Restricted Simple 1-Designs

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ABSTRACT. Given a partition $\{P_1, \ldots, P_m\}$ of a v-set, a restricted simple 1-design is a collection of distinct subsets (blocks) such that every element occurs in same number of blocks but any two elements from the same part do not occur together in the same block. We give a construction of restricted simple 1-designs to show that the necessary conditions are sufficient for the existence of restricted simple 1-designs.

1. Introduction

Let v, k and r be positive integers. A simple 1-(v, k, r) design[2] is a collection $\mathscr{B}$ of k-subsets (blocks) on a v-set V, so that no two blocks are the same (simple) and each element/point of V occurs exactly r times (1-design). Billington[1] gave an elegant proof for the existence of a simple 1-(v, k, r) design. Are there any interesting questions remain open for the simplest of balanced designs? We think that the problem discussed here is one such question. Challenge is to get a solution as appealing as Billington's technique. Suppose that a v-set is partitioned into $\mathscr{P} = \{P_1, P_2, \ldots, P_m\}$. A restricted simple 1-$(v, k, r)$ design, $\mathscr{D} = (V, \mathscr{P}, \mathscr{B})$, is a simple 1-$(v, k, r)$ design where any two elements from the same part in $\mathscr{P}$ do not occur together in a block.

Note that a restricted simple 1-(v, k, r) design in which all parts are size one is a simple 1-(v, k, r) design. Certain restricted simple 1-designs can be constructed using simple balanced incomplete block designs (BIBDs) when a parallel class exist (The blocks from the parallel class can be considered as the parts). Every simple group divisible design with all groups of same size or of different sizes also gives a restricted simple 1-design where the groups play the role of parts of the partition.

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Naturally for the same v-set, we can have many partitions and for certain partition a restricted simple 1-design may exists and for some other partition the design may not exist. The following example is instructive.

EXAMPLE 1. *Let* $V = \{1,2,3,4,5,6\}$, *we consider the existence or non-existence of a restricted simple 1-(6,3,3) design for all possible non-isomorphic partitions:*

(1) First consider the partition of V *into* $\{\{1,2\},\{3,4\},\{5,6\}\}$ *(each part of the same size* 2*). The set of blocks* $\{\{1,3,5\},\{1,3,6\},\{1,4,5\},\{2,3,6\},\{2,4,5\},\{2,4,6\}\}$ *gives a restricted simple 1-(6,3,3) design. All 6 elements occur 3 times in the blocks of size 3 and any two elements from the same part do not occur together in any block. Interestingly, for the partitions* $(\{\{1,2\},\{3,4\},\{5\},\{6\}\}$ *or* $\{\{1,2\},\{3\},\{4\},\{5\},\{6\}\}$ *or* $\{\{1\},\{2\},\{3\},\{4\},\{5\},\{6\}\})$, *the same set of blocks also give restricted simple 1-designs.*

(2) On the other hand, suppose V *is partitioned into* $\{\{1,2,3\},\{4,5,6\}\}$. *Then a restricted simple 1-(6,3,3) design does not exist, because each block must have 3 elements that come from different parts but this design has only 2 parts.*

(3) Lastly, suppose V *is partitioned into* $\{\{1\},\{2\},\{3\},\{4,5,6\}\}$. *A restricted simple 1-(6,3,3) design does not exist, because for each element to occur 3 times in blocks of size 3, the number of blocks required is* $\frac{6(3)}{3} = 6$, *but at least 9 blocks are required to take care of the three elements from the part of size three,* $\{4,5,6\}$.

The following example shows some of the difficulty of proving the existence of restricted simple 1-designs but it also gives a direct construction of a family of restricted simple 1-designs for $k = 2$.

EXAMPLE 2. *Let* V *be a* v*-set, suppose that* V *is partitioned into only 2 parts with* $p = |P_1| \neq |P_2| = q$, *a restricted simple 1-*$(v,2,r)$ *design does not exist. If such a design exist with* b *blocks, as the elements from the same part do not occur in the same block,* $b = pr = qr$ *and hence* $p = q$. *On the other hand, if* $|P_1| = |P_2|$, *a restricted simple 1-*$(v,2,r)$ *design exists for any* $r \leq |P_1|$. *To see this, note that* $vr = 2b$ *and a restricted simple 1-*(v,k,r) *design requires* $b \leq |P_1||P_2| = v^2/4$, $vr \leq v^2/2$, *so* $r \leq v/2$. *For* $r = v/2$, *the required design is the set of all subsets of size two with the first element from* P_1 *and the second element from* P_2. *Let us denote the elements of* P_1 *by* $1,3,5,\ldots,v-1$ *and of* P_2 *by* $2,4,\ldots,v$. *To get a design with replication* $v/2-1$ *from the design with replication* $v/2$, *delete sets* $\{1,2\},\{3,4\},\ldots,\{v-1,v\}$. *In general, to get a design with replication* $v/2-i$, *where* $i = 1,2,\ldots,v/2-1$ *from the design with replication* $v/2$, *delete sets* $\{1,2\},\{1,4\},\ldots,\{1,2i\};\{3,4\},\ldots,\{3,2(i+1)\};\ldots;\{v-1,v\},\ldots\{v-1,(2i+(v-2))/2\}$ *where the second element of each block is an even integer modulo* v.

