javaid [22] the dimension of *Jahangir graph* J_{2n}.

Theorem 1. *([3], [5], [22]) Let W_n be a wheel of order $n \geq 3$, f_n be fan of order $n \geq 1$ and J_{2n} be a Jahangir graph. Then*
(i) For $n \geq 7$, $dim(W_n) = \lfloor \frac{2n+2}{5} \rfloor$;
(ii) For $n \geq 7$, $dim(f_n) = \lfloor \frac{2n+2}{5} \rfloor$;
(iii) For $n \geq 4$, $dim(J_{2n}) = \lfloor \frac{2n}{3} \rfloor$.

The metric dimension of all these plane graphs depends upon the number of vertices in the graph.
On the other hand, we say that a family $\mathcal{G}$ of connected graphs is a family with constant metric dimension if dim(G) is finite and does not depend upon the choice of G in $\mathcal{G}$. In [6] it was shown that a graph has metric dimension 1 if and only if it is a *path*, hence paths on n vertices constitute a family of graphs with constant metric dimension. Similarly, *cycles* with

$n(\geq 3)$ vertices also constitute such a family of graphs as their metric dimension is 2 and does not depend upon on the number of vertices n. In [4] it was proved that

$$dim(P_m \times C_n)=\begin{cases} 2, & \text{if } n \text{ is odd;} \\ 3, & \text{if } n \text{ is even.} \end{cases}$$

Since *prisms* D_n are the trivalent plane graphs obtained by the cross product of path P_2 with a cycle C_n. So prisms constitute a family of 3-*regular graphs* with constant metric dimension. Javaid *et al.* proved in [13] that the plane graph *antiprism* A_n constitute a family of regular graphs with constant metric dimension as $dim(A_n) = 3$ for every $n \geq 5$. The prism and the antiprism are *Archimedean* convex polytopes defined e.g. in [15]. The metric dimension of cartesian product of graphs has been discussed in [4, 19].

The metric dimension of some classes of convex polytopes has been determined in [8–12] where it was shown that these classes of convex polytopes have constant metric dimension 3 and an open problem was raised in [10]:

Open problem [10]: *Let G be the graph of a convex polytope which is obtained by joining the graph of two different convex polytopes G_1 and G_2 (such that the outer cycle of G_1 is the inner cycle of G_2) both having constant metric dimension. Is it the case that G will always have the constant metric dimension?*

Note that the problem of determining whether $dim(G) < k$ is an *NP*-complete problem [7]. Some bounds for this invariant, in terms of the diameter of the graph, are given in [17] and it was shown in [6, 17–19] that the metric dimension of trees can be determined efficiently. It appears unlikely that significant progress can be made in determining the dimension of a graph unless it belongs to a class for which the distances between vertices can be described in some systematic manner.

The metric dimension of some classes of convex polytopes which are obtained by the combination of two different graphs of convex polytopes has been recently studied in [10]. In this paper, we extend this study to an infinite class of convex polytopes which are obtained by combination (such that the outer cycle of graph G_1 is the inner cycle of graph G_2) of graph of convex polytope $\mathbb{D}_n$ [1] and graph of an antiprism [1]. We prove that this infinite class of convex polytopes has constant metric dimension and only three vertices appropriately chosen suffice to resolve all the vertices of these classes of convex polytopes.

In what follows all indices i which do not satisfy the given inequalities will be taken modolu n.

