Colored-independence on Paths

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Abstract

We consider a storage/scheduling problem which, in addition to the standard restriction involving pairs of elements that cannot be placed together, considers pairs of elements that must be placed together. A set S is a colored-independent set if, for each color class V_i, $S \cap V_i = V_i$ or $S \cap V_i = \emptyset$. In particular, $\beta_{PRT}(G)$, the independence-partition number, is determined for all paths of order n. Finally, we show that the resulting decision problem for graphs is NP-complete even when the input graph is a path.

1 Introduction

Suppose we have ten items to be stored/scheduled but certain pairs of the items cannot be stored/scheduled in the same place/time period. Each item can be represented by the vertex of a graph G, and the conflicts can be represented by edges. For example, ten items can be identified with the vertices of graph P in Figure 1. There are $15 = |E(P)|$ conflicts indicated by the graph $P = (V(P), E(P))$ where, for example, items/vertices v_1 and v_6 can not be stored/scheduled together because $\{v_1, v_6\} \in E(P)$. Note that items v_1, v_8 and v_9 can be placed together because $S = \{v_1, v_8, v_9\}$ is an independent set in P, that is, no two vertices in S are adjacent. We denote the independence number of a graph G, the maximum order of an independent vertex set in G, by $\beta(G)$. While S is a maximal independent set in P, it is not maximum. We have $\beta(P) = 4 = |\{v_1, v_3, v_9, v_{10}\}|$. Note that $\{\{v_1, v_3, v_9, v_{10}\}, \{v_2, v_4, v_6\}, \{v_5, v_7, v_8\}\}$ shows that we can place the ten items into three storage/time units with no conflicts.

In this paper we restrict our attention to the independence number, the maximum number of items that can be placed into one unit. We can think of each vertex as representing a person with the objective of forming as large a committee/team as possible, where each edge represents a pair of people who will not be together on the team.

In addition to the conflicts, suppose that there are sets of people each of whom will be on the team only if all of the members of the set containing him or her are also on the team. Suppose the conflicts among ten people are those of graph P in Figure 1. In addition, suppose the ten people are actually five couples A, B, C, D and E, and no individual will serve without the spouse. If we have couples $A = \{v_1, v_3\}, B = \{v_2, v_8\}, C = \{v_4, v_7\}, D = \{v_5, v_6\}$ and $E = \{v_9, v_{10}\}$, then we can form a team $A \cup E = \{v_1, v_3, v_9, v_{10}\}$. However, if $A = \{v_1, v_8\}, B = \{v_2, v_9\}, C = \{v_3, v_{10}\}, D = \{v_4, v_6\}$ and $E = \{v_5, v_7\}$, then any two couples have at least one conflict between them, and the maximum team size is two.

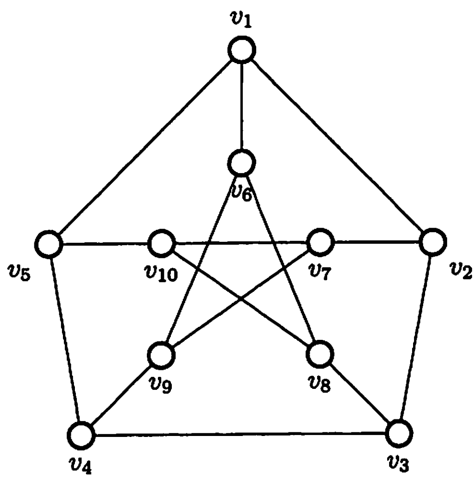

Figure 1: Fifteen conflicts among ten people.

We model these storage/scheduling/team forming problems with both conflicts and bonds as "colored-independence" problems for graphs. Colored-independence and the related colored-domination were discussed by Slater in [6]. See also[3, 4, 5, 7, 8].

2 Colored-Independence

Given a graph $G = (V, E)$, let $\mathfrak{S} = \{V_1, V_2, \ldots, V_t\}$ be a partition of the vertex set $V(G)$. For $I \subseteq \{1, 2, \ldots, t\} = [t]$, let $\mathfrak{S}_I = \cup_{i \in I} V_i$. A vertex set $R \subseteq V(G)$ is called *$\mathfrak{S}$-independent* if R is independent and $R = \mathfrak{S}_I$ for some $I \subseteq [t]$. The *$\mathfrak{S}$-independence number* of G is $\beta(G; \mathfrak{S}) = \max\{|\mathfrak{S}_I| : I \subseteq [t], \mathfrak{S}_I \text{ is independent}\}$. That is, $\beta(G; \mathfrak{S})$ is the maximum cardinality of an independent set $S \subseteq V(G)$ such that either $S \cap V_i = \emptyset$ or $S \cap V_i = V_i$ for $1 \leq i \leq t$. Noting that if any two vertices in a V_i are adjacent then one must have $S \cap V_i = \emptyset$, for the following parameters we assume that each V_i is independent. That is, thinking of $\mathfrak{S}$ as being a coloring of G with V_i being color class i, only proper colorings are being allowed.

The independence-partition number of graph G is $\beta_{PRT}(G) = \min\{\beta(G; \mathfrak{S}) : \mathfrak{S}$ is a partition of $V(G)$ into independent sets$\}$. For the graph P in Figure 1 we considered the case where there were couples involved. The coupled-independence number of G is $\beta_{cpl}(G) = \min\{\beta(G; \mathfrak{S}) : \mathfrak{S}$ is a partition of $V(G)$ into independent sets V_i with each $|V_i| \leq 2\}$.