2. Simple constructions

In this section, we give two simple methods of constructing new restricted simple 1-designs from the existing one. The first construction may be called a *refinement construction* and the second construction may be called a *sum construction* .

Any partition set $\mathscr{A}$ on a set X is a *refinement* of a partition set $\mathscr{P}$ on X, if every element of $\mathscr{A}$ is a subset of some element of $\mathscr{P}$.

THEOREM 1. *Suppose there exists a restricted simple 1-(v,k,r) design, $(V,\mathscr{P},\mathscr{B})$. Then there also exists a restricted simple 1-(v,k,r) design with any refinement of $\mathscr{P}$.*

PROOF. Since points in a block come from different parts in $\mathscr{P}$, they also come from different parts in any refinement of $\mathscr{P}$. □

COROLLARY 1. *If for some positive integer t the union of t parts from a partition $\{P_1, P_2, \ldots, P_m\}$ of X is equal to the size of the union of the remaining $m-t$ parts, then a restricted simple 1-design on X with block size 2 exists.*

PROOF. Above theorem and the construction given in Example 2 of a restricted simple 1-design with block size 2 for a partition $\{P_1, P_2\}$ with $|P_1| = |P_2|$ give the result. □

Unfortunately, the converse of the above corollary is not true. For example, for $\mathscr{P} = \{\{1\},\{2\},\{3\},\{4\},\{5,6,7\}\}$, we can easily construct a restricted simple 1-$(7,2,4)$ design and a restricted simple 1-$(7,2,2)$ design, but as v is odd,no partition with two same size parts is possible. On the other hand as $v = 7$ is odd, for simple design r has to be 2 or 4.
A set of blocks of a restricted simple 1-$(7,2,4)$ design is
$\{\{5,1\},\{5,2\},\{5,3\},\{5,4\},\{6,1\},\{6,2\},\{6,3\},\{6,4\},\{7,1\},\{7,2\},\{7,3\},\{7,4\},\{1,2\},\{3,4\}\}$ and the blocks of a restricted simple 1-$(7,2,2)$ design are $\{5,1\},\{5,2\},\{6,3\},\{6,4\},\{7,1\},\{7,2\}$, and $\{3,4\}$.

THEOREM 2. *Suppose that there exist a restricted simple 1-(v_1,k,r) design, $(V_1,\mathscr{P}_1,\mathscr{B}_1)$ and a restricted simple 1-(v_2,k,r) design $(V_2,\mathscr{P}_2,\mathscr{B}_2)$ with $V_1 \cap V_2 = \emptyset$. Then there exists a restricted simple 1-(v_1+v_2,k,r) design.*

PROOF. $(V_1 \cup V_2, \mathscr{P}_1 \cup \mathscr{P}_2, \mathscr{B}_1 \cup \mathscr{B}_2)$ is a restricted simple 1-$(v_1 + v_2, k, r)$ design. □

3. General Necessary conditions

It is known that if a simple 1-(v,k,r) design exists with b blocks, then $b \le \binom{v}{k}$ and $vr = bk$ ([**1**]).

THEOREM 3. *If a restricted simple 1-(v,k,r) design, $\mathscr{D} = (V, \mathscr{P}, \mathscr{B})$, where $\mathscr{P} = \{P_1, P_2, \ldots, P_m\}$ with $|P_i| = p_i$ for all $i = 1, 2, \ldots, m$ exists, then, assuming with out loss of generality $p_1 \geq p_2 \geq \cdots \geq p_m$, the following conditions hold.*

(1) $vr = bk$,

(2) $p_1 r \leq b$,

(3) $k \leq min\{\frac{v}{p_1}, m\}$, *so* $p_1 k \leq v$,

(4) $b \leq \sum_{\{\alpha_1,\alpha_2,\ldots,\alpha_k\}\subseteq\{1,\ldots,m\}} p_{\alpha_1} p_{\alpha_2} \cdots p_{\alpha_k}$,

(5) $r \leq \sum_{\{\alpha_1,\alpha_2,\ldots,\alpha_{k-1}\}\subseteq\{2,\ldots,m\}} p_{\alpha_1} p_{\alpha_2} \cdots p_{\alpha_{k-1}}$, *and*

(6) *If* $V = \{1, 2, \ldots, v\}$, *then for all* $j = 1, \ldots, m$, $\sum_{\forall i \in P_j} r_i = p_j r$ *where* r_i *is the replication number of point* i *for* $i \in V$.

