EDGE MAXIMAL NON-INTERVAL GRAPHS

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ABSTRACT. Let $\mathcal{P}$ be a graph property and G a graph. G is said to be $\mathcal{P}$-saturated if G does not have property $\mathcal{P}$ but the addition of any edge between non-adjacent vertices of G results in a graph with property $\mathcal{P}$. If $\mathcal{P}$ is a bipartite graph property and G is a bipartite graph not in $\mathcal{P}$, but the addition of any edge between non-adjacent vertices in different parts results in a graph in $\mathcal{P}$, then G is $\mathcal{P}$-bisaturated. We characterize all $\mathcal{P}$-saturated graphs, for which $\mathcal{P}$ is the family of interval graphs, and show that this family is precisely the family of maximally non-chordal graphs. We also present a conjectured characterization of all $\mathcal{P}$-bisaturated graphs, in the case where $\mathcal{P}$ is the family of interval bigraphs, and prove it as far as current forbidden subgraph characterizations allow. We demonstrate that extremal noninterval graphs and extremal non-interval bigraphs are highly related, in that the former is simply a complete graph with $2K_2$ removed and the latter is a complete bipartite graph with $3K_2$ removed.

1. INTRODUCTION

Definition 1.1. *Let $\mathcal{S}$ be a family of open intervals on the real line, and G a graph defined by the elements of $\mathcal{S}$ in the following way. For each interval $s \in \mathcal{S}$ let v_s be a vertex of G and join vertices v_s, v_t in G if and only if the intervals $s, t \in \mathcal{S}$ have a non-empty intersection. We say that $\mathcal{S}$ is an* interval representation *of the graph G. Every graph for which an interval representation exists is called an* interval graph.

Definition 1.2. *A bipartite graph $G = (X, Y)$ is an* interval bigraph *if every vertex can be associated with an open interval on the real line in which two vertices $x \in X, y \in Y$ are adjacent in G if and only if their associated intervals have a non-empty intersection.*

Note that we retain the bipartite property by not joining $x_1, x_2 \in X$ or $y_1, y_2 \in Y$ even if their associated intervals intersect.

Interval graphs, and related families including interval bigraphs, proper interval graphs, and circular arc graphs, have been studied extensively, ([1], [2], [3], [6], [8], [9], and [10]). Identification of interval graphs and bigraphs has been a topic of interest in graph theory for some time. Since Lekkerkerker and Boland determined a

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simple forbidden subgraph characterization of interval graphs in 1962 [9], it has been assumed that a similar characterization could be found for interval bigraphs. Early in their study [6] interval bigraphs were thought to be similar enough to interval graphs that a simple translation of the forbidden family of asteroidal triples and induced cycles of length at least four into the family of asteroidal triples of edges and induced cycles of length at least six would suffice for determining whether or not a graph was an interval bigraph. This family was shown to be insufficient in [10], and we are currently left with the graphs in Theorems 3.4 and 3.5. This family has not yet proven to be complete, and graphs like those in [7] may in fact confirm that it is not.

We are primarily concerned with properties of these families related to edge-maximality, which has thus far received little attention. In [4] Eckhoff examines r-extremal interval graphs, those interval graphs with the maximum number of edges among those with fixed order and clique number r, and characterizes them. We extend the study of extremal interval graphs to those graphs that are not interval graphs and have the maximum number of edges, and additionally we examine related families. Although there is a known, simple forbidden subgraph characterization for interval graphs, and an as yet incomplete and very different forbidden subgraph characterization for interval bigraphs, we show that the family of extremal noninterval graphs and a sub-family of extremal noninterval bigraphs are very similar.

We will use $\mathcal{P}$ to represent both a graph property and the complete family of all graphs with property $\mathcal{P}$.

Definition 1.3. *Let $\mathcal{P}$ be a graph property and let G be a graph not in $\mathcal{P}$ such that for any edge $\epsilon = xy \in G^c$ the graph $G + \epsilon$ is in $\mathcal{P}$. Then G is* $\mathcal{P}$-saturated. *If $H = (X, Y)$ is a bipartite graph not in $\mathcal{P}$ such that for any edge $\epsilon = xy \in H^c$ with $x \in X, y \in Y$ the graph $H + \epsilon$ is in $\mathcal{P}$. We say that H is* $\mathcal{P}$-bisaturated.

For n a positive integer we let $Sat(n, \mathcal{P})$ be the family of all $\mathcal{P}$-saturated graphs on n vertices.