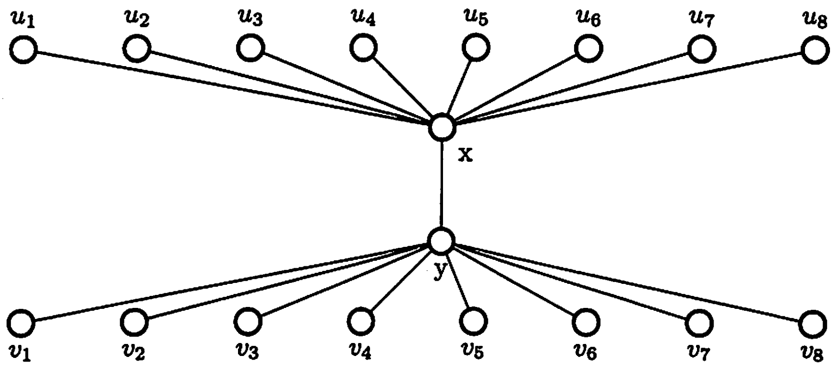

Figure 2: Double star $S_{8,8}$.

For the double star $S_{8,8}$ in Figure 2, $|V(S_{8,8})| = n = 18$ and $\beta(S_{8,8}) = n - 2 = 16$. For $\beta_{cpl}(S_{8,8})$ let $\mathfrak{S}$ be any proper partition containing $\{x, v_1\}$ and $\{y, u_1\}$, and we have $\beta_{cpl}(S_{8,8}) = \beta(S_{8,8}; \mathfrak{S}) = 14$. For $\mathfrak{S}_9 = \{\{x, v_1, v_2, \ldots, v_8\}, \{y, u_1, u_2, \ldots, u_8\}\}$ we have $\beta_{PRT}(S_{8,8}) \leq \beta(S_{8,8}; \mathfrak{S}_9) = 9$. But one can do better! In fact, $\beta_{PRT}(S_{8,8}) = \beta(S_{8,8}; \{\{x, v_1, v_2, \ldots, v_5\}, \{y, u_1, u_2, \ldots, u_5\}, \{v_6, v_7, v_8, u_6, u_7, u_8\}\} = 6$.

In general, we let $\sigma_k(G)$ be the collection of all partitions $\mathfrak{S} =$

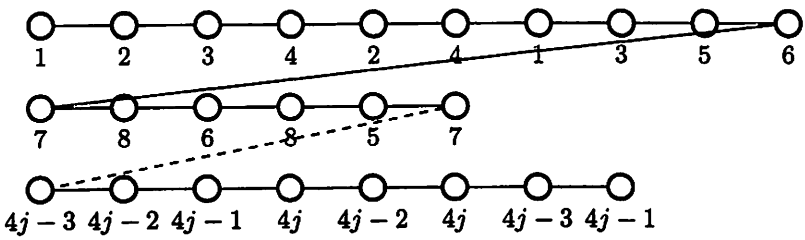

Figure 3: $\beta_{cpl}(P_{8j}) = 2j$.

$\{V_1, V_2, \ldots, V_t\}$ of $V(G)$ with $|V_i| \leq k$ for $1 \leq i \leq t$. Then, $\beta_{PRT(k)}(G) = \min\{\beta(G;\mathfrak{S}) : \mathfrak{S} \in \sigma_k(G)$ is a proper coloring$\}$. Clearly $\beta_{PRT(1)}(G) = \beta(G), \beta_{PRT(2)}(G) = \beta_{cpl}(G)$, and the next theorem is obvious.

Theorem 1 *(Slater [7]) For every graph G, $\beta(G) = \beta_{PRT(1)}(G) \geq \beta_{cpl}(G) = \beta_{PRT(2)}(G) \geq \beta_{PRT(3)}(G) \geq \ldots \geq \beta_{PRT(k)}(G) \geq \beta_{PRT(k+1)}(G) \geq \ldots \geq \beta_{PRT}(G)$.*

Theorem 2 *(Slater [7]) For the path P_n on n vertices, $\beta_{cpl}(P_{8j}) = 2j$, $\beta_{cpl}(P_{8j+1}) = \beta_{cpl}(P_{8j+2}) = 2j+1, \beta_{cpl}(P_{8j+3}) = \beta_{cpl}(P_{8j+4}) = \beta_{cpl}(P_{8j+5}) = \beta_{cpl}(P_{8j+6}) = 2j+2$, and $\beta_{cpl}(P_{8j+7}) = 2j+3$.*

Note that $\beta_{cpl}(P_{8j+7}) = 2j + 3 > \beta_{cpl}(P_{8j+8}) = 2j + 2$.

3 Colored-independence for paths.

In this section we determine $\beta_{PRT}(P_n)$ for all values of n. Nevertheless, we show that deciding if $\beta(G;\mathfrak{S}) \geq K$ is an NP-complete problem for a given partition $\mathfrak{S}$ of $V(G)$ even when G is restricted to be a path.

Theorem 3 *For a path P_n of order n, $\beta_{PRT(k)}(P_n) \geq \frac{n}{2k}$ and $\beta_{PRT(k)}(P_{2k^2j}) = kj = \frac{n}{2k}$.*

Proof. Consider a linear forest LF_n of order n with some partition $\mathfrak{S} = \{S_1, S_2, \ldots, S_t\}$ where $max_{1\leq i \leq t}|S_i| \leq k$. We will construct an independent set $A = S_I \subseteq V(LF_n)$.