PROOF. Condition (1) follows from the equireplicate property of a simple 1-(v,k,r) design.

Since each point in the part P_1 must occur r times in blocks, the number of distinct blocks is at least $p_1 r$.

Since each block must have k points that come from different parts by definition, $k \leq m$. Using conditions (1) and (2), we get $k \leq \frac{v}{p_1}$. Note, $k = vr/b \leq vr/p_1 r$

Since the number of distinct k-subsets such that each point comes from different parts is

$$\sum_{\{\alpha_1,\alpha_2,\ldots,\alpha_k\}\subseteq\{1,\ldots,m\}} p_{\alpha_1} p_{\alpha_2} \cdots p_{\alpha_k},$$

it follows that the maximum number of blocks is

$$\sum_{\{\alpha_1,\alpha_2,\ldots,\alpha_k\}\subseteq\{1,\ldots,m\}} p_{\alpha_1} p_{\alpha_2} \cdots p_{\alpha_k}.$$

The maximum number of r may be achieved if each element of P_1 comes in each of the blocks of size $k-1$ from any $k-1$ parts from $P_2, P_3, \ldots, P_m$. Essentially this is the same argument as for the condition (4).

Since each point $i \in P_j$ is contained in exactly r blocks, the sum of replication numbers of $i \in P_j$ is $\underbrace{r + r + \cdots + r}_{p_j} = p_j r$. □

4. A construction for restricted simple 1-designs

Billington [1] gave an elegant proof for the existence of a simple 1-(v,k,r) design. Based on the idea in the proof by Billington, we first introduce a new design and then present a construction for a restricted simple 1-(v,k,r) design when the size of partition is arbitrary. Let v, k be positive integers and $r_1, r_2, \ldots, r_v$ be nonnegative integers. A restricted simple $(k; r_1, r_2, \ldots, r_v)$ design is a set of b k-subsets (blocks) of a v-set $V=\{1,\ldots,v\}$ such that the elements of k-subsets come from different parts of a partition $\mathscr{P}=\{P_1, P_2, \ldots, P_m\}$ of V, (In other words, no two blocks are the same and no two elements from a block come from the same part of $\mathscr{P}$.) and for all $i \in V$ the number of blocks containing i is r_i. Hence a restricted simple 1-(v,k,r) design is a restricted simple $(k; r, r, \ldots, r)$ design in which $r_i = r$ for all $i \in V$. When there is no confusion, we use $\mathscr{D} = (V, \mathscr{P}, \mathscr{B})$ to denote the restricted simple $(k; r_1, r_2, \ldots, r_v)$ design without mentioning the parameters of the design.

EXAMPLE 3. , *Let* $V = \{1, 2, \ldots, 15\}$, $\mathscr{P} = \{\{1,2,3,4,5\}, \{6,7,8,9\}, \{10,11,12,13\}, \{14,15\}\}$ *and* $\mathscr{B} = \{\{1,6,10\}, \{2,6,10\}, \{3,6,10\}, \{4,6,10\}, \{5,6,10\}, \{1,7,10\}, \{1,8,10\}, \{1,9,10\}, \{1,6,11\}, \{1,6,12\}, \{1,6,13\}, \{2,7,10\}, \{1,6,14\}, \{2,6,14\}, \{3,6,14\}, \{4,6,14\}, \{1,10,14\}, \{2,10,14\}, \{3,10,14\}, \{4,10,14\}\}$. *Then* $(V, \mathscr{P}, \mathscr{B})$ *is a restricted simple* $(3; 9,4,3,3,1,12,2,1,1,13,1,1,1,8,0)$ *design.*

In what follows, the next proposition shows how a restricted simple $(k; r_1, r_2, \ldots, r_v)$ design can be made "more regular". Then a construction of a restricted simple $(k; r_1, r_2, \ldots, r_v)$ design with certain property is given as a Lemma and finally, a procedure is described in a theorem to construct a restricted simple 1-(v,k,r) design.

PROPOSITION 1. *Let* v, m *and* k *be positive integers such that* $2 \le k \le m \le v$ *and* $r_1, r_2, \ldots, r_v$ *be nonnegative integers. Let* $V = \{1, 2, \ldots, v\}$ *be partitioned into* m *parts* $\mathscr{P} = \{P_1, P_2, \ldots, P_m\}$. *Suppose that a restricted simple* $(k; r_1, r_2, \ldots, r_v)$ *design exists. If* $x, y \in P_w$ *for some integer* w, $1 \le w \le m$ *with* $r_x > r_y$, *then there exists a restricted simple* $(k; r_1, r_2, \ldots, r_x - 1, \ldots, r_y + 1, \ldots, r_v)$ *design.*

