The overlap chromatic numbers of wheel graphs

Fred Holroyd

Department of Mathematics and Statistics, The Open University,
Walton Hall, Milton Keynes MK7 6AA, United Kingdom
F.C.Holroyd@open.ac.uk

Ivor Watts

Department of Mathematics and Statistics, The Open University,
Walton Hall, Milton Keynes MK7 6AA, United Kingdom
illw@talktalk.net

Abstract

An (r, λ) *overlap colouring* of a graph G allocates r colours to each vertex subject to the condition that any pair of adjacent vertices shares exactly λ colours. The (r, λ) *overlap chromatic number* of G is the least number of colours required for such a colouring. The overlap chromatic numbers of bipartite graphs are easy to find; those of odd cycle graphs have been already been established. In this paper we find the overlap chromatic numbers of the wheel graphs.

1 Definitions and introduction

Throughout this paper, graphs are assumed to be simple and finite. The multichromatic numbers of graphs have been studied for some decades, the concept having been originated by Hilton, Rado and Scott [1] and independently by Stahl [5]; see also the PhD thesis of Scott [4]. The r-fold chromatic number of a graph G is denoted by $\chi_r(G)$ and is the least number of colours that are required in order that r colours may be assigned to each vertex of G such that adjacent vertices are always assigned disjoint colour sets.

Later, Johnson and Holroyd [3] discussed the concept of an *overlap colouring* of a graph; that is, a vertex colouring μ with a fixed number r of colours per vertex, such that any pair of adjacent vertices shares exactly λ colours. The *palette* $P(\mu)$ of such a colouring is the set of all colours used. The (r, λ) *chromatic number* of G, denoted by $\chi_{r,\lambda}(G)$, is the least possible palette size $|P(\mu)|$ for an (r, λ) colouring μ of G.

We use the notation $\mathbb{Z}^+$ for the positive integers and $\mathbb{N}$ for the non-negative integers. Throughout the paper we assume $r \in \mathbb{Z}^+$, $\lambda \in \mathbb{N}$, $\lambda \leq r$. (Of course, $\chi_{r,r}(G) = r$ for any non-empty graph G.)

2 Odd-order wheel graphs

We denote by C_n the n-cycle graph and by W_{n+1} the graph of order $n+1$ formed from C_n by adjoining a *hub* vertex h, adjacent to each vertex $v_0, v_1, \ldots, v_{n-1}$ of C_n; these are the *rim* vertices. We label them in cyclic order, considering the subscripts modulo n.

Before proceeding to the overlap chromatic numbers of wheel graphs, it is worth noting the results for bipartite graphs and for odd cycles. These are given both in [6] and in [2] and are as follows.

Theorem 1 *Let G be any bipartite graph; then*

$$\chi_{r,\lambda}(G) = 2r - \lambda.$$

Theorem 2 *Let $p \in \mathbb{Z}^+$. Then*

$$\chi_{r,\lambda}(C_{2p+1}) = \max\{\left\lceil \frac{(2p+1)(r-\lambda)}{p} \right\rceil, 2r - \lambda\}.$$

Corollary 3 *Let $p \in \mathbb{Z}^+$. Then*

$$\chi_{r,\lambda}(W_{2p+1}) = \max\{3(r-\lambda), 2r - \lambda\}.$$

Proof W_{2p+1} contains at least one triangle; thus, $\chi_{r,\lambda}(W_{2p+1}) \geq \chi_{r,\lambda}(C_3) = \max\{3(r-\lambda), 2r-\lambda\}$.

Let μ be an (r, λ) colouring of C_3 with $|P(\mu)| = \max\{3(r-\lambda), 2r-\lambda\}$. There is a homomorphism $\phi: W_{2p+1} \to C_3$; then $\mu \circ \phi$ is an (r, λ) colouring of W_{2p+1} using the same palette as μ.

□

3 Even-order wheel graphs

The even-order wheels are considerably more difficult to deal with. It turns out that the expression for $\chi_{r,\lambda}(W_{2p+2})$ depends on the ratio λ/r, four distinct expressions being required, and that the (2, 1) chromatic number nevertheless requires separate treatment. The full result is as follows.