2 The graph of convex polytope V_n

The graph of convex polytope V_n is obtained as a combination of graph of convex polytope $\mathbb{D}_n$ [1] and graph of an antiprism A_n [1] such that the outer cycle of graph of an antiprism A_n is the inner cylce of graph of convex polytope $\mathbb{D}_n$.
For our purpose, we call the cycle induced by $\{a_i : 1 \leq i \leq n\}$, the inner

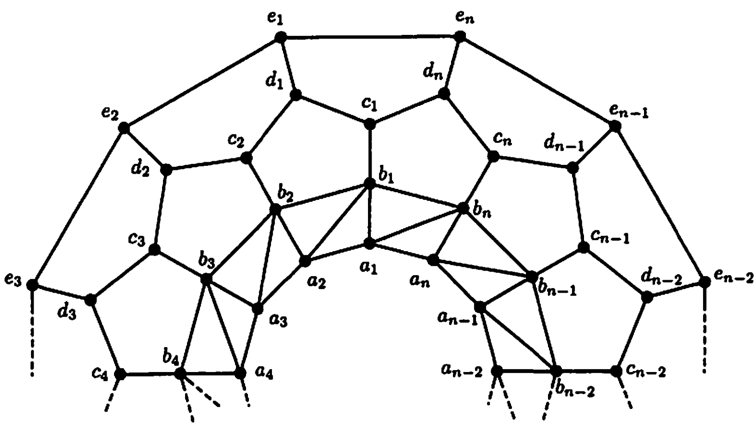

Fig. 1. The graph of convex polytope V_n

cycle, cycle induced by $\{b_i : 1 \leq i \leq n\}$, the interior cycle, cycle induced by $\{c_i : 1 \leq i \leq n\} \cup \{d_i : 1 \leq i \leq n\}$, the exterior cycle and cycle induced by $\{e_i : 1 \leq i \leq n\}$, the outer cycle.
The metric dimension of graph of convex polytope $\mathbb{D}_n$ and graph of an antiprism A_n have been studied in [8] and [13]. In the next theorem, we show that the metric dimension of the graph of convex polytope V_n is 3. Note that, the choice of appropriate landmarks is very important.

Theorem 2. *Let V_n denotes the graph of convex polytope; then $dim(V_n) = 3$ for every $n \geq 6$.*

Proof. We will prove the above equality by double inequalities. We consider the two cases.
Case(i) When n is even.
In this case, we can write $n = 2k$, $k \geq 3$, $k \in \mathbf{Z}^+$. Let $W = \{a_1, a_2, a_{k+1}\} \subset V(V_n)$, we show that W is a resolving set for V_n in this case. For this we give representations of any vertex of $V(V_n)\backslash W$ with respect to W.

Representations for the vertices of inner cycle are

$$r(a_i|W) = \begin{cases} (i-1, i-2, k-i+1), & 3 \le i \le k; \\ (2k-i+1, 2k-i+2, i-k-1), & k+2 \le i \le 2k. \end{cases}$$

Representations for the vertices of interior cycle are

$$r(b_i|W) = \begin{cases} (1,1,k), & i = 1; \\ (i, i-1, k-i+2), & 2 \le i \le k; \\ (k+1, k+1, 1), & i = k+1; \\ (2k-i+1, 2k-i+2, i-k), & k+2 \le i \le 2k. \end{cases}$$

Representations for the vertices of exterior cycle are

$$r(c_i|W) = \begin{cases} (2,2,k+1), & i = 1; \\ (i+1, i, k-i+2), & 2 \le i \le k; \\ (k+1, k+1, 2), & i = k+1; \\ (2k-i+2, 2k-i+3, i-k+1), & k+2 \le i \le 2k. \end{cases}$$

and

$$r(d_i|W) = \begin{cases} (3,3,k+1), & i = 1; \\ (i+2, i+1, k-i+2), & 2 \le i \le k-1; \\ (k+2, k+1, 3), & i = k; \\ (k+1, k+2, 3), & i = k+1; \\ (2k-i+2, 2k-i+3, i-k+2), & k+2 \le i \le 2k-1; \\ (3,3,k+2), & i = 2k. \end{cases}$$

