# Twofold 2-perfect 8-cycle systems with an extra property

Elizabeth J. Billington
School of Mathematics and Physics
The University of Queensland, Qld 4072, Australia
ejb@maths.uq.edu.au

Abdollah Khodkar
Department of Mathematics, University of West Georgia
Carrollton, GA 30118, U.S.A.
akhodkar@westga.edu

**Abstract**

A twofold 8-cycle system is an edge-disjoint decomposition of a twofold complete graph (which has two edges between every pair of vertices) into 8-cycles. The order of the complete graph is also called the order of the 8-cycle system. A twofold 2-perfect 8-cycle system is a twofold 8-cycle system such that the collection of distance 2 edges in each 8-cycle also cover the complete graph, forming a (twofold) 4-cycle system. Existence of 2-perfect 8-cycle systems for all admissible orders was shown in [1], although $\lambda$-fold existence for $\lambda > 1$ has not been done.

In this paper we impose an extra condition on the twofold 2-perfect 8-cycle system. We require that the two paths of length two between each pair of vertices, say $x, a_{xy}, y$ and $x, b_{xy}, y$, should be distinct, that is, with $a_{xy} \neq b_{xy}$; thus they form a 4-cycle $(x, a, y, b)$.

We completely solve the existence of such twofold 2-perfect 8-cycle systems with this "extra" property. All admissible orders congruent to 0 or 1 modulo 8 can be achieved, apart from order 8.

# 1 Introduction

An 8-cycle system of order $n$ is a pair $(X, C)$ where $X$ is the vertex set of a complete graph $K_n$ of order $n$ and $C$ is a collection of 8-cycles with vertices in $X$, such that every edge of $K_n$ is contained in exactly one 8-cycle. In other words, the 8-cycles form a partition of the edge set of $K_n$. If the graph $2K_n$ is taken instead of $K_n$ (there are two edges between every pair of vertices in $2K_n$), and if its edges are partitioned into 8-cycles, then the collection $(X, C)$ is a *twofold* 8-cycle system of order $|X| = n$.

An 8-cycle system $(X, C)$ is said to be *2-perfect* if every pair of distinct vertices of $X$ is joined by a path of length two in exactly one of the 8-cycles in $C$. Existence of 2-perfect 8-cycle systems was settled in [1]: a 2-perfect 8-cycle system of order $n$ exists if and only if $n \equiv 1 \pmod{16}$. Similarly, we say that a *twofold* 8-cycle system $(X, C)$ is *2-perfect* if every pair of distinct vertices of $X$ is joined by a path of length two in exactly *two* of the 8-cycles in $C$.

Let $(X, C)$ be a twofold 2-perfect 8-cycle system, and suppose that for every pair $x, y$ of distinct vertices in $X$ the two paths of length 2 joining $x$ and $y$ are $x$, $a_{xy}$, $y$ and $x$, $b_{xy}$, $y$. The midpoints of these paths, $a_{xy}$ and $b_{xy}$, may not necessarily be distinct. However, if for *every* pair of points $x, y$ from $X$, these midpoints *are* distinct, then we refer to the twofold 2-perfect 8-cycle system as an "EXTRA twofold 2-perfect 8-cycle system", or an ET8CS for short.

In what follows we denote the 8-cycle on vertex set $\{x_1, x_2, \ldots, x_8\}$ having edge set $\{\{x_i, x_{i+1} \mid 1 \leqslant i \leqslant 7\} \cup \{x_1, x_8\}$, by $(x_1, x_2, \ldots, x_8)$ or $(x_8, x_7, \ldots, x_1)$ or by any cyclic shift of these.

We remark that for each pair of vertices $x, y$ in $X$, in an ET8CS the two paths $x$, $a_{xy}$, $y$ and $x$, $b_{xy}$, $y$ together form a 4-cycle $(x, a_{xy}, y, b_{xy})$, and it is straightforward to check that the collection of all such 4-cycles in an ET8CS of order $n$ will form a fourfold 4-cycle system of order $n$.