Theorem 4 $\chi_{2,1}(W_{2p+2}) = 5$. *For all other values of* (r, λ)*:*

$$\chi_{r,\lambda}(W_{2p+2}) = \max\left\{r+\left\lceil\frac{(2p+1)(r-2\lambda)}{p}\right\rceil, 3(r-\lambda), 3r-2\lambda-\left\lfloor\frac{pr}{2p+1}\right\rfloor, 2r-\lambda\right\}.$$

That is, for $(r, \lambda) \neq (2, 1)$*:*

1. *if* $0 \leq \lambda \leq \frac{r}{p+2}$*, then* $\chi_{r,\lambda}(W_{2p+2}) = r + \left\lceil\frac{(2p+1)(r-2\lambda)}{p}\right\rceil$*;*
2. *if* $\frac{r}{p+2} \leq \lambda \leq \frac{pr}{2p+1}$*, then* $\chi_{r,\lambda}(W_{2p+2}) = 3(r-\lambda)$*;*
3. *if* $\frac{pr}{2p+1} \leq \lambda \leq \frac{(p+1)r}{2p+1}$*, then* $\chi_{r,\lambda}(W_{2p+2}) = 3r - 2\lambda - \left\lfloor\frac{pr}{2p+1}\right\rfloor$*;*
4. *if* $\frac{(p+1)r}{2p+1} \leq \lambda \leq r$*, then* $\chi_{r,\lambda}(W_{2p+2}) = 2r - \lambda$*.*

Note that W_{2p+2} is the graph-theoretic join $C_{2p+1} + \{h\}$ where $\{h\}$ is a singleton vertex. Of course, for any two graphs G, H, there is the straightforward result $\chi_r(G+H) = \chi_r(G) + \chi_r(H)$, so it is of interest that the position for overlap colourings is considerably more complex, even when H is a singleton vertex.

Let μ_1, μ_2 be, respectively, an (r_1, λ_1) and an (r_2, λ_2) overlap colouring of a graph G, using disjoint palettes. The *sum* $\mu_1 + \mu_2$ is the colouring resulting from placing the colour set $\mu_1(v) \cup \mu_2(v)$ at each vertex $v \in V(G)$; this is an $(r_1 + r_2, \lambda_1 + \lambda_2)$ colouring. More generally, if $\mu_1, \mu_2, \ldots, \mu_k$ are overlap colourings of G and $a_1, \ldots, a_k \in \mathbb{N}$, then $\sum_{i=1}^{k} a_i\mu_i$ represents (up to choice of palette) the overlap colouring arising from taking a_i isomorphic copies of μ_i $(1 \leq i \leq k)$, all palettes being disjoint, and placing the appropriate unions of colour sets at each vertex of G.

Now let μ be an (r, λ) overlap colouring of W_{2p+2}. The restriction μ_{rim} of μ to the rim vertices is an (r, λ) overlap colouring of C_{2p+1}, and we consider μ_{rim} to be the sum $\mu_{\mathrm{H}} + \mu_{\mathrm{R}}$ of two colourings, the first involving only colours that are on the hub and the second involving only colours that are not found on the hub. Thus, μ_{H} and μ_{R} allocate, respectively, λ and $r - \lambda$ colours to each vertex of C_{2p+1}. Then,

$$|P(\mu)| = r + |P(\mu_{\mathrm{R}})|,$$

despite the possibility that $|P(\mu_{\mathrm{H}})| < r$.

In most cases, μ_{H} and μ_{R} are true overlap colourings as defined above, but occasionally it is necessary to allow that, considered separately, μ_{H} and μ_{R} do not have constant overlap. For example, consider the following (5, 2) colouring of W_4.