Take a vertex v such that $deg(v) \leq 1$. Vertex v is in some color class S_i. Put S_i in A. Delete from LF_n all vertices of the set $S_i \cup \{S_j : \exists u \in S_j, v \in S_i$

and $uv \in E(LF_n)\}$. That is, delete every vertex in color class S_i and every vertex in a color class that contains an element that is a neighbor of a vertex in color class S_i. Notice that $|S_i|$ elements have been added to A and at most $|S_i| + k(2|S_i| - 1)$ total vertices have been deleted from LF_n: $|S_i|$ from color class S_i and the remaining $k(2|S_i| - 1)$ vertices from other color classes. This leads to the ratio $\frac{|S_i|}{k(2|S_i|-1)+|S_i|} = \frac{|S_i|}{2k|S_i|-k+|S_i|} \geq \frac{1}{2k}$.

Pick another vertex u in the deleted linear forest such that the degree of u in the deleted linear forest is no more than 1. Vertex u is in some color class S_m. Add S_m to A and delete vertices from the deleted linear forest as above. Continue this process until no more vertices are remaining. Since for each color class S_j included in A we have the ratio $\frac{|S_j|}{k(2|S_j|-1)+|S_j|} \geq \frac{1}{2k}$, for any vertex in A at most $2k$ vertices were deleted from LF_n. This implies that for the given partition, $\beta(LF_n; \mathfrak{S}) \geq \frac{n}{2k}$. But, this holds for any partition where $max_{1 \leq i \leq t}|S_i| \leq k$. Thus, $\beta_{PRT(k)}(LF_n) \geq \frac{n}{2k}$. In particular, $\beta_{PRT(k)}(P_n) \geq \frac{n}{2k}$.

Now we will show that $\beta_{PRT(k)}(P_{2k^2j}) = kj$. By the above work, we already know that $\beta_{PRT(k)}(P_{2k^2j}) \geq \frac{n}{2k} = kj$, so it will be shown that $\beta_{PRT(k)}(P_{2k^2j}) \leq \frac{n}{2k}$. Suppose $j = 1$. Given k, we can use $2k$ color classes, each of order k, to construct a partition $\mathfrak{S}$. We will construct the partition $\mathfrak{S}$ so that for $1 \leq i < j \leq 2k$ there exists some element $v_i \in S_i$ and $v_j \in S_j$ such that $v_iv_j \in E(P_{2k^2})$. This will ensure that only one color class can be in any independent set and so $\beta_{PRT(k)}(P_{2k^2}) \leq k = \frac{n}{2k}$. Let $V(P_{2k^2}) = \{v_1, v_2, \ldots, v_{2k^2}\}$ where $v_iv_{i+1} \in E(P_{2k^2})$ for $1 \leq i \leq 2k^2 - 1$, let and $V(H) = \{u_1, u_2, \ldots, u_{2k}\}$ where H is a multigraph constructed by starting with a complete graph on $2k$ vertices and adding the $k - 1$ edges $u_2u_{2k-1}, u_3u_{2k-2}, \ldots, u_{k-1}u_{k+2}$ and u_ku_{k+1}. Note that degree of u_1 and the degree of u_{2k} are both odd while the degrees of the vertices $u_2, u_3, \ldots, u_{2k-2}$ and u_{2k-1} are all even. This implies that an Eulerian trail exists on H that begins at u_1 and ends at u_{2k}. This Eulerian trail contains exactly $2k^2$ vertices each of which is repeated exactly k times. This trail will be used to define the partition $\mathfrak{S}$ on the path. Each vertex v_i on the path will be placed into the color class S_{u_j} where u_j is the i^{th} vertex in the trail on H. For example, consider the case $k = 3$. The path has the vertex set $V(P_{18}) = \{v_1, v_2, \ldots, v_{18}\}$ and the multigraph H has the vertex set $V(H) = \{u_1, u_2, \ldots, u_6\}$. H is constructed by starting with the complete graph K_6 on six vertices and adding the edges u_2u_5 and u_3u_4. We use the Eulerian trail $u_1u_2u_3u_4u_5u_6u_2u_5u_3u_1u_4u_3u_6u_4u_2u_5u_1u_6$ of H to construct the partition $\mathfrak{S}$ on the path. In fact, $\mathfrak{S} = \{S_1, S_2, \ldots, S_6\}$. Since u_1 is the first vertex in the trail, the first vertex in the path, v_1, is placed into the color class S_1. Similarly, since u_2 is the second vertex in the trail, v_2 is placed into the color class S_2. By placing each vertex v_i into the color class defined by the i^{th} vertex in the trail, $S_1 = \{v_1, v_{10}, v_{17}\}, S_2 = \{v_2, v_7, v_{15}\}, S_3 = \{v_3, v_9, v_{12}\}, S_4 = \{v_4, v_{11}, v_{14}\}, S_5 = \{v_5, v_8, v_{16}\}$ and $S_6 = \{v_6, v_{13}, v_{18}\}$.

For color classes S_i and S_j with $1 \leq i < j \leq 2k$, there are vertices $v_i \in S_i$ and $v_j \in S_j$ such that v_iv_j is an edge in the path since in the multigraph H the edge u_iu_j was traversed by the Eulerian trail. Thus, any $\mathfrak{S}$-independent set can contain no more than one color class, and so $\beta(P_{2k^2}; \mathfrak{S}) \leq k$.