**Example 1.1** *An* ET8CS, $(X, C)$, *of order* 9.

$(X, C)$ is given by $X = \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$ and

$$\begin{aligned} C = \{&(0,1,2,5,7,3,6,4), && (1,2,3,6,8,4,7,5), && (2,3,4,7,0,5,8,6), \\ &(3,4,5,8,1,6,0,7), && (4,5,6,0,2,7,1,8), && (5,6,7,1,3,8,2,0), \\ &(6,7,8,2,4,0,3,1), && (7,8,0,3,5,1,4,2), && (8,0,1,4,6,2,5,3)\}. \end{aligned}$$

□

Previous work on twofold 2-perfect cycle systems with this "extra" property has dealt with existence in the case of 5-cycles [5], 6-cycles [2], and also for twofold 2-perfect bowtie systems, [3].

In this paper our main result is the following.

**MAIN THEOREM** *There exists a twofold 2-perfect 8-cycle system of order $n$ having the "extra" property if and only if $n \equiv 0$ or 1 (mod 8), $n \geqslant 9$.*

An exhaustive computer search has shown an ET8CS of order 8 cannot exist.

In the subsequent constructions we make frequent use of group divisible designs with block size 4; in particular we use 4-GDDs of type $a^u b^v$. Details about existence of such group divisible designs is collected in [4], Section IV.4.

# 2 Constructions

In the following we shall use the terminology "ET8CS of order $n$ with a hole of size $h$" to mean a partition of the edge-set of $2(K_n \setminus K_h)$ into 2-perfect 8-cycles with the "extra" property, that is, such that every pair of distinct vertices (not both in the hole $K_h$) is joined by a path of length two in two of the 8-cycles, with these paths having distinct midpoints.

We divide our constructions into various cases, according to the congruence class of the order, modulo 24. Isolated cases, not essential for the recursive constructions, yet not following from the constructions, are listed in the Appendix. In our constructions we make use of the following ET8CS on the twofold 4-partite graph $2K_{2,2,2,2}$.

**Example 2.1** *A twofold 2-perfect 8-cycle system on $2K_{2,2,2,2}$ having the "extra" property.*

We take the vertex set of $2K_{2,2,2,2}$ to be

$$\{0_1, 1_1\} \cup \{0_2, 1_2\} \cup \{0_3, 1_3\} \cup \{0_4, 1_4\}.$$

The following six 8-cycles then form an ET8CS on the graph $2K_{2,2,2,2}$.

$$\begin{array}{ll} (0_1, 0_2, 0_3, 0_4, 1_1, 1_2, 1_3, 1_4), & (0_1, 1_2, 0_3, 1_4, 1_1, 0_2, 1_3, 0_4), \\ (0_1, 0_3, 0_2, 0_4, 1_1, 1_3, 1_2, 1_4), & (0_1, 1_3, 0_2, 1_4, 1_1, 0_3, 1_2, 0_4), \\ (0_1, 0_2, 0_4, 0_3, 1_1, 1_2, 1_4, 1_3), & (0_1, 1_2, 0_4, 1_3, 1_1, 0_2, 1_4, 0_3). \end{array}$$

□

## 2.1 Orders 0 and 1 (mod 24)

We begin with two crucial examples.