In section 2 we characterize all $\mathcal{P}_i$-saturated graphs for $\mathcal{P}_i$ the family of interval graphs. We then characterize all $\mathcal{P}_b$-bisaturated graphs, up to a conjectured forbidden subgraph characterization, where $\mathcal{P}_b$ is the family of interval bigraphs in section 3. In section 4 we address the family of edge maximal split non-interval graphs. We end by examining unit interval and circular arc graphs.

2. Edge-Maximal Non-Interval Graphs

Let G be a graph. An *asteroidal triple* in G is a set A of three vertices such that between any two vertices in A there is a path within G from one to the other that avoids all neighbors of the third. An example is in the 3-sun in Fig. 1. Lekkerkerker and Boland showed that all interval graphs are completely characterized by the absence of both asteroidal triples and induced cycles of length greater than 3 in [9].

Lemma 2.1. *Any graph containing an asteroidal triple contains an induced P_4.*

Proof. Let $A = \{x, y, z\} \subseteq G$ be an asteroidal triple and assume that G does not contain an induced P_4. Between any two vertices in A, say x, y, there is a

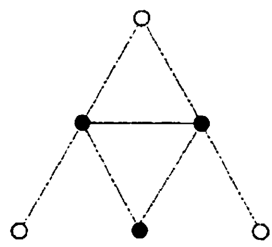

FIGURE 1. The 3-sun in which the white vertices form an AT

shortest path $P(x,y)$ in G avoiding the neighbors of z. $P(x,y)$ cannot have length 1 since there is a z,y path that avoids all neighbors of x and thus y cannot be a neighbor of x. If $P(x,y)$ has length 3 or greater then it contains an induced P_4. Therefore $P(x,y), P(x,z), P(y,z)$ all have length 2. These three paths are internally disjoint so G contains a 6-cycle $xaybzca$. If $\{a,b,c\}$ are not mutually adjacent then G contains an induced P_4. Thus G contains the 3-sun as a subgraph. If $\{a,b,c,x,y,z\}$ does not induce the 3-sun then one of the asteroidal paths $P(u,v)$ between two vertices in A contains a neighbor of the third. However, the 3-sun, and thus G, contains the induced P_4 $yacz$. □

Let $\mathcal{P}_i$ denote the set of interval graphs.

Theorem 2.2. $Sat(n, \mathcal{P}_i) = \{C_4 \vee K_{n-4}\}, n \geq 4$

Proof. Say $G \in Sat(n, \mathcal{P}_i)$. By Lemma 2.1 G cannot contain an asteroidal triple since the endpoints of an induced P_4 can be joined to create an induced C_4, which is another of our forbidden subgraphs. Therefore, G must contain an induced cycle of length at least 4. If G contains an induced $C_k, k > 4$ then there is a pair of vertices that when joined create an induced C_4. So, G must contain an induced 4-cycle C. If there is an edge $\epsilon \in G^c - C^c$ then $G + \epsilon$ still contains the induced 4-cycle, and thus is not an interval graph. Therefore, G must be a 4-cycle joined to a complete graph. □

Note that the family of maximally non-interval graphs is the collection of cliques with an isolated pair of edges removed, precisely the same as the family of maximally non-chordal graphs. While there are non-interval chordal graphs containing asteroidal triples, we have shown that no such graph is edge-maximally non-interval.

3. Edge-Maximal Non-Interval Bigraphs

Throughout this section we are concerned with bipartite graphs $G = (X, Y)$. We sometimes choose to discuss part X or Y, but since there is no distinction between parts X and Y this choice is merely made for convenience, and generality should be assumed.

We begin by introducing some new structures.

Definition 3.1. [10] *A set $A = \{a, c, e\}$ of three edges of a graph G form an* asteriodal triple of edges *(ATE) if for any two, say a, c, there is a path from one*

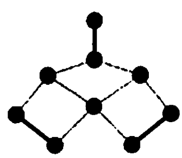

FIGURE 2. The bold edges form an asteriodal triple of edges

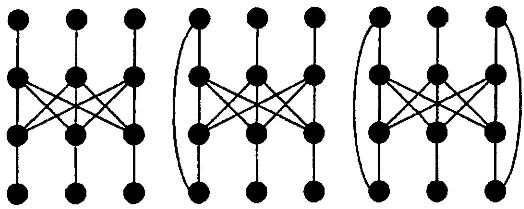

FIGURE 3. Three graphs, called insects, that contain an exobiclique

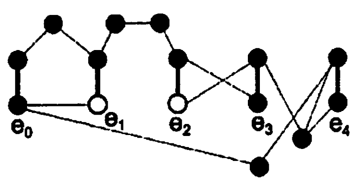

FIGURE 4. An edge asteroid of order 5. Note that joining either vertex in e_4 to any vertex on the path between the two white vertices, inclusive, eliminates this property.

endpoint of a to an endpoint of c that avoids all neighbors of the endpoints of e, (see Fig. 2).