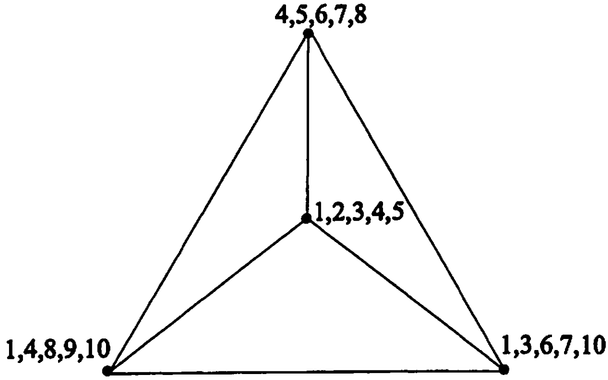

Figure 1

Then, μ_H allocates the colour sets $\{4,5\}$, $\{1,3\}$, $\{1,4\}$ while μ_R allocates colour sets $\{6,7,8\}$, $\{6,7,10\}$, $\{8,9,10\}$. It is not difficult to check that any $(5,2)$ colouring μ of W_4 where μ_H and μ_R are true overlap colourings requires at least 11 colours.

We shall see that W_4 is the only wheel for which, to minimize the palette size, μ_H and μ_R are sometimes required to have the above property.

The proof of Theorem 4

We deal first with the case $(r,\lambda) = (2,1)$. The remainder of the proof is then divided into lower-bound and upper-bound stages.

Proof that $\chi_{2,1}(W_{2p+2}) = 5$

Consider the following $(2,1)$ colouring μ of W_{2p+2}:

$\mu(h) = \{1,2\}$; $P(\mu_H) = \{1\}$; $P(\mu_R) = \{3,4,5\}$,

where μ_H and μ_R are respectively $(1,1)$ and $(1,0)$ colourings.

Thus, $\chi_{2,1}(W_{2p+2}) \leq 5$.

To show the lower bound, suppose μ is a $(2,1)$ colouring of W_{2p+2} with $|P(\mu)| \leq 4$, and let $\mu(h) = \{1,2\}$. It is clear that at least two further colours are needed; suppose $P(\mu) = \{1,2,3,4\}$. We may assume $\mu(v_0) = \{1,3\}$. Then it is straightforward to check that, for $1 \leq i \leq p$, we have $\mu(v_{2i}) = \{1,3\}$ or $\{2,4\}$, contradicting the requirement that $|\mu(v_{2p}) \cap \mu(v_0)| = 1$. □

Before dealing with the remaining cases, we prove the following straightforward property of finite sets.

Lemma 5 *Let $Q_0, Q_1, \ldots, Q_n$ be finite sets (where $n \geq 2$). Then* $|Q_n \backslash Q_0| \leq \sum_{i=1}^{n} |Q_i \backslash Q_{i-1}|$.

Proof For each $x \in Q_n \backslash Q_0$, let $j(x)$ be the least i such that $x \in Q_i$; then $1 \le j(x) \le n$ and $x \in Q_{j(x)} \backslash Q_{j(x)-1}$. Thus $Q_n \backslash Q_0 \subseteq \bigcup_{i=1}^{n}(Q_i \backslash Q_{i-1})$, and the result follows. □

The remaining cases: proof of lower bound

Note first that in Cases 2 and 4 ($\frac{r}{p+2} \le \lambda \le \frac{pr}{2p+1}$ and $\frac{(p+1)r}{2p+1} \le \lambda \le r$), by Corollary 3 the theorem states that $\chi_{r,\lambda}(W_{2p+2}) = \chi_{r,\lambda}(C_3)$. Since W_{2p+2} contains triangles, the lower bound follows immediately in these cases.

To obtain the lower bound in Cases 1 and 3, we use an extension of a lower-bound argument in Stahl's early paper [5].

Case 1: $0 \le \lambda \le \frac{r}{p+2}$

Let μ be an (r, λ) colouring of W_{2p+2} with $|P(\mu)| = \chi_{r,\lambda}(W_{2p+2})$ $= 2r - \lambda + s$; note that $s \ge 0$ since W_{2p+2} contains edges. Consider the colourings $\mu_{\mathrm{H}}, \mu_{\mathrm{R}}$ (we allow them to be 'improper' colourings), with palettes P_{H} and P_{R}; thus $|P_{\mathrm{H}}| \le r$, $|P_{\mathrm{R}}| = r - \lambda + s$.