Now let $j = 2$. Define a second multigraph H_2 constructed from a complete graph on $2k$ vertices with the same $k-1$ added edges. However, let $V(H_2) = \{u_{2k+1}, u_{2k+2}, \ldots, u_{4k}\}$. Assign the first $2k^2$ vertices on the path the same way as before. For the second $2k^2$ vertices, assign their color classes in the same manner using an Eulerian trail on H_2 that begins at u_{2k+1} and ends at u_{4k}. Only one color class from each of the two groups of $2k^2$ vertices can be in an independent set so $\beta(P_{(2k^2)2}; \mathfrak{S}) \leq 2k$.

Let j be any value. For each set of $2k^2$ vertices use a new multigraph H_i where $V(H_i) = \{u_{2k(i-1)+1}, u_{2k(i-1)+2}, \ldots, u_{4k(i-1)}\}$ for $1 \leq i \leq j$ to assign the vertices in the path to color classes. Again, for each group of $2k^2$ vertices only one color class can be in an independent set so that $\beta(P_{2k^2j}, \mathfrak{S}) \leq kj = \frac{n}{2k}$. Since there exists a partition $\mathfrak{S}$ which achieves $\beta(P_{2k^2j}; \mathfrak{S}) \leq kj$, $\beta_{PRT(k)}(P_{2k^2j}) = kj = \frac{n}{2k}$.

□

Proposition 4 *For a path of order $2t^2$, we have $\beta_{PRT}(P_{2t^2}) = t$.*

Proof. From Theorem 3, there exists a partition $\mathfrak{S}$ such that $\beta(P_{2t^2}; \mathfrak{S}) = t$. Then, $\beta_{PRT}(P_{2t^2}) \leq \beta(P_{2t^2}; \mathfrak{S}) = t$. So, we will show that $\beta_{PRT}(P_{2t^2}) \geq t$.

Let $\mathfrak{S} = \{S_1, S_2, \ldots, S_r\}$ be a partition of $V(P_{2t^2})$. Let $J = \max_{1 \leq i \leq r} |S_i|$. If $J \geq t$, then clearly $\beta(P_{2t^2}; \mathfrak{S}) \geq t$. Otherwise $J < t$. By previous work, $\beta(P_{2t^2}; \mathfrak{S}) \geq \beta_{PRT(J)}(P_{2t^2}) \geq \frac{n}{2J} > \frac{n}{2t} = \frac{2t^2}{2t} = t$. $\mathfrak{S}$, however, was an arbitrary partition, so $\beta_{PRT}(P_{2t^2}) \geq t$. Therefore, $\beta_{PRT}(P_{2t^2}) = t$.

□

We would now like to determine $\beta_{PRT}(P_n)$ for all values of n. The following Lemma is an essential step for proving these values.

Lemma 5 *For any path P_n of order n where $n > 2(t-1)^2$, we have $\beta_{PRT}(P_n) \geq t$.*

Proof. Let $\mathfrak{S} = \{S_1, S_2, \ldots, S_r\}$ be a partition of $V(P_n)$ such that $\beta(P_n; \mathfrak{S}) = \beta_{PRT}(P_n)$. Let $J = \max_{1 \le i \le r}|S_i|$. If $J \ge t$, then $\beta(P_n; \mathfrak{S}) \ge t$. Otherwise, $J \le t-1$. Then $\beta_{PRT}(P_n) = \beta(P_n; \mathfrak{S}) \ge \beta_{PRT(J)}(P_n) \ge \frac{n}{2J} \ge \frac{n}{2(t-1)} > \frac{2(t-1)^2}{2(t-1)} = t-1$. So, $\beta_{PRT}(P_n) \ge t$.

□

Theorem 6 *For $t \ge 3$ and a path P_n of order $n = 2(t-1)^2 + b$, we have*

$$\beta_{PRT}(P_{2(t-1)^2+b}) = \begin{cases} t-1, & \text{if } b = 0 \\ t, & \text{if } 1 \le b \le 3t-2 \\ t+1, & \text{if } 3t-1 \le b \le 4t-3 \end{cases}.$$

Proof. Notice that this has already been proven in Theorem 3 for the case when $b = 0$. By Lemma 5, $\beta_{PRT}(P_{2(t-1)^2+b}) \ge t$ for $1 \le b \le 4t-3$. Consider cases for different values of b.

Case 1. $1 \le j \le 2(t-1) = 2t-2$.

We will show that $\beta_{PRT}(P_{2(t-1)^2+b}) \le t$. By previous work, there is a partition $\mathfrak{S} = \{S_1, S_2, \ldots, S_r\}$ such that the first $2(t-1)^2$ vertices are partitioned in such a way that for color classes S_i and S_j there exist vertices $v_i \in S_i$ and $v_j \in S_j$ such that v_iv_j is an edge in the path for $1 \le i < j \le 2(t-1)$, $\max_{1 \le i \le r}|S_i| \le t-1$ and vertex $v_{2(t-1)^2}$ is an element of color class $S_{2(t-1)}$. Partition the remaining b vertices as $1-2-3-\ldots-b$. Each color class is increased by no more than one element so that $\max_{1 \le i \le r}|S_i| \le t$. But then $\beta_{PRT}(P_{2(t-1)^2+b}) \le \beta(P_{2(t-1)^2+b}; \mathfrak{S}) \le t$. Thus, $\beta_{PRT}(P_{2(t-1)^2+b}) = t$.