PROOF. Let $\mathscr{D} = (V, \mathscr{P}, \mathscr{B})$ be a restricted simple $(k; r_1, r_2, \ldots, r_v)$ design. Let $B_1, B_2, \ldots, B_i$ be all blocks of $\mathscr{B}$ which contain point x. Let $C_1, C_2, \ldots, C_j$ be all blocks of $\mathscr{B}$ which contain point y. First step, we set $\mathscr{B}^x = \{B_1{}^x, B_2{}^x, \ldots, B_i{}^x\}$ where $B_a{}^x = B_a - \{x\}$ and $\mathscr{C}^y = \{C_1{}^y, C_2{}^y, \ldots, C_j{}^y\}$ where $C_b{}^y = C_b - \{y\}$. Second step, we can choose a block $B \in \mathscr{B}^x - \mathscr{C}^y$ which has property that if $B^* = B \cup \{y\}$ then $B^* \notin \{C_1, C_2, \ldots, C_j\}$ (note that, $|\mathscr{B}^x - \mathscr{C}^y| \ge r_x - r_y$ or there are at least $r_x - r_y$ such B's). Final step, let $\mathscr{B}^* = [\mathscr{B} - (B \cup \{x\})] \cup \{B^*\}$, i.e., we delete $B \cup \{x\}$ from $\mathscr{B}$ and replace it by B^*. This implies that, $\mathscr{D}^* = (V, \mathscr{P}, \mathscr{B}^*)$ is a design with the replication number of x and y in $\mathscr{D}^*$ changed to $r_x - 1$ and $r_y + 1$, respectively.The design is simple as $y \in B^*$ and $B^* \notin \{C_1, C_2, \ldots, C_j\}$ implies B^* is different from other blocks. Also, no two elements form the same part come in the same block as x and any points in B come from different parts and $x, y \in P_w$, y and any points in B^* must come from different parts of $\mathscr{P}$ as well. $\square$

EXAMPLE 4. *As an illustration of Proposition 1, consider Example 3. Points* 6, *and* 7 *in the part* $\{6,7,8,9\}$ *occur with frequencies* $r_6 = 12$ *and* $r_7 = 2$ *respectively.* $B_1 = \{1,6,10\}, B_2 = \{2,6,10\}, B_3 = \{3,6,10\}, B_4 = \{4,6,10\}, B_5 = \{5,6,10\}, B_6 = \{1,6,11\}, B_7 = \{1,6,12\}, B_8 = \{1,6,13\}, B_9 = \{1,6,14\}, B_{10} = \{2,6,14\}, B_{11} = \{3,6,14\}, B_{12} = \{4,6,14\}$ *are the blocks in* $\mathscr{B}$ *containing the point* 6. *Also* $C_1 = \{1,7,10\}$, *and* $C_2 = \{2,7,10\}$ *are the only two blocks in* $\mathscr{B}$ *containing point* 7. *First step, we set* $\mathscr{B}^6 = \{B_1{}^6, B_2{}^6, \ldots, B_{12}{}^6\}$ *where* $B_1{}^6 = B_1 - \{6\} = \{1,10\}, B_2{}^6 = B_2 - \{6\} = \{2,10\}, \ldots, B_{12}{}^6 = B_{12} - \{6\} = \{4,14\}$ *and* $\mathscr{C}^7 = \{C_1{}^7, C_2{}^7\}$ *where* $C_1{}^7 = C_1 - \{7\} = \{1,10\}, C_2{}^7 = C_2 - \{7\} = \{2,10\}$. *Second step, we choose* $B = \{3,10\} \in \mathscr{B}^6 - \mathscr{C}^7$ *since* $B^* = B \cup \{7\} = \{3,7,10\}$ *and* $\{3,7,10\} \notin \{C_1, C_2\}$. *Final step, we delete* $B = \{3,6,10\}$ *from* $\mathscr{B}$ *and replace it by* $B^* = \{3,7,10\}$ *to obtain a restricted* $(3;9,4,3,3,1,11,3,1,1,13,1,1,1,8,0)$ *design* $(V, \mathscr{P}, \mathscr{B}^*)$ *with* $\mathscr{B}^* = \{\{1,6,10\},\{2,6,10\},\{3,7,10\},\{4,6,10\},\{5,6,10\}, \{1,7,10\},\{1,8,10\},\{1,9,10\},\{1,6,11\},\{1,6,12\},\{1,6,13\},\{2,7,10\}, \{1,6,14\},\{2,6,14\},\{3,6,14\},\{4,6,14\},\{1,10,14\},\{2,10,14\},\{3,10,14\}, \{4,10,14\}\}$.