For $0 \le i \le 2p$, let $S_i = \mu_{\mathrm{R}}(v_i)$, $T_i = \mu_{\mathrm{H}}(v_i)$, $W_i = S_i \cap S_{i+1}$, $X_i = T_i \cap T_{i+1}$ (see Figure 2 for $i = 0$), and for $0 \le i \le p-1$ let $Y_i = S_{2i+2} \backslash S_{2i}$, $Z_i = T_{2i+2} \backslash T_{2i}$. We use the corresponding lower-case symbols for the sizes of these sets. Thus $s_i = r - \lambda$, $t_i = \lambda$, $w_i + x_i = \lambda$ $(0 \le i \le 2p)$. We now find a bound on y_0.

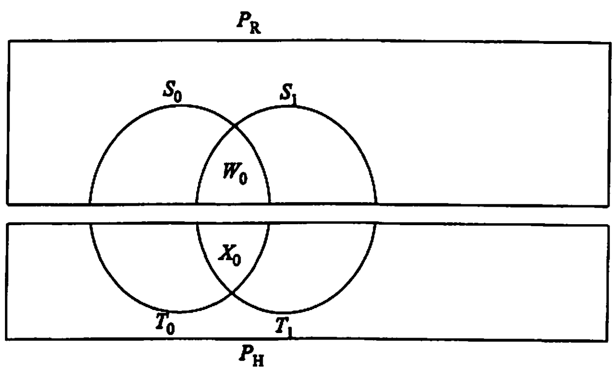

Figure 2

Let $A = P_{\mathrm{R}} \backslash (S_0 \cup S_1)$, $B = S_1 \backslash S_0$; then $Y_0 = (Y_0 \cap A) \cup (Y_0 \cap B)$. Now $|A| = r - \lambda + s - 2(r - \lambda) + w_0 = \lambda - r + s + w_0$, and $Y_0 \cap B \subseteq W_1$, giving $y_0 \le \lambda - r + s + w_0 + w_1$.

The argument is independent of i; thus

$$y_i \le \lambda - r + s + w_{2i} + w_{2i+1} \quad (0 \le i \le p-1). \tag{1}$$

Now v_{2p} is adjacent to v_0; thus $|S_{2p}\backslash S_0| = r - \lambda - w_{2p}$; moreover, by Lemma 5,

$$|S_{2p}\backslash S_0| \le \sum_{i=0}^{p-1} |S_{2i+2}\backslash S_{2i}|,$$

that is,

$$r - \lambda - w_{2p} \le \sum_{i=0}^{p-1} y_i \le p(\lambda - r + s) + \sum_{j=0}^{2p-1} w_j. \tag{2}$$

A little rearrangement gives

$$ps \ge (p+1)(r-\lambda) - \sum_{i=0}^{2p} w_i \ge (p+1)(r-\lambda) - (2p+1)\lambda.$$

Thus, $|P_R| = r - \lambda + s \ge \lceil \frac{(2p+1)(r-2\lambda)}{p} \rceil$, and so

$$\chi_{r,\lambda}(W_{2p+2}) \ge r + \left\lceil \frac{(2p+1)(r-2\lambda)}{p} \right\rceil,$$

giving the required lower bound for Case 1.

Case 3: $\frac{pr}{2p+1} \le \lambda \le \frac{(p+1)r}{2p+1}$

Note that

$$r \ge |T_i \cup T_{i+1}| = 2\lambda - x_i \ (0 \le i \le 2p),$$

so that $x_i \ge 2\lambda - r$. Proceeding as above, the analogues of equations (1) and (2) are

$$z_i \le r - 2\lambda + x_{2i} + x_{2i+1} \ (0 \le i \le p-1),$$

and

$$\lambda - x_{2p} \le \sum_{i=0}^{p-1} z_i,$$

yielding

$$\sum_{i=0}^{2p} x_i \ge (2p+1)\lambda - pr. \tag{3}$$

Thus, for some index j we have $x_j \ge \lambda - \frac{pr}{2p+1}$, and so

$$w_j \le \frac{pr}{2p+1}.$$

Thus, $|S_j \cup S_{j+1}| \ge 2(r-\lambda) - \frac{pr}{2p+1}$, giving, as required,

$$\chi_{r,\lambda}(W_{2p+2}) \ge 3r - 2\lambda - \left\lfloor \frac{pr}{2p+1} \right\rfloor.$$

The remaining cases: proof of upper bound

We deal with the simple cases 1, 2, 4 first.