Representations for the vertices of outer cycle are

$$r(e_i|W) = \begin{cases} (4,4,k+2), & i = 1; \\ (i+3, i+2, k-i+3), & 2 \le i \le k-1; \\ (k+3, k+2, 4), & i = k; \\ (2k-i+3, 2k-i+4, i-k+3), & k+1 \le i \le 2k-1; \\ (4,4,k+3), & i = 2k. \end{cases}$$

We note that there are no two vertices having the same representations implying that $dim(V_n) \le 3$.
On the other hand, we show that $dim(V_n) \ge 3$. Suppose on contrary that $dim(V_n) = 2$, then there are following possibilities to be discussed.
(1) Both vertices are in the inner cycle. Without loss of generality we suppose that one resolving vertex is a_1. Suppose that the second resolving vertex is a_t $(2 \le t \le k+1)$. Then for $2 \le t \le k$, we have $r(a_n|\{a_1, a_t\}) = r(b_1|\{a_1, a_t\}) = (1, t)$ and for $t = k+1$, $r(a_2|\{a_1, a_{k+1}\}) = r(a_n|\{a_1, a_{k+1}\}) = (1, k-1)$, a contradiction.
(2) Both vertices are in the interior cycle. Without loss of generality we suppose that one resolving vertex is b_1. Suppose that the second resolving

vertex is b_t $(2 \le t \le k+1)$. Then for $2 \le t \le k$, we have $r(a_1|\{b_1, b_t\}) = r(b_n|\{b_1, b_t\}) = (1, t)$ and for $t = k+1$, $r(b_2|\{b_1, b_{k+1}\}) = r(b_n|\{b_1, b_{k+1}\}) = (1, k-1)$, a contradiction.

(3) Both vertices are in the exterior cycle. Here are the two subcases.

• Both vertices are in the set $\{c_i : 1 \le i \le n\}$. Without loss of generality we suppose that one resolving vertex is c_1. Suppose that the second resolving vertex is c_t $(2 \le t \le k+1)$. Then for $2 \le t \le k$, we have $r(a_1|\{c_1, c_t\}) = r(b_n|\{c_1, c_t\}) = (2, t+1)$ and for $t = k+1$, $r(d_1|\{c_1, c_{k+1}\}) = r(d_n|\{c_1, c_{k+1}\}) = (1, k+3)$, a contradiction.

• Both vertices are in the set $\{d_i : 1 \le i \le n\}$. Without loss of generality we suppose that one resolving vertex is d_1. Suppose that the second resolving vertex is d_t $(2 \le t \le k+1)$. Then for $2 \le t \le k+1$, we have $r(b_1|\{d_1, d_t\}) = r(e_n|\{d_1, d_t\}) = (2, t+1)$, a contradiction.

• One vertex is in the set $\{c_i : 1 \le i \le n\}$ and other in the set $\{d_i : 1 \le i \le n\}$. Without loss of generality we suppose that one resolving vertex is c_1. Suppose that the second resolving vertex is d_t $(2 \le t \le k+1)$. Then for $t = 1$, we have $r(b_1|\{c_1, d_1\}) = r(d_n|\{c_1, d_1\}) = (1, 2)$. If $2 \le t \le k$, we have $r(a_2|\{c_1, d_t\}) = r(b_1|\{c_1, d_t\}) = (1, t+1)$ and for $t = k+1$, $r(a_n|\{c_1, d_{k+1}\}) = r(b_1|\{c_1, d_{k+1}\}) = (1, k+1)$, a contradiction.

(4) Both vertices are in the outer cycle. Without loss of generality we suppose that one resolving vertex is e_1. Suppose that the second resolving vertex is e_t $(2 \le t \le k+1)$. Then for $2 \le t \le k$, we have $r(d_1|\{e_1, e_t\}) = r(e_n|\{e_1, e_t\}) = (1, t)$ and for $t = k+1$, $r(e_2|\{e_1, e_{k+1}\}) = r(e_n|\{e_1, e_{k+1}\}) = (1, k-1)$, a contradiction.