LEMMA 1. *Let* v, m, k *and* r *be positive integers such that* $2 \le k \le m \le v$. *Let* $V = \{1,2,\ldots,v\}$ *be partitioned into* m *parts* $\mathscr{P} = \{P_1, P_2, \ldots, P_m\}$ *of size* $p_1, p_2, \ldots, p_m$, *respectively such that* $p_1 \ge p_2 \ge \cdots \ge p_m$. *Suppose* $vr \equiv 0 \pmod{k}$, $p_1 k \le v$ *and* $r \le \sum_{\{\alpha_1,\alpha_2,\ldots,\alpha_{k-1}\} \subseteq \{2,\ldots,m\}} p_{\alpha_1} p_{\alpha_2} \cdots p_{\alpha_{k-1}}$.
Then there are nonnegative integers $r_1, r_2, \ldots, r_v$ *such that* $\sum_{i \in P_j} r_i = p_j r$ *for all* $j = 1,2,\ldots,m$ *and a restricted simple* $(k; r_1, r_2, \ldots, r_v)$ *design exists with* $\frac{vr}{k}$ *blocks where* r_i *is the replication number of point* i *for* $i \in V$.

PROOF. Without loss of generality, let $P_i = \{x_{ij} \in V \mid j = 1,\ldots,p_i\}$ for $i = 1,\ldots,m$. First, we construct a restricted $(k; p_1 r, \underbrace{0,\ldots,0}_{(p_1-1)\text{ terms}}, p_2 r, \underbrace{0,\ldots,0}_{(p_2-1)\text{ terms}}, \ldots, p_m r, \underbrace{0,\ldots,0}_{(p_m-1)\text{ terms}})$ design, say $\mathscr{D}_1$ by constructing blocks of size k, $B_1, B_2, \ldots, B_{\frac{vr}{k}}$ as described below. Place point x_{11} in $p_1 r$ blocks $B_1, \ldots, B_{p_1 r}$. Then continue placing point x_{21} in $p_2 r$ blocks $B_{p_1 r+1}, \ldots, B_{p_1 r + p_2 r}$ where the subscripts are added modulo $\frac{vr}{k}$. The same method is applied to distribute the points $x_{31}, \ldots, x_{m1}$ in the blocks. Hence each point $x_{11}, x_{21}, \ldots, x_{m1}$ occurs $p_1 r, p_2 r, \ldots, p_m r$ times, respectively in different blocks. Since the total of replication numbers of $x_{11}, x_{21}, \ldots, x_{m1}$ is $p_1 r + p_2 r + \cdots + p_m r = (p_1 + p_2 + \cdots + p_m) r = vr$ and $vr \equiv 0 \pmod{k}$, it follows that $\mathscr{D}_1$ is a restricted $(k; p_1 r, \underbrace{0,\ldots,0}_{(p_1-1)\text{ terms}}, p_2 r,$

$\underbrace{0,\ldots,0}_{(p_2-1)\text{ terms}},\ldots,p_m r,\underbrace{0,\ldots,0}_{(p_m-1)\text{ terms}}$) design, with $\frac{vr}{k}$ blocks, but $\mathscr{D}_1$ may not be simple. Note that for all $j=1,2,\ldots,m$, $\sum_{i\in P_j} r_i = p_j r + \underbrace{0+0+\cdots+0}_{(p_j-1)\text{ terms}} = p_j r$.

Assume $\mathscr{D}_1$ contains exactly l distinct blocks occurring with frequencies $\mu_1,\mu_2,\ldots,\mu_l$. Let $\mathscr{A}_1,\mathscr{A}_2,\ldots,\mathscr{A}_l$ be the collections of $\mu_1,\mu_2,\ldots,\mu_l$ repeated blocks, respectively. Without loss of generality, suppose that $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_l$. Next, construct a restricted simple $(k; s_{11},\ldots,s_{1p_1},s_{21},\ldots, s_{2p_2},\ldots,s_{m1},\ldots,s_{mp_m})$ design, where $s_{i1}+\cdots+s_{ip_i} = p_i r$, for $i=1,\ldots,m$ as follows.

Let $\{a_{\beta_1 1}, a_{\beta_2 1},\ldots,a_{\beta_k 1}\}$ be the block repeated μ_1 times in $\mathscr{A}_1$, where $a_{\beta_i 1}\in P_{\beta_i}$, for $i=1,\ldots,k$ and $\{\beta_1,\beta_2,\ldots,\beta_k\}\subseteq\{1,\ldots,m\}$. Without loss of generality, assume $p_{\beta_1}\geq p_{\beta_2}\geq\cdots\geq p_{\beta_k}$.