**Example 2.2** *An* ET8CS *of order* 24*:*

With vertex set $\mathbb{Z}_{23} \cup \{\infty\}$, the following three starter cycles (mod 23) provide a suitable ET8CS of order 24.

$$(0,1,2,4,6,3,7,10), \quad (0,9,15,1,17,5,11,18), \quad (0,11,6,14,1,9,13,\infty).$$

□

**Example 2.3** *An* ET8CS *of order* 25*:*

With vertex set $\mathbb{Z}_{25}$, the following three starter cycles (mod 25) provide a suitable ET8CS of order 25.

$$(0,7,1,17,20,2,13,5), (0,8,18,12,13,1,4,14), (0,20,8,12,3,1,2,4).$$

□

For order 0 (mod 24), we let the vertex set be $\{(i,j) \mid i \in \mathbb{Z}_{12x},\ j = 1,2\}$, and on the set $\mathbb{Z}_{12x}$ we take a 4-GDD of type $12^x$ which exists for $x \geqslant 4$ ([4], Section IV.4). Then for each group of the GDD, say $\{g_1, g_2, \ldots, g_{12}\}$, we place a copy of Example 2.2 on the vertex set $\{(g_i,1),(g_i,2) \mid i = 1,2,\ldots,12\}$. And for each block of size 4 of the GDD, say $\{a,b,c,d\}$, we place on the vertex set $\{(a,1),(a,2)\} \cup \{(b,1),(b,2)\} \cup \{(c,1),(c,2)\} \cup \{(d,1),(d,2)\}$ a copy of Example 2.1.

Examples of orders 48 and 72 (when $x = 2$ and 3 in the above construction) are given in the Appendix.

For order 1 (mod 24), we adjoin the vertex $\{\infty\}$ to the vertex set, and use Example 2.3 instead of Example 2.2 in the construction.

Once again, when $x = 2$ and 3, since the GDD used above requires $x \geqslant 4$, the construction does not work; so examples of orders 49 and 73 are given in the Appendix.

## 2.2 Order 8 (mod 24)

We begin with two examples for the construction in this case.

**Example 2.4** *An* ET8CS *of order* 32*:*

With vertex set $\mathbb{Z}_{31} \cup \{\infty\}$, the following four starter cycles (mod 31) provide a suitable ET8CS of order 32.

$(0,1,10,14,29,3,2,25)$, $(0,2,11,28,17,19,16,20)$,
$(0,3,15,1,7,20,10,26)$, $(0,18,28,4,11,30,7,\infty)$.

□

**Example 2.5** *An* ET8CS *of order* 32 *with a hole of size* 8*:*
The hole elements are {A, B, C, D, E, F, G, H}, and the remaining 24 elements are $\{i_j \mid 0 \leqslant i \leqslant 11,\ j = 1,2\}$.

The number of 8-cycles needed is 117; we take the following nine cycles, and then a further nine starter cycles modulo 12 (subscripts and hole elements fixed).

Fixed cycles:

$(0_1,0_2,3_1,3_2,6_1,6_2,9_1,9_2)$, $(1_1,1_2,4_1,4_2,7_1,7_2,10_1,10_2)$,
$(2_1,2_2,5_1,5_2,8_1,8_2,11_1,11_2)$, $(0_1,9_1,3_2,0_2,6_1,3_1,9_2,6_2)$,
$(1_1,10_1,4_2,1_2,7_1,4_1,10_2,7_2)$, $(2_1,11_1,5_2,2_2,8_1,5_1,11_2,8_2)$,
$(3_1,0_1,6_2,3_2,9_1,6_1,0_2,9_2)$, $(4_1,1_1,7_2,4_2,10_1,7_1,1_2,10_2)$,
$(5_1,2_1,8_2,5_2,11_1,8_1,2_2,11_2)$.

Starter cycles, mod 12:

$(0_1,3_2,2_2,A,7_1,10_2,6_2,B)$, $(0_1,2_1,6_1,A,5_2,9_1,1_2,B)$,
$(0_1,5_1,0_2,C,8_2,4_2,2_2,D)$, $(0_1,8_2,4_1,C,8_1,1_1,6_2,D)$,
$(0_1,7_2,2_1,E,1_1,3_1,5_2,H)$, $(5_2,10_2,0_1,G,3_1,6_1,7_2,E)$,
$(1_1,0_1,11_2,G,9_2,10_2,8_1,F)$, $(2_1,3_2,3_1,7_1,1_1,0_1,9_2,H)$,
$(11_2,0_1,10_2,0_2,3_2,9_2,4_2,F)$.