(5) One vertex is in the inner cycle and other in the interior cycle. Without loss of generality we suppose that one resolving vertex is a_1. Suppose that the second resolving vertex is b_t $(1 \le t \le k+1)$. Then for $1 \le t \le k$, we have $r(b_n|\{a_1, b_t\}) = r(c_1|\{a_1, b_t\}) = (2, t)$ and when $t = k+1$, $r(a_2|\{a_1, b_{k+1}\}) = r(a_n|\{a_1, b_{k+1}\}) = (1, k)$, a contradiction.

(6) One vertex is in the inner cycle and other in the exterior cycle. Here are the two subcases.

• One vertex is in the inner cycle and other in the set $\{c_i : 1 \le i \le n\}$. Without loss of generality we suppose that one resolving vertex is a_1. Suppose that the second resolving vertex is c_t $(1 \le t \le k+1)$. Then for $t = 1$, we have $r(a_2|\{a_1, c_1\}) = r(a_n|\{a_1, c_1\}) = (1, 3)$. If $2 \le t \le k+1$, $r(a_2|\{a_1, b_t\}) = r(b_1|\{a_1, b_t\}) = (1, t)$, a contradiction.

• One vertex is in the inner cycle and other in the set $\{d_i : 1 \le i \le n\}$. Without loss of generality we suppose that one resolving vertex is a_1. Suppose that the second resolving vertex is d_t $(1 \le t \le k+1)$. Then for $t = 1$, we have $r(d_2|\{a_1, d_1\}) = r(e_n|\{a_1, d_1\}) = (4, 2)$. If $2 \le t \le k$, $r(a_2|\{a_1, d_t\}) = r(b_1|\{a_1, d_t\}) = (1, t+1)$ and when $t = k+1$, $r(a_n|\{a_1, d_{k+1}\}) = r(b_1|\{a_1, d_{k+1}\}) = (1, k+1)$, a contradiction.

(7) One vertex is in the inner cycle and other in the outer cycle. Without

loss of generality we suppose that one resolving vertex is a_1. Suppose that the second resolving vertex is e_t $(1 \leq t \leq k+1)$. Then for $t=1$, we have $r(c_2|\{a_1,e_1\}) = r(d_n|\{a_1,e_1\}) = (1,3)$. If $2 \leq t \leq k$, $r(a_2|\{a_1,e_t\}) = r(b_1|\{a_1,e_t\}) = (1,t+2)$ and when $t=k+1$, $r(a_n|\{a_1,e_{k+1}\}) = r(b_1|\{a_1,e_{k+1}\}) = (1,k+3)$, a contradiction.

(8) One vertex is in the interior cycle and other in the exterior cycle. Here are the two subcases.

• One vertex is in the interior cycle and other in the set $\{c_i : 1 \leq i \leq n\}$. Without loss of generality we suppose that one resolving vertex is b_1. Suppose that the second resolving vertex is c_t $(1 \leq t \leq k+1)$. Then for $1 \leq t \leq k$, we have $r(a_1|\{b_1,c_t\}) = r(b_n|\{b_1,c_t\}) = (1,t+1)$ and when $t=k+1$, $r(b_2|\{b_1,c_{k+1}\}) = r(b_n|\{b_1,c_{k+1}\}) = (1,k)$, a contradiction.

• One vertex is in the interior cycle and other in the set $\{d_i : 1 \leq i \leq n\}$. Without loss of generality we suppose that one resolving vertex is b_1. Suppose that the second resolving vertex is d_t $(1 \leq t \leq k+1)$. Then for $1 \leq t \leq k-1$, we have $r(a_1|\{b_1,d_t\}) = r(b_n|\{b_1,d_t\}) = (1,t+2)$. For $t=k,k+1$, $r(a_{n-2}|\{b_1,d_t\}) = r(e_n|\{b_1,d_t\}) = (3,t)$, a contradiction.