Since there are at most rp_{β_k} repeated blocks in $\mathscr{D}_1$ containing $a_{\beta_k 1}$, $\mu_1 \leq rp_{\beta_k}$. Since there are $p_{\beta_1}p_{\beta_2}\cdots p_{\beta_k}$ k-subsets for which each point comes from the parts $P_{\beta_1},P_{\beta_2},\ldots,P_{\beta_k}$ and $r \leq \sum_{\{\alpha_1,\alpha_2,\ldots,\alpha_{k-1}\}\subseteq\{2,\ldots,m\}} p_{\alpha_1}p_{\alpha_2}\cdots p_{\alpha_{k-1}}$, it follows that $\mu_1 \leq rp_{\beta_k} \leq p_{\beta_k}\left(\sum_{\{\alpha_1,\alpha_2,\ldots,\alpha_{k-1}\}\subseteq\{2,\ldots,m\}} p_{\alpha_1}p_{\alpha_2}\cdots p_{\alpha_{k-1}}\right)$. Replace μ_1 repeated blocks by any distinct k-subsets, each consisting of exactly one point from each of the parts $P_{\beta_1},P_{\beta_2}\ldots,P_{\beta_k}$. Note that the replication number of points in each part P_i from the parts $P_{\beta_1},P_{\beta_2},\ldots,P_{\beta_k}$ is changed from $p_i r,0,\ldots,0$ to $s_{i1}{}^{(1)},\ldots,s_{ip_i}{}^{(1)}$ such that $s_{i1}{}^{(1)}+\cdots+s_{ip_i}{}^{(1)} = p_i r$, for all $i=1,\ldots,k$. Apply the same process to $\mathscr{A}_2,\mathscr{A}_3$, until $\mathscr{A}_l$. Of course, the replication numbers of each part P_i whenever changed will still satisfy the condition that the sum of the replications of all elements in $P_i = p_i r$. Since $\mathscr{A}_1,\mathscr{A}_2,\ldots,\mathscr{A}_l$ are different collections of repeated blocks, and even if an element comes in several of $\mathscr{A}$'s, it follows that each time the repeat blocks can be replaced by different new k-subsets because of the condition on r and the last design will be a restricted 1-design with replication numbers as desired. □

EXAMPLE 5. *Let* $v=15$ *and* $k=3$. *Let* $V=\{1,2,\ldots,15\}$ *be partitioned into* 4 *parts* $\mathscr{P}=\{P_1,P_2,P_3,P_4\}$ *of size* $p_1=5, p_2=4, p_3=4, p_4=2$, *respectively. Note that* $p_1 k = 5\cdot 3 \leq 15 = v$ *and* $r \leq p_2p_3+p_2p_4+p_3p_4 = 32$. *Since* $vr = 15r \equiv 0 \pmod{k}$, *choose* $r=4$. *Without loss of generality, suppose that* $P_1=\{1,2,3,4,5\}=\{x_{11},x_{12},x_{13},x_{14},x_{15}\}$, $P_2=\{6,7,8,9\}=\{x_{21},x_{22},x_{23},x_{24}\}$, $P_3=\{10,11,12,13\}=\{x_{31},x_{32},x_{33},x_{34}\}$ *and* $P_4=\{14,15\}=\{x_{41},x_{42}\}$. *First, we construct a restricted* $(3;20,0,0,0,0,16,0,0,0,16,0,0,0,8,0)$ *design, say* $\mathscr{D}_1$ *with blocks:*
$\{x_{11},x_{21},x_{31}\},\{x_{11},x_{21},x_{31}\},\{x_{11},x_{21},x_{31}\},\{x_{11},x_{21},x_{31}\},\{x_{11},x_{21},x_{31}\}$,

$\{x_{11},x_{21},x_{31}\},\{x_{11},x_{21},x_{31}\},\{x_{11},x_{21},x_{31}\},\{x_{11},x_{21},x_{31}\},\{x_{11},x_{21},x_{31}\},$ $\{x_{11},x_{21},x_{31}\},\{x_{11},x_{21},x_{31}\},\{x_{11},x_{21},x_{41}\},\{x_{11},x_{21},x_{41}\},\{x_{11},x_{21},x_{41}\},$ $\{x_{11},x_{21},x_{41}\},\{x_{11},x_{31},x_{41}\},\{x_{11},x_{31},x_{41}\},\{x_{11},x_{31},x_{41}\},\{x_{11},x_{31},x_{41}\}.$
Note that $r_1+r_2+r_3+r_4+r_5 = 20+0+0+0+0 = 5\times 4$, $r_6+r_7+r_8+r_9 = 16+0+0+0 = 4\times 4$, $r_{10}+r_{11}+r_{12}+r_{13} = 16+0+0+0 = 4\times 4$ *and* $r_{14}+r_{15} = 8+0 = 2\times 4$. $\mathscr{D}_1$ *is not simple, let* $\mathscr{A}_1 = 12\{x_{11},x_{21},x_{31}\}$ *be the collection of* $\mu_1 = 12$ *repeated blocks. Let* $\mathscr{A}_2 = 4\{x_{11},x_{21},x_{41}\}$ *be a collection of* $\mu_2 = 4$ *repeated blocks and finally let* $\mathscr{A}_3 = 4\{x_{11},x_{31},x_{41}\}$ *be a collection of* $\mu_3 = 4$ *repeated blocks.*
Replace the repeated blocks in $\mathscr{A}_1$ *with any different* 3*-subsets from the possible eighty* 3*-subsets where each subset contains one point from each of* P_1, P_2 *and* P_3.