□

Now we take the vertex set

$$\{(i,j) \mid i \in \mathbb{Z}_{12x},\ j = 1,2\} \cup \{\infty_k \mid k = 1,2,\ldots,8\}.$$

We use a 4-GDD of type $12^x$ (as in the case 0 (mod 24) above). For *one* group of size 12, say $P = \{g_1, g_2, \ldots, g_{12}\}$, we place on the vertex set $P \times \{1,2\} \cup \{\infty_k \mid k = 1,2,\ldots,8\}$ a copy of Example 2.4. Then for all the remaining groups, such as say $Q$, of size 12, on $Q \times \{1,2\} \cup \{\infty_k \mid k = 1,2,\ldots,8\}$ we place a copy of Example 2.5 with the hole being of course $\{\infty_k \mid k = 1,2,\ldots,8\}$. Then we use copies of Example 2.1 on each of the blocks of size 4, as described in the case 0 (mod 24) above.

Again, the GDD requires $x \geqslant 4$, so cases when $x = 2$ and 3, of orders 56 and 80, were found by machine and appear in the Appendix.

## 2.3 Order 9 (mod 24)

In this easy case we take the vertex set $\{(i,j) \mid i \in \mathbb{Z}_{12x+4},\ j = 1,2\} \cup \{\infty\}$, and we use a 4-GDD of type $4^{3x+1}$ on $\mathbb{Z}_{12x+4}$ (see [4]; this also arises from a resolvable balanced incomplete block design with blocks of size 4 and order $12x+4$). Then for each group $\{g_1, g_2, g_3, g_4\}$ of the GDD, we place a copy of Example 1.1 on the vertex set $\{(g_i, j) \mid i = 1,2,3,4,\ j = 1,2\} \cup \{\infty\}$; and for each block $\{a,b,c,d\}$ we place a copy of Example 2.1 as described in the case of order 0 (mod 24).

## 2.4 Order 16 (mod 24)

We split this into two cases, 16 (mod 48) and 40 (mod 48).

**Example 2.6** *An* ET8CS *of order* 16*:*

With vertex set $\mathbb{Z}_{15} \cup \{\infty\}$, the following two starter cycles (mod 15) provide a suitable ET8CS of order 16.

$$(0,1,2,4,6,3,11,8), \qquad (0,5,9,13,7,1,6,\infty).$$

□

For order $48x+16$, we take the vertex set $\{(i,j) \mid i \in \mathbb{Z}_{6x+2},\ 1 \leqslant j \leqslant 8\}$. Then we use a 4-GDD of type $2^{3x+1}$ for $x \geqslant 1$, copies of Example 2.6 for the GDD groups, and copies of an ET8CS on $2K_{8,8,8,8}$ for the GDD blocks.

We remark that an ET8CS on $2K_{8,8,8,8}$ arises easily by combining 16 appropriate copies of Example 2.1. (We can use a transversal design with group and block size 4, where each element of the transversal design (TD) is a pair of elements in $K_{8,8,8,8}$, and where the groups of the TD are the partite sets of the graph.)

For order $48x+40$, we use the vertex set $\{(i,j) \mid i \in \mathbb{Z}_{6x+5},\ 1 \leqslant j \leqslant 8\}$. A 4-GDD of type $2^{3x}5^1$ exists for $x \geqslant 2$, so we use Example 2.7 once (with the group of size 5), Example 2.6 $3x$ times, and then copies of an ET8CS on $2K_{8,8,8,8}$ for the blocks of the GDD.

The isolated case of order 88 is given in the Appendix; order 40 is below.