Case 2. $2t-1 \le b \le 3t-2$

Again, we will show that $\beta_{PRT}(P_{2(t-1)^2+b}) \le t$. We will use $2t-1$ color classes to construct the partition $\mathfrak{S}$. Again, we will construct $\mathfrak{S}$ so that for color classes S_i and S_j there exist $v_i \in S_i$ and $v_j \in S_j$ such that $v_iv_j \in E(P_{2(t-1)^2+b})$. This will ensure that only one color class can be in any $\mathfrak{S}$-independent set and thus $\beta_{PRT}(P_{2(t-1)^2+b}) \le t$. Let $V(P_{2(t-1)^2+b}) = \{v_1, v_2, \ldots, v_{2(t-1)^2+b}\}$ and $V(H) = \{u_1, u_2, \ldots, u_{2t-1}\}$ where H is the complete graph on $2t-1$ vertices. Since every vertex in H has even degree, there is an Eulerian cycle on H that can be listed as $u_1, u_2, \ldots, u_{2t-1}, u_1$ that begins and ends at vertex u_1. In this list vertex u_1 is repeated t times and the remaining vertices $u_2, u_3, \ldots, u_{2t-2}$ and u_{2t-1} are repeated $t-1$ times. Furthermore, there are $\binom{2t-1}{2} + 1 = 2t^2 - 3t + 2$ vertices in this cycle list. For the first $2t^2 - 3t + 2$ vertices in the path, each v_i will be assigned color class S_{u_j} where u_j is the i^{th} vertex of the Eulerian cycle list on H. Note that at this point $|S_1| = t$ and $|S_i| = t-1$ for $2 \le i \le 2t-1$. Note that $n - (2t^2 - 3t + 2) = 2(t-1)^2 + b - (2t^2 - 3t + 2)$ vertices remain to be assigned a color class. Thus, at least $2(t-1)^2+2t-1-(2t^2-3t+2)=t-1$ vertices and at most $= 2(t-1)^2+3t-2-(2t^2-3t+2)=2t-2$ vertices remain. For these remaining vertices, assign in order the color classes S_2 through S_{2t-1}. At most, each of these $2t-2$ color classes receives one more element. Thus $\max_{1\leq i\leq 2t-1}|S_i| \leq t$. As usual, notice that for color classes S_i and S_j there are vertices $v_i \in S_i$ and $v_j \in S_j$ such that v_iv_j is an edge in the path for $1 \leq i < j \leq 2(t-1)$ so that only one color class can be in an $\mathfrak{S}$-independent set. So, $\beta_{PRT}(P_{2(t-1)^2+b}) \leq \beta(P_{2(t-1)^2+b};\mathfrak{S}) = t$. Thus, $\beta_{PRT}(P_{2(t-1)^2+b}) = t$.

Case 3. $3t-1 \leq b \leq 4t-3$

We will begin by showing that $\beta_{PRT}(P_{2(t-1)^2+b}) \leq t+1$. Let $V(P_{2(t-1)^2+b}) = \{v_1, v_2, \ldots, v_{2(t-1)^2+b}\}$ and $V(H) = \{u_1, u_2, \ldots, u_{2t-1}\}$ where H is the complete graph on $2t-1$ vertices. Since every vertex in H has even degree, there exists an Eulerian cycle on H that can be listed as $u_1, u_2, \ldots, u_{2t-1}, u_1$ so that it begins and ends at vertex u_1 and vertex u_1 is repeated t times in the list and the remaining vertices $u_2, u_3, \ldots, u_{2t-2}$ and u_{2t-1} are repeated $t-1$ times in the list. As before, for the first $2t^2-3t+2$ vertices, assign vertex v_i in the path the color class S_{u_j} where u_j is the i^{th} vertex of the Eulerian cycle list on H. At this point, $|S_1| = t$ and $|S_i| = t-1$ for $2 \leq i \leq 2t-1$. Again, there are $n-(2t^2-3t+2) = 2(t-1)^2+b-(2t^2-3t+2)$ vertices remaining. Thus, there are at least $(2(t-1)^2+3t-1)-(2t^2-3t+2) = 2t-1$ vertices and at most $(2(t-1)^2+4t-3)-(2t^2-3t+2) = 3t-3$ vertices remaining to be assigned to a color class. Assign the remaining vertices in order to the color classes S_2 through S_{2t-1} and then to S_1 through S_{2t-1} or until reaching the end of the path. With this partition $\mathfrak{S}$, $\max_{1\leq i\leq 2t-1}|S_i| = t+1$ and for color classes S_i and S_j there are vertices $v_i \in S_i$ and $v_j \in S_j$ such that v_iv_j is an edge in the path for $1 \leq i < j \leq 2(t-1)$ so that at most one color class can be in an $\mathfrak{S}$-independent set, and $\beta_{PRT}(P_{2(t-1)^2+b}) \leq \beta(P_{2(t-1)^2+b};\mathfrak{S}) = t+1$.