Cases 1 and 2: $0 \le \lambda \le \frac{pr}{2p+1}$

Let μ_1 be a $(\lambda, 0)$ colouring of C_{2p+1} with $|P(\mu_1)| \le r$; this is possible by Theorem 2, since $r \ge \lceil \frac{(2p+1)\lambda}{p} \rceil$. Next, let μ_2 be an $(r-\lambda, \lambda)$ colouring of C_{2p+1} with $|P(\mu_2)| = \chi_{r-\lambda,\lambda}(C_{2p+1})$. Thus, $|P(\mu_2)| = \lceil \frac{(2p+1)(r-2\lambda)}{p} \rceil$ if $0 \le \lambda \le \frac{r}{p+2}$; $|P(\mu_2)| = 2r - 3\lambda$ if $\frac{r}{p+2} \le \lambda \le \frac{pr}{2p+1}$. Then, there is an (r, λ) colouring μ of W_{2p+2} such that $\mu_{\mathrm{H}} = \mu_1$ and $\mu_{\mathrm{R}} = \mu_2$. Thus,

$$|P(\mu)| = \max\left\{r + \left\lceil \frac{(2p+1)(r-2\lambda)}{p} \right\rceil, 3(r-\lambda)\right\},$$

as required.

Case 4: $\frac{(p+1)r}{2p+1} \le \lambda \le r$

Let μ_1 be the trivial $(r-\lambda, r-\lambda)$ colouring of C_{2p+1} with $|P(\mu_1)| = r - \lambda$, and let μ_2 be a $(\lambda, 2\lambda - r)$ colouring of C_{2p+1} with $|P(\mu_2)| \le r$; this is possible by Theorem 2. Again, there is an (r, λ) colouring μ of W_{2p+2} with $\mu_{\mathrm{H}} = \mu_1$ and $\mu_{\mathrm{R}} = \mu_2$, giving

$$|P(\mu)| = 2r - \lambda.$$

Case 3: $\frac{pr}{2p+1} \le \lambda \le \frac{(p+1)r}{2p+1}$

When $r < 3$, the only relevant pair for Case 3 is $(r, \lambda) = (2, 1)$, which has been considered separately. Thus we may now assume $r \ge 3$.

There are two subcases.

Subcase 3(a): $p \ge 2$

We require the following technical lemma.

Lemma 6 *Let* $r \ge 3$, $p \ge 2$, $L = \lfloor \frac{pr}{2p+1} \rfloor$ *and* $\lambda \ge \frac{pr}{2p+1}$; *then*

$$L \ge \frac{r}{2p+1}, \tag{4}$$

$$2(r-\lambda) - L \ge \left\lceil \frac{(2p+1)(r-\lambda-L)}{p} \right\rceil. \tag{5}$$

Proof

(i) Note that

$$\frac{pr}{2p+1} - \frac{r}{2p+1} \geq 1$$

whenever $p \geq 2$ and $r \geq 3$, except for the cases $(p,r) = (2,3)$, $(2,4)$, $(3,3)$. Thus (4) holds except possibly for these three instances, which may be checked by hand.

(ii) From (4) we obtain $(p+1)L \geq \frac{(p+1)r}{2p+1}$.