(9) One vertex is in the interior cycle and other in the outer cycle. Without loss of generality we suppose that one resolving vertex is b_1. Suppose that the second resolving vertex is e_t $(1 \leq t \leq k+1)$. Then for $1 \leq t \leq k-1$, we have $r(a_1|\{b_1,e_t\}) = r(b_n|\{b_1,e_t\}) = (1,t+3)$. For $t=k$, $r(b_2|\{b_1,e_k\}) = r(c_1|\{b_1,e_k\}) = (1,k+1)$ and when $t=k+1$, $r(b_n|\{b_1,e_{k+1}\}) = r(c_1|\{b_1,e_{k+1}\}) = (1,k+1)$, a contradiction.

(10) One vertex is in the exterior cycle and other in the outer cycle. Here are the two subcases.

• One vertex is in the set $\{c_i : 1 \leq i \leq n\}$ and other in outer cycle. Due to the symmetry of the graph, this subcase is analogous to second subcase of case **(8)**.

• One vertex is in the set $\{d_i : 1 \leq i \leq n\}$ and other in outer cycle. This subcase is analogous to first subcase of case **(8)**.

Hence, from above it follows that there is no resolving set with two vertices for $V(V_n)$ implying that $dim(V_n) = 3$ in this case.

Case(ii) When n is odd.

In this case, we can write $n = 2k+1$, $k \geq 3$, $k \in \mathbf{Z}^+$. Let $W = \{a_1, a_2, a_{k+1}\} \subset V(V_n)$, we show that W is a resolving set for V_n in this case. For this we give representations of any vertex of $V(V_n)\backslash W$ with respect to W.

Representations for the vertices of inner cycle are

$$r(a_i|W) = \begin{cases} (i-1, i-2, k-i+1), & 3 \leq i \leq k; \\ (k,k,1), & i = k+2; \\ (2k-i+2, 2k-i+3, i-k-1), & k+3 \leq i \leq 2k+1. \end{cases}$$

Representations for the vertices of interior cycle are

$$r(b_i|W) = \begin{cases} (1,1,k), & i=1; \\ (i, i-1, k-i+1), & 2 \le i \le k; \\ (k+1, k, 1), & i = k+1; \\ (2k-i+2, 2k-i+3, i-k), & k+2 \le i \le 2k+1. \end{cases}$$

Representations for the vertices of exterior cycle are

$$r(c_i|W) = \begin{cases} (2,2,k+1), & i=1; \\ (i+1, i, k-i+2), & 2 \le i \le k; \\ (k+2, k+1, 2), & i = k+1; \\ (2k-i+3, 2k-i+4, i-k+1), & k+2 \le i \le 2k+1. \end{cases}$$

and

$$r(d_i|W) = \begin{cases} (3,3,k+1), & i=1; \\ (i+2, i+1, k-i+2), & 2 \le i \le k-1; \\ (k+2, k+1, 3), & i = k; \\ (k+2, k+2, 3), & i = k+1; \\ (2k-i+3, 2k-i+4, i-k+2), & k+2 \le i \le 2k; \\ (3,3,k+2), & i = 2k+1. \end{cases}$$

Representations for the vertices of outer cycle are

$$r(e_i|W) = \begin{cases} (4,4,k+2), & i=1; \\ (i+3, i+2, k-i+3), & 2 \le i \le k-1; \\ (k+3, k+2, 4), & i = k; \\ (k+3, k+3, 4), & i = k+1; \\ (2k-i+4, 2k-i+5, i-k+3), & k+2 \le i \le 2k; \\ (4,4,k+3), & i = 2k+1. \end{cases}$$

Again we see that there are no two vertices having the same representations which implies that $dim(V_n) \le 3$ in this case.
On the other hand, suppose that $dim(V_n) = 2$, then there are the same subcases as in case (i) and contradiction can be obtained analogously. This implies that $dim(V_n) = 3$ in this case, which completes the proof.

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