**Example 2.7** *An* ET8CS *of order* 40*:*

With vertex set $\mathbb{Z}_{39} \cup \{\infty\}$, the following five starter cycles (mod 39) provide a suitable ET8CS of order 40.

$$(0,1,29,8,27,5,3,38), \qquad (0,2,16,24,4,12,29,23),$$
$$(0,3,8,11,24,17,28,12), \qquad (0,4,13,3,8,2,15,24),$$
$$(0,15,25,11,4,31,13,\infty).$$

□

## 2.5 Order 17 (mod 24)

Again we split this into two cases, orders 17 (mod 48) and 41 (mod 48). In the former case, noting that $48x+17 = 8[2(3x+1)]+1$, we use ET8CSs of $2K_{8,8,8,8}$, and of $2K_{17}$, together with a 4-GDD of type $2^{3x+1}$, which exists for $x \geqslant 1$.

**Example 2.8** *An* ET8CS *of order* 17*:*

The vertex set is $\mathbb{Z}_{17}$; two starter cycles (mod 17) are

$$(0,1,4,2,14,7,16,13), \quad (0,2,8,16,4,14,10,11).$$

□

In the latter case, noting that $48x+41 = 8[2(3x)+5]+1$, we again use ET8CSs of $2K_{8,8,8,8}$ and of $2K_{17}$, as well as an ET8CS of order 41, together with a 4-GDD of type $2^{3x}5^1$, which exists for $x \geqslant 2$. The isolated case of order 89 (when $x = 1$) is given in the Appendix. Order 41 appears below.

**Example 2.9** *An* ET8CS *of order* 41*:*

With the vertex set $\mathbb{Z}_{41}$, the following five starters (mod 41) provide a suitable ET8CS of order 41.

$$(0,2,6,9,1,11,38,18), \quad (0,3,9,13,21,30,20,19),$$
$$(0,5,11,18,29,3,12,24), \quad (0,5,18,19,21,33,8,27),$$
$$(0,7,32,12,36,13,26,15).$$

□

# 3 Conclusion

Combining the constructions in Section 2 above with the examples in the Appendix below, we have the following result.

**MAIN THEOREM** *There exists a twofold* 2*-perfect* 8*-cycle system of order* $n$ *having the "extra" property if and only if* $n \equiv 0$ *or* 1 (mod 8), $n \geqslant 9$.

Future work involves extending the concept of "extra" twofold 2-perfect to "extra" twofold 3-perfect, where a twofold 3-perfect 8-cycle system is "extra" if the two distance *three* paths between each distinct pair of vertices are vertex disjoint, so they form a 6-cycle when placed together. In other words, if $x$ and $y$ are any two distinct vertices, in a 3-perfect 8-cycle system there are two paths of length 3 joining $x$ and $y$ in two of the 8-cycles, such as $x, a_1, b_1, y$ and $x, a_2, b_2, y$, and these form the 6-cycle $(x, a_1, b_1, y, b_2, a_2, x)$.

We conclude this section with the following problem.

> Does there exist a twofold 8-cycle system of order $n$ which is 2-perfect, 3-perfect, "extra" with respect to 2-perfect, and also "extra" with respect to 3-perfect, for all orders $n \equiv 0$ or 1 (mod 8), ($n \geqslant 16$)?

# Appendix

In this appendix we list further necessary examples.

**Order 48**

Vertex set is $\mathbb{Z}_{47} \cup \{\infty\}$. Six starter cycles mod 47:

$$(0,1,3,6,10,15,21,\infty),\quad (0,7,43,27,10,3,4,32),$$
$$(0,8,27,7,5,31,17,25),\quad (0,9,29,7,10,39,26,37),$$
$$(0,9,32,44,40,8,3,17),\quad (0,10,34,5,21,42,7,41).$$

**Order 72**

Vertex set is $\mathbb{Z}_{71} \cup \{\infty\}$. Nine starter cycles mod 71:

$$(0,1,3,6,10,15,21,\infty),\quad (0,7,17,27,49,43,28,67),$$
$$(0,8,23,41,58,60,39,62),\quad (0,11,47,54,17,69,48,1),$$
$$(0,12,31,58,24,38,61,8),\quad (0,13,2,70,31,69,49,25),$$
$$(0,5,14,1,60,30,16,42),\quad (0,16,36,10,54,23,63,38),$$
$$(0,16,38,21,64,29,70,42).$$