For example, we obtain a restricted $(3;15,2,1,1,1,12,2,1,1,13,1,1,1,8,0)$ *design with blocks:*
$\{x_{11},x_{21},x_{31}\},\{x_{12},x_{21},x_{31}\},\{x_{13},x_{21},x_{31}\},\{x_{14},x_{21},x_{31}\},\{x_{15},x_{21},x_{31}\},$ $\{x_{11},x_{22},x_{31}\},\{x_{11},x_{23},x_{31}\},\{x_{11},x_{24},x_{31}\},\{x_{11},x_{21},x_{32}\},\{x_{11},x_{21},x_{33}\},$ $\{x_{11},x_{21},x_{34}\},\{x_{12},x_{22},x_{31}\},\{x_{11},x_{21},x_{41}\},\{x_{11},x_{21},x_{41}\},\{x_{11},x_{21},x_{41}\},$ $\{x_{11},x_{21},x_{41}\},\{x_{11},x_{31},x_{41}\},\{x_{11},x_{31},x_{41}\},\{x_{11},x_{31},x_{41}\},\{x_{11},x_{31},x_{41}\}.$
Note that $r_1+r_2+r_3+r_4+r_5 = 15+2+1+1+1 = 5\times 4$, $r_6+r_7+r_8+r_9 = 12+2+1+1 = 4\times 4$, $r_{10}+r_{11}+r_{12}+r_{13} = 13+1+1+1 = 4\times 4$ *and* $r_{14}+r_{15} = 8+0 = 2\times 4$. *Replace the repeated blocks in* $\mathscr{A}_2 = 4\{x_{11},x_{21},x_{41}\}$ *with any different four* 3*-subsets from the possible forty* 3*-subsets where each subset contains one point from each of* P_1, P_2 *and* P_4.

We obtain a restricted $(3;12,3,2,2,1,12,2,1,1,13,1,1,1,8,0)$ *design with blocks:*
$\{x_{11},x_{21},x_{31}\},\{x_{12},x_{21},x_{31}\},\{x_{13},x_{21},x_{31}\},\{x_{14},x_{21},x_{31}\},\{x_{15},x_{21},x_{31}\},$ $\{x_{11},x_{22},x_{31}\},\{x_{11},x_{23},x_{31}\},\{x_{11},x_{24},x_{31}\},\{x_{11},x_{21},x_{32}\},\{x_{11},x_{21},x_{33}\},$ $\{x_{11},x_{21},x_{34}\},\{x_{12},x_{22},x_{31}\},\{x_{11},x_{21},x_{41}\},\{x_{12},x_{21},x_{41}\},\{x_{13},x_{21},x_{41}\},$ $\{x_{14},x_{21},x_{41}\},\{x_{11},x_{31},x_{41}\},\{x_{11},x_{31},x_{41}\},\{x_{11},x_{31},x_{41}\},\{x_{11},x_{31},x_{41}\}.$
Note that $r_1+r_2+r_3+r_4+r_5 = 12+3+2+2+1 = 5\times 4$, $r_6+r_7+r_8+r_9 = 12+2+1+1 = 4\times 4$, $r_{10}+r_{11}+r_{12}+r_{13} = 13+1+1+1 = 4\times 4$ *and* $r_{14}+r_{15} = 8+0 = 2\times 4$.

Finally, replace the repeated blocks in $\mathscr{A}_3 = 4\{x_{11},x_{31},x_{41}\}$ *with any four different* 3*-subsets from the possible forty* 3*-subsets where each subset contains one point from each of* P_1, P_3 *and* P_4.

We obtain a restricted simple $(3;9,4,3,3,1,12,2,1,1,13,1,1,1,8,0)$ *design with blocks :*
$\{x_{11},x_{21},x_{31}\},\{x_{12},x_{21},x_{31}\},\{x_{13},x_{21},x_{31}\},\{x_{14},x_{21},x_{31}\},\{x_{15},x_{21},x_{31}\},$ $\{x_{11},x_{22},x_{31}\},\{x_{11},x_{23},x_{31}\},\{x_{11},x_{24},x_{31}\},\{x_{11},x_{21},x_{32}\},\{x_{11},x_{21},x_{33}\},$ $\{x_{11},x_{21},x_{34}\},\{x_{12},x_{22},x_{31}\},\{x_{11},x_{21},x_{41}\},\{x_{12},x_{21},x_{41}\},\{x_{13},x_{21},x_{41}\},$ $\{x_{14},x_{21},x_{41}\},\{x_{11},x_{31},x_{41}\},\{x_{12},x_{31},x_{41}\},\{x_{13},x_{31},x_{41}\},\{x_{14},x_{31},x_{41}\},$
which correspond to $\{1,6,10\},\{2,6,10\},\{3,6,10\},\{4,6,10\},\{5,6,10\},\{1,7,$