Definition 3.2. [8] *Two sets* A, B *are* incomparable *if* $A \not\subseteq B$ *and* $B \not\subseteq A$. *An* exobiclique *is a bipartite graph* $H = (X, Y)$ *containing a biclique with nonempty parts* $M \subset X$ *and* $N \subset Y$ *such that each of* $X - M$ *and* $Y - N$ *contain three vertices with incomparable neighborhoods in the biclique, (see Fig. 3).*

Definition 3.3. [8] *An* edge-asteroid *of order* $2k+1$ *is a set of edges* $e_0, e_1, ..., e_{2k}$ *such that, for each* $i = 0, 1, \dots, 2k$, *there is a path containing both* e_i *and* e_{i+1} *that avoids the neighbors of* e_{i+k+1}*; the subscript addition is modulo* $2k+1$, *(see Fig. 4).*

Note that the cycle C_6 contains an asteroidal triple of edges but is not an edge-asteroid of order 3, since for any 3 pairwise non-incident edges any pair have an endpoint in the neighborhood of the third, (see Fig. 5). Thus, although their definitions are similar, an edge-asteroid is not simply a generalized asteroidal triple.

None of the structures in Definitions 3.1, 3.2, 3.3 are permitted as induced subgraphs in an interval bigraph, as seen by the following theorems of Muller, Hell and Huang, and Harary et al.

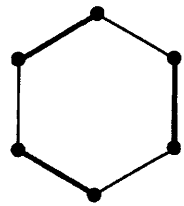

FIGURE 5. An ATE that is not an EA of order 3

Theorem 3.4. *A bipartite graph containing an induced asteroidal triple of edges or an induced cycle with length greater than 4 is not an interval bigraph.* [6]

Theorem 3.5. *A bipartite graph containing an induced exobiclique or an edge asteroid is not an interval bigraph.* [8]

Let $\mathcal{P}_b$ be the family of interval bigraphs.

There is currently no forbidden subgraph characterization of interval bigraphs. However, the preceding theorems provide the most extensive known collection of forbidden subgraphs. Though there are currently proposed bipartite graphs [7] that are not interval but do not fall into one of the cases addressed in Theorems 3.4 and 3.5 the authors are not aware of any examples without induced paths of length at least 5. Since joining the endpoints of an induced P_5 creates an induced C_6 no graph containing an induced P_5 is $\mathcal{P}_b$-saturated.

We will return to this issue at the end of this section.

Definition 3.6. *Let $G = (A, B), H = (C, D)$ be bipartite graphs. The* bipartite joins *of G and H are the graphs consisting of a copy of G, a copy of H, and either all adjacencies between A and C and between B and D, or all adjacencies between A and D and between B and C. Denote by $G * H$ the family of bipartite joins of G and H.*

Lemma 3.7. *Let $G = (X, Y)$ be a $\mathcal{P}_b$-bisaturated bipartite graph of order n. If G contains an induced C_6 then there is an integer $m < n$ such that G is a graph in $C_6 * K_{m,n-m-6}$.*

Proof. Assume not, and let C be an induced C_6 in G. There is a non-adjacent pair of vertices $x \in X, y \in Y$ such that without loss of generality $x \notin C$. The addition of edge xy to G does not destroy the induced C_6 and hence G is not an interval bigraph. So, G must be of the form described. □

Lemma 3.8. *No $\mathcal{P}_b$-bisaturated bipartite graph contains an induced P_6.*

Proof. Let G be a bipartite graph with $v_1, v_2, \ldots, v_6$ an induced P_6. The addition of edge v_1v_6 creates an induced C_6 in the new graph, and hence G is not $\mathcal{P}_b$-bisaturated. □

Lemma 3.9. *Every bigraph containing an asteroidal triple of edges contains either an induced P_6 or an induced C_6.*