Next, we will show that $\beta_{PRT}(P_{2(t-1)^2+b}) \geq t+1$. We can assume that for each color class S_i there is a vertex $v_i \in S_i$ and $v_j \in S_j$ such that v_iv_j is an edge in the path for $1 \leq i < j \leq r$. If not, any two independent color classes can be combined into a single color class. If there exists a partition $\mathfrak{S}$ such that $\beta(P_n;\mathfrak{S}) = t$ where $2t^2-t+1 \leq n \leq 2t^2-1$ then there must be at least $2t$ color classes. Otherwise, $n \leq 2t^2-t$ which implies that $b \leq 3t-2$. If the number of color classes is greater than $2t$ then $n \geq \binom{2t+1}{2}$ $= 2t^2+t$ which implies that $b \geq 5t-2$. Thus, if such a partition exists it must have $2t$ color classes. However, since only one color class can be in an $\mathfrak{S}$-independent set there must exists an Eulerian trail on the multigraph H constructed from the complete graph on $2t$ vertices. Thus, we must add the $t-1$ edges $u_2u_{2t-1}, u_3u_{2t-2}, \ldots, u_{t-1}u_{t+2}$ and u_tu_{t+1} to H which implies that $n \geq (2t^2 - t) + (t-1) + 1$ where $n = 2(t-1)^2 + b$. Thus, $b \geq 4t-2$. Since $b \leq 4t-3$, no such partition $\mathfrak{S}$ exists. Thus, $\beta_{PRT}(P_{2(t-1)^2+b}) \geq t+1$ and so $\beta_{PRT}(P_{2(t-1)^2+b}) = t+1$.

Hence, the theorem is proved.

□

It is important to note the nonmonotonicity when $n = 2t^2$ of the sequence for $\beta_{PRT}(P_n)$. In particular, note that $\beta_{PRT}(P_{2t^2-1}) = t+1 > t = \beta_{PRT}(P_{2t^2})$.

To construct the NP-completeness theorem, we will define the decision problem of colored-independence and show a reduction from independence. Figure 4 shows the construction of a colored-independence problem on a path from a small graph, namely H_6, while Figure 5 shows the general construction.

Colored-Independence (COLIND)
INSTANCE: Graph $G = (V(G), E(G))$, partition $\mathfrak{S} = \{S_1, S_2, \dots, S_t\}$ of $V(G)$, $K \in \mathbb{Z}^+$, $K \leq |V(G)|$
QUESTION: Is $\beta(G; \mathfrak{S}) \geq K$?

Independence (IND)
INSTANCE: Graph $H = (V(H), E(H))$, $J \in \mathbb{Z}^+$, $J \leq |V(H)|$
QUESTION: Is $\beta(H) \geq J$?

Theorem 7 *The colored-independence problem, $\beta(G; \mathfrak{S})$ is NP-complete, even when G is restricted to be a path.*

Proof. It is easy to see that $\beta(G; \mathfrak{S})$ is in NP since a nondeterministic algorithm need only choose $I' \subseteq [t]$, a collection of color classes, $\mathfrak{S}_i$, and verify that $\cup_{i \in I'} S_i$ contains no more than one endpoint of every edge and that $|\cup_{i \in I'} S_i| \geq K$.

We will reduce the known NP-complete independence problem IND to the problem COLIND. Now consider a graph H with $V(H) = \{v_1, v_2, \dots, v_n\}$ and $E(H) = \{e_1, e_2, \dots, e_m\}$ with $e_i = \{u_i, w_i\}$ and $J \leq |V(H)| = n$. Assume that $deg(v_1) = \Delta(H)$. We will construct the path G with a partition that has $n+3 = t$ elements. Denote the color classes by $S_1, S_2, \dots, S_n, S_x, S_y$, and S_z. Write the edges of H as a set of K_2's. Add a P_7 in front of $u_1 w_1$ and color them as $y - x - z - y - z - y - x - u_1$. To connect the K_2's add a P_3 from w_i to u_{i+1} colored $x - y - z$ for $i = 1$ to $m - 1$. We will make every $S_i, 1 \leq i \leq n$, have $|S_i| = \Delta(H) = d_1$. Note that v_1 appears in d_1 of the K_2's. Consider $d_1 - d_2$, the difference between the degrees of v_1 and v_2. Extend G by adding the path colored $x - y - z - v_2$ a total of $d_1 - d_2$ times.