**Order 49**

Vertex set is $\mathbb{Z}_{49}$. Six starter cycles mod 49:

$$(0,1,47,5,29,13,27,2),\quad (0,4,13,8,40,1,31,18),$$
$$(0,6,39,36,48,11,28,5),\quad (0,6,41,39,48,14,32,11),$$
$$(0,8,12,2,25,32,10,29),\quad (0,8,37,22,11,33,34,13).$$

Order 73

Vertex set is $\mathbb{Z}_{73}$. Nine starter cycles mod 73:

$$(0,1,50,66,63,51,2,3), \quad (0,2,11,21,57,45,72,22),$$
$$(0,4,22,63,46,6,60,39), \quad (0,5,48,37,27,9,65,15),$$
$$(0,6,19,32,53,59,70,38), \quad (0,7,22,36,55,64,11,44),$$
$$(0,7,34,50,8,43,72,25), \quad (0,14,42,44,49,41,37,45),$$
$$(0,20,63,26,68,42,64,25).$$

Order 56

Vertex set is $\mathbb{Z}_{55} \cup \{\infty\}$. Seven starter cycles mod 55:

$$(0,1,3,6,10,15,21,\infty), \quad (0,7,10,2,38,34,21,16),$$
$$(0,9,30,1,32,44,43,45), \quad (0,11,27,4,45,18,53,43),$$
$$(0,15,33,53,23,14,52,28), \quad (0,15,51,21,35,2,31,37),$$
$$(0,21,54,47,34,23,40,8).$$

Order 80

Vertex set is $\mathbb{Z}_{79} \cup \{\infty\}$. Ten starter cycles mod 79:

$$(0,1,3,6,10,15,21,\infty), \quad (0,2,10,24,45,9,17,43),$$
$$(0,5,18,3,9,44,65,24), \quad (0,7,21,30,33,64,48,31),$$
$$(0,11,36,63,26,76,25,51), \quad (0,12,39,10,45,65,16,32),$$
$$(0,13,37,5,25,62,32,78), \quad (0,18,52,29,10,64,4,61),$$
$$(0,22,60,48,52,19,53,70), \quad (0,39,32,21,36,26,66,56).$$

Order 88

Vertex set is $\mathbb{Z}_{87} \cup \{\infty\}$. Eleven starter cycles mod 87:

$$(0,1,3,6,10,15,21,\infty), \quad (0,1,5,29,3,33,68,55),$$
$$(0,3,13,40,51,57,15,58), \quad (0,5,19,11,39,74,12,51),$$
$$(0,7,30,38,65,15,58,17), \quad (0,7,40,80,10,41,86,15),$$
$$(0,9,24,45,16,70,34,53), \quad (0,9,34,83,72,11,32,50),$$
$$(0,10,66,28,50,18,81,53), \quad (0,12,25,3,70,6,20,39),$$
$$(0,12,58,38,54,24,42,40).$$

**Order 89**

Vertex set is $\mathbb{Z}_{89}$. Eleven starter cycles mod 89:

$$
\begin{array}{ll}
(0,1,3,31,48,9,47,22), & (0,1,5,19,43,9,4,49), \\
(0,2,5,12,20,47,4,29), & (0,3,14,30,57,21,68,45), \\
(0,4,13,7,30,60,88,38), & (0,5,37,7,87,45,55,33), \\
(0,6,23,58,79,38,87,13), & (0,7,25,1,36,82,56,19), \\
(0,8,19,31,64,1,16,36), & (0,10,24,4,17,33,74,19), \\
(0,12,72,41,9,77,40,58). &
\end{array}
$$

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