10}, {1, 8, 10}, {1, 9, 10}, {1, 6, 11}, {1, 6, 12}, {1, 6, 13}, {2, 7, 10}, {1, 6, 14}, {2, 6, 14}, {3, 6, 14}, {4, 6, 14}, {1, 10, 14}, {2, 10, 14}, {3, 10, 14}, {4, 10, 14}. *Note that* $r_1 + r_2 + r_3 + r_4 + r_5 = 9 + 4 + 3 + 3 + 1 = 5 \times 4$, $r_6 + r_7 + r_8 + r_9 = 12 + 2 + 1 + 1 = 4 \times 4$, $r_{10} + r_{11} + r_{12} + r_{13} = 13 + 1 + 1 + 1 = 4 \times 4$ *and* $r_{14} + r_{15} = 8 + 0 = 2 \times 4$. *Therefore, we obtain a restricted simple* $(3; 9, 4, 3, 3, 1, 12, 2, 1, 1, 13, 1, 1, 1, 8, 0)$ *design.*

Now, we are ready to show the existence of a restricted simple 1-(v, k, r) design.

THEOREM 4. *Let* v, m, k *and* r *be positive integers such that* $2 \leq k \leq m \leq v$. *Let* $V = \{1, 2, \ldots, v\}$ *be partitioned into* m *parts* $\mathscr{P} = \{P_1, P_2, \ldots, P_m\}$ *of size* $p_1, p_2, \ldots, p_m$, *respectively such that* $p_1 \geq p_2 \geq \cdots \geq p_m$. *Suppose that* $vr \equiv 0 \pmod{k}$, $p_1 k \leq v$ *and there exists a restricted simple* $(k; r_1, r_2, \ldots, r_v)$ *design and* $\frac{1}{p_j} \sum_{i \in P_j} r_i = r$, *for all* $j = 1, \ldots, m$ *where* r_i *is the replication number of point* i *for* $i \in V$. *Then there exists a restricted simple 1-*(v, k, r) *design.*

PROOF. Let $(V, \mathscr{P}, \mathscr{B})$ be a restricted simple $(k; r_1, r_2, \ldots, r_v)$ design and for each $j = 1, \ldots, m$, $\sum_{i \in P_j} r_i = rp_j$ and $bk = rp_1 + rp_2 + \cdots + rp_m = (p_1 + p_2 + \cdots + p_m)r = vr$. In order to change the replication number r_i to r for all $i \in V$, any two points are considered at a time, by Proposition 1 and this theorem may be applied $\frac{1}{2}\sum_{i=1}^{v} |r - r_i|$ times. Since $vr = r_1 + r_2 + \cdots + r_v$, it follows that $\mathscr{B}$ is transformed into a restricted simple $(k; r, r, \ldots, r)$ design, say $\mathscr{B}'$. Hence, $(V, \mathscr{P}, \mathscr{B}')$ forms a restricted simple 1-(v, k, r) design and so the proof is complete. □

EXAMPLE 6. *From Example 5, we obtain a restricted simple* $(3; 9, 4, 3, 3, 1, 12, 2, 1, 1, 13, 1, 1, 1, 8, 0)$ *design, say* $(V, \mathscr{P}, \mathscr{B})$ *where a collection* $\mathscr{B}$ *of twenty blocks of size three is* {{1, 6, 10}, {2, 6, 10}, {3, 6, 10}, {4, 6, 10}, {5, 6, 10}, {1, 7, 10}, {1, 8, 10}, {1, 9, 10}, {1, 6, 11}, {1, 6, 12}, {1, 6, 13}, {2, 7, 10}, {1, 6, 14}, {2, 6, 14}, {3, 6, 14}, {4, 6, 14}, {1, 10, 14}, {2, 10, 14}, {3, 10, 14}, {4, 10, 14}}. *Since for all* $j = 1, \ldots, 5$, $\sum_{i \in P_j} r_i = 4p_j$, *Proposition 1 may be applied* $\frac{1}{2}\sum_{i=1}^{15} |4 - r_i| = 26$ *times to change the replication number* r_i *to* 4 *for all* $i \in V$. *From the last transformation, we obtain a restricted simple* $(3; 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4)$ *design, say* $(V, \mathscr{P}, \mathscr{B}')$ *where a collection* $\mathscr{B}'$ *of twenty blocks of size three is* {{1, 8, 11}, {2, 8, 11}, {3, 8, 11}, {4, 7, 12}, {5, 7, 12}, {3, 7, 12}, {4, 8, 13}, {5, 9, 13}, {5, 9, 11}, {5, 9, 12}, {1, 9, 13}, {2, 7, 13}, {1, 6, 15}, {2, 6, 15}, {3, 6, 15}, {4, 6, 15}, {1, 10, 14}, {2, 10, 14},

$\{3,10,14\},\{4,10,14\}\}$. *Therefore,* $(V,\mathscr{P},\mathscr{B}')$ *is a restricted simple 1-* $(15,3,4)$ *design.*

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