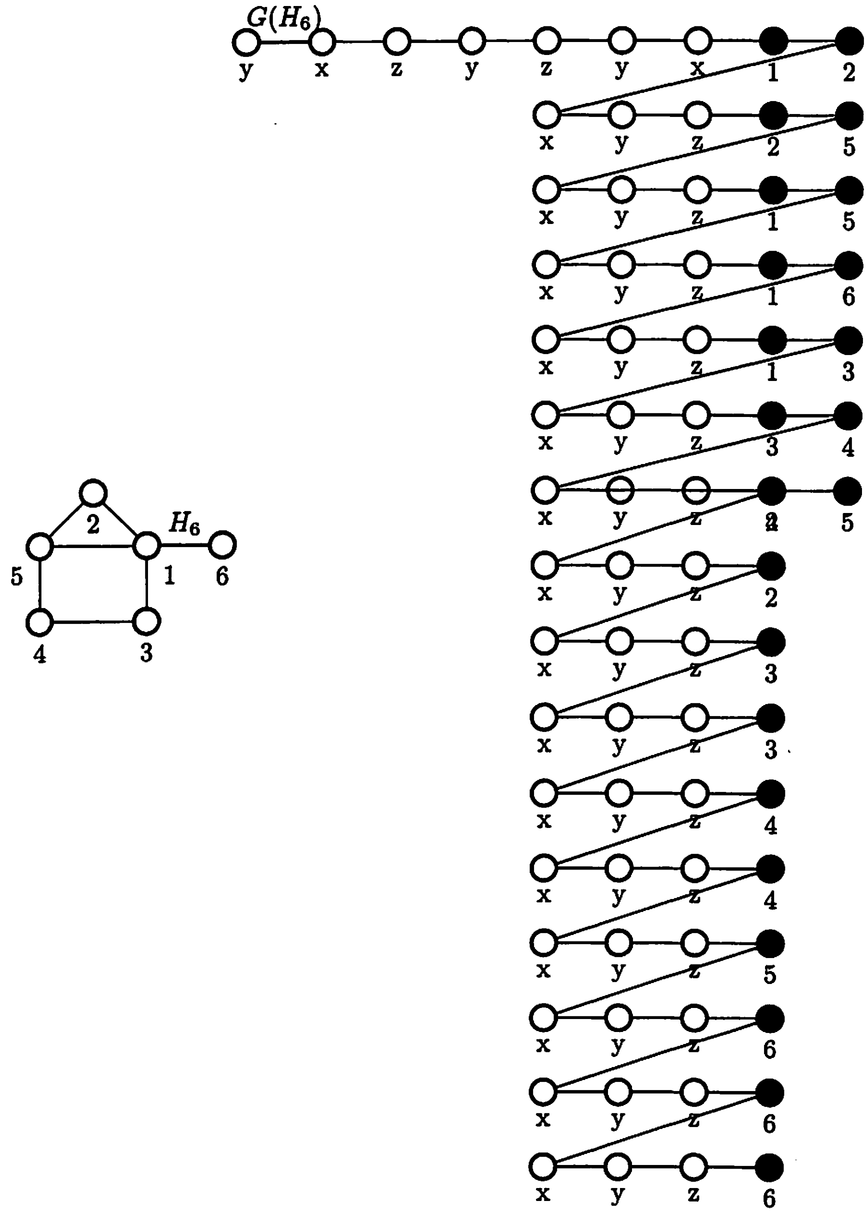

Figure 4: The graph H_6 transformed to a colored path.

Now add the path colored $x - y - z - v_3$ a total of $d_1 - d_3$ times. Continue with this process so that the colored path $x - y - z - v_i$ is added $d_1 - d_i$ times. When $x - y - z - v_n$ has been added $d_1 - d_n$ times, G is completely constructed as shown in Figure 5. Notice that G has $p = 4(d_1 n + 1) - 3m$ vertices and each color class $S_1, S_2, \ldots, S_n$ has cardinality d_1. Furthermore, classes S_x and S_z have cardinality $2 + (m-1) + \sum_{i=2}^{n}(d_1 - d_i) = c$ and class S_y has cardinality $3 + (m-1) + \sum_{i=2}^{n}(d_1 - d_i) = c + 1$. Finally, G can clearly be constructed in time polynomial in n.

Let $K = Jd_1 + 3 + (m-1) + \sum_{i=2}^{n}(d_1 - d_i) = Jd_1 + d_1 n + 2 - m$. We will show that $\beta(H) \geq J$ if and only if $\beta(G; \mathfrak{S}) \geq K$.

Suppose that there is a set $S \subseteq V(H)$ that is an independent set and $|S| \geq J$. For each vertex $v_i \in S$, add the color class S_i to the set $R \subseteq V(G)$. These vertices in G must be independent since any two vertices v_i and v_j were independent in H. Since the color class S_y is independent of all of the color classes $S_1, S_2, \ldots, S_n$, the color class S_y can also be added to the set R. Clearly R is a colored independent set in G with $|R| = Jd_1 + (3 + (m-1) + \sum_{i=2}^{n}(d_1 - d_i)) = Jd_1 + d_1 n + 2 - m$ and so $\beta(G; \mathfrak{S}) \geq Jd_1 + d_1 n + 2 - m$.

Conversely, suppose that $\beta(G; \mathfrak{S}) \geq Jd_1 + d_1 n + 2 - m$. Let R be a $\beta(G; \mathfrak{S})$-set. Notice that only one color class from the classes S_x, S_y and S_z can be in a colored independent set since each of these color classes is adjacent to both of the others. Since the cardinality of the color class S_y is greater than the cardinality of color class S_x or S_z, and S_y is not adjacent to color class S_i with $1 \leq i \leq n$, any colored independent set that uses color class S_x or color class S_z can be revised to use color class S_y. This implies that at least J color classes are used from the set $\{S_1, S_2, \ldots, S_n\}$ to contribute at least Jd_1 vertices to the independent set. Let $S = \{v_i \in V(H) : S_i \subseteq R\}$. Clearly, $|S| \geq J$. Since any two color classes S_i and S_j in the independent set in G imply that all vertices colored v_i and v_j are independent, then there does not exist edge $e_k \in E(G)$ such that $e_k = \{v_i, v_j\}$. However, every edge in H is also an edge in G, thus v_i and v_j are independent in the graph H. Thus, S is an independent set and $\beta(H) \geq J$.

So $\beta(H) \geq J$ if and only if $\beta(G; \mathfrak{S}) \geq K$, and colored-independence is NP-complete even for paths.

□

As noted by Garey and Johnson in [1], the independence problem (IND) is NP-complete for cubic (planar) graphs as shown by Garey, Johnson, and Stockmeyer (see [2]). Note that if we restrict the input graph H for IND to be cubic, each color class i for $1 \leq i \leq |V(H)|$ has a cardinality of at most three in G. That is, the partition $\mathfrak{S}$ can be restricted so that $\mathfrak{S} \in \sigma_3$.