Now $r - \lambda \leq \frac{(p+1)r}{2p+1}$, and hence

$$(p+1)L \geq r - \lambda,$$

$$2p(r-\lambda) + (p+1)L \geq (2p+1)(r-\lambda),$$

$$2p(r-\lambda) - pL \geq (2p+1)(r-\lambda-L),$$

$$2(r-\lambda) - L \geq \frac{(2p+1)(r-\lambda-L)}{p},$$

and since $2(r-\lambda) - L \in \mathbb{N}$, (5) follows. □

Let $L = \lfloor \frac{pr}{2p+1} \rfloor$. Since $\frac{pr}{2p+1} \leq \lambda$, we may consider a $(\lambda, \lambda - L)$ colouring μ_1 of C_{2p+1}. Then $|P(\mu_1)| \leq r$ by Theorem 2, since $r \geq \max\{\lceil \frac{(2p+1)L}{p} \rceil, \lambda + L\}$.

Next, let μ_2 be an $(r - \lambda, L)$ colouring of C_{2p+1} with $|P(\mu_2)| = 2(r-\lambda) - L$; this is possible by Theorem 2 and (5).

Arguing as in the previous cases, there is an (r, λ) colouring μ of W_{2p+2} with $\mu_{\mathrm{H}} = \mu_1$ and $\mu_{\mathrm{R}} = \mu_2$, thus giving the required upper bound.

Case 3(b): $p = 1$

In this case, W_{2p+2} is the complete graph K_4. We note the existence of the following colourings of W_4 (Figure 3).

- Trivially there is a $(1,1)$ colouring μ_{11} with $|P(\mu_{11})| = 1$.
- By Case 2 there is a $(3,1)$ colouring μ_{31} with $|P(\mu_{31})| = 6$.
- By Case 4 there is a $(3,2)$ colouring μ_{32} with $|P(\mu_{32})| = 4$.

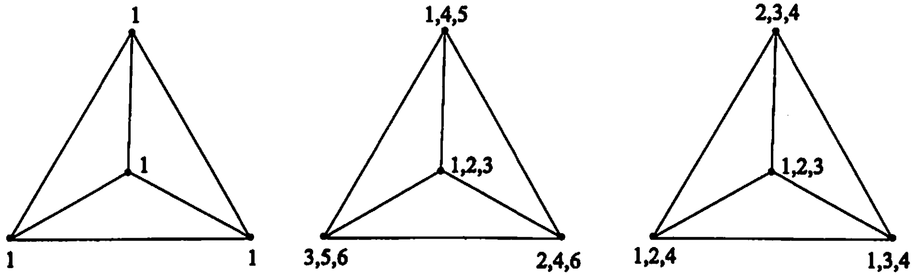

Figure 3

Finally, recall that the $(5,2)$ colouring presented in Figure 1 (which we now denote by μ_{52}) has $|P(\mu_{52})| = 10$.

Now let $r = 3L + q$ (where $0 \leq q \leq 2$).

Subcase 1: $q = 0$

Then $\lambda = L + k$ where $0 \leq k \leq L$.

For each such value of k, the colouring $\mu = (L-k)\mu_{31} + k\mu_{32}$ is a $(3L, L+k)$ colouring of W_4 with $|P(\mu)| = 6L - 2k = 3r - 2\lambda - L$.

Subcase 2: $q = 1$

Then $\lambda = L + k$ where again $1 \leq k \leq L$.

For each such value, the colouring $\mu = \mu_{11} + (L-k+1)\mu_{31} + (k-1)\mu_{32}$ is a $(3L+1, L+k)$ colouring of W_4 with $|P(\mu)| = 6L-2k+3 = 3r-2\lambda-L$.

Subcase 3: $q = 2$

Then $\lambda = L + k$ where $1 \leq k \leq L+1$.

If $k = 1$, take $\mu = (L-1)\mu_{31} + \mu_{52}$; this is a $(3L+2, L+1)$ colouring of W_4 with $|P(\mu)| = 6L + 4 = 3r - 2\lambda - L$.

Otherwise, take $\mu = 2\mu_{11} + (L-k+2)\mu_{31} + (k-2)\mu_{32}$; this is a $(3L+2, L+k)$ colouring of W_4 with $|P(\mu)| = 6L-2k+6 = 3r-2\lambda-L$. □

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