Theorem 8 *The colored-independence problem, $\beta(G; \mathfrak{S})$ is NP-complete when G is restricted to be a path and $\mathfrak{S} \in \sigma_3$.*

To prove Theorem 8, one can modify the proof of Theorem 7 as follows. Figure 6 illustrates the transformation of one cubic graph, H, into a colored path where each color class has cardinality no greater than three. This construction is quite similar to the one in Theorem 7 and begins in the same manner; each edge in the cubic graph H is listed as a K_2. These K_2's are connected by P_8's colored $x_j - y_j - x_j - z_j - y_j - z_j - y_j - z_j$ where $1 \leq j \leq m-1$. Notice that $|S_{x_j}| = 2$ and $|S_{y_j}| = |S_{z_j}| = 3$. Furthermore, since H is a cubic graph, each vertex is listed exactly three times in the K_2's so that each color class S_i for $1 \leq i \leq |V(H)|$ already has the same cardinality, namely three. The resulting graph G is a path on $10m - 8$ vertices. Letting $K = 3J + 3(m-1)$, we have $\beta(H) \geq J$ if and only if $\beta(G; \mathfrak{S}) \geq K$.

Last, note that the P_8's connecting the K_2's could be extended to P_9's colored $x_j - y_j - x_j - z_j - y_j - z_j - x_j - y_j - z_j$ for $1 \leq j \leq m-1$ so that every color class has cardinality exactly three, and we have the following theorem.

EXACT-3-COLORED-INDEPENDENCE (X3COLIND)
INSTANCE: Graph $G = (V(G), E(G))$, partition $\mathfrak{S} = \{S_1, S_2, \ldots, S_t\}$ of $V(G)$ where $|S_i| = 3$ for $1 \leq i \leq t$, $K \in \mathbb{Z}^+, K \leq |V(G)|$
QUESTION: Is $\beta(G; \mathfrak{S}) \geq K$?

Theorem 9 *The decision problem X3COLIND is NP-complete even when G is restricted to be a path.*

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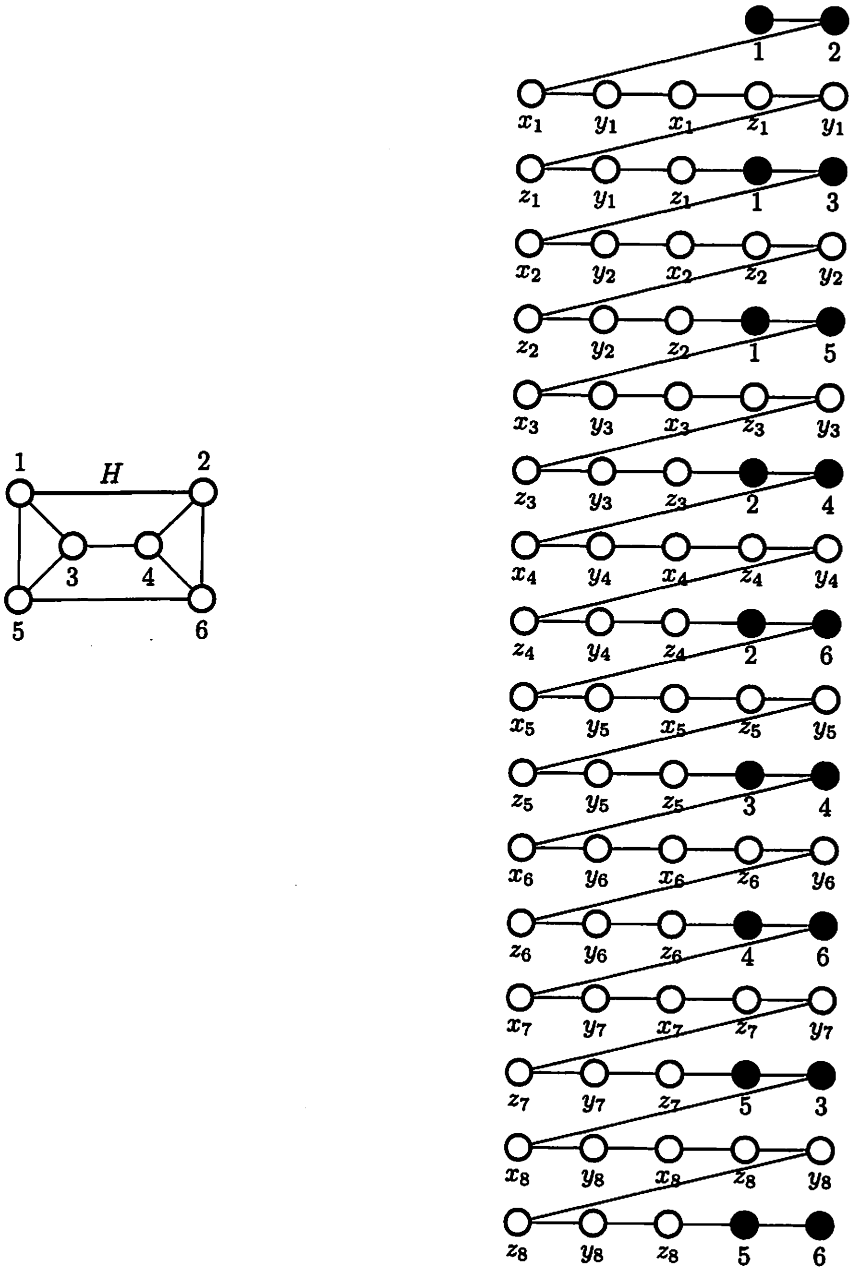

Figure 6: A cubic graph transformed into a colored path where the partition $\mathfrak{S} \in \sigma_3$.