Proof. Let $G = X \cup Y$ be a bigraph with an asteroidal triple of edges, $a = a_1a_2$, $b = b_1b_2$, and $c = c_1c_2$ such that $a_1, b_1, c_1 \in X$ and $a_2, b_2, c_2 \in Y$. Assume there is no induced P_6 or C_6 in G, so a, b, and c cannot be alternating edges of a C_6. Therefore, we assume there exists at least one minimal asteroidal triple path, P^1, with length at least 2. Without loss of generality assume this path is between edges a and b. If P^1 has 3 or more edges then it comprises an induced P_6 with edges a and b, so assume this path between a and b has length 2, and label the vertices a_1, d, b_1. There exists a minimal path from edge b to c that avoids the neighborhood of a, call it P^2. Again if P^2 has 3 or more edges then G contains an induced P_6, so P^2 has 1 or 2 edges. If P^2 has length 1 and the path is b_1c_2, then the edge c_1a_2 either exists, creating an induced C_6 with $a_2a_1db_1c_1c_1a_2$, or does not, in which case the vertices form an induced P_6. Now assume the edge comprising P^2 is b_2c_1. In this situation, there are no more edges among these 6, and $a_2a_1db_1b_2c_1$ is an induced P_6. Therefore, let us assume that P^2 has length 2. If $P^2 = b_1ec_1$ then, since there are no more edges among these 6 vertices, $a_1db_1ec_1c_2$ is an induced P_6. So assume the path is b_2ec_2. Now consider possible adjacencies between e, d and a_2, c_1. If $ed, a_2c_1 \in E(G)$, then $c_1a_2a_1dec_2c_1$ is an induced C_6. Otherwise, if neither edge is in G then $a_2a_1db_1b_2e$ is an induced P_6. Finally, if $ed \in E(G)$, but $a_2c_1 \notin E(G)$, then $a_2a_1dec_2c_1$ is an induced P_6. If $a_2c_1 \in E(G)$, but $ed \notin E(G)$, then the 6 vertices create an induced C_8, which contains an induced P_6. Therefore, G has an induced P_6 or C_6. □

By Lemmas 3.8 and 3.9 no bipartite graph with an asteroidal triple of edges is $\mathcal{P}_b$-bisaturated unless it is of the form $C_6 * K_{l,m}$ for integers l, m.

Lemma 3.10. *Every maximal exobiclique contains an induced C_6.*

Proof. Let $G = (A, B)$ be an exobiclique with the property that for any edge $\epsilon \in G^c, G + \epsilon$ is not an exobiclique. There are sets $X, N \subseteq A$ and $Y, M \subseteq B$ such that $H = (M, N)$ is a biclique and X, Y are sets of size at least 3 with incomparable neighborhoods. There exist $x_1, x_2 \in X$ with neighbors $m_1, m_2 \in M$, respectively, such that $x_1 \nsim m_2$ and $x_2 \nsim m_1$. Similarly there exist $y_1, y_2 \in Y$ with neighbors $n_1, n_2 \in N$, respectively, such that $y_1 \nsim n_2$ and $y_2 \nsim n_1$. If x_1y_2 or x_2y_2 is an edge in G, then $m_1n_1m_2x_2y_2x_1m_1$ is an induced C_6. Otherwise, one of these edges can be added without eliminating the property that G is an exobiclique, and thus G remains a non-interval bigraph. □

By Lemmas 3.8 and 3.10 no bipartite graph containing an exobiclique is $\mathcal{P}_b$-bisaturated unless it is of the form $C_6 * K_{l,m}$ for integers l, m.

Now consider $G = (X, Y)$, a bipartite graph containing an edge asteroid $\{e_0, e_1, \ldots, e_{2k}\}$ of order $2k+1$. For each i e_i is the edge joining $x_i \in X, y_i \in Y$. If the distance between the sets $\{x_i, y_i\}$ and $\{x_j, y_j\}$ is greater than 2 for some $0 \leq i, j \leq 2k$ then G contains an induced P_6, so we need only consider the case in which the distance between any pair of edges in an edge asteroid is at most 2. Note that for all $0 \leq i \leq 2k$ we have that no endpoints of either e_i or e_{i+1} is a neighbor of either endpoint of e_{i+k+1}.

Lemma 3.11. *If $k = 1$ then G contains an induced P_6.*

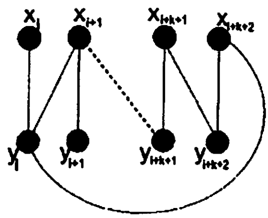

FIGURE 6. The addition of edge $\{x_{i+1}, y_{i+k+1}\}$ creates an ATE

Proof. Let $\{x_0y_0, x_1y_1, x_2y_2\}$ be an edge asteroid of order 3 in G and assume that G contains no induced P_6. The endpoints x_i and y_j are not adjacent for any distinct pair $i, j \in \{0, 1, 2\}$, so without loss of generality there is a path y_0ay_1 in G for some vertex a that is not the endpoint of an edge in the EA. Similarly there is a distinct vertex b that is not an endpoint of an edge in the EA such that either y_1by_2 or x_1bx_2 is a path in G. Therefore, $x_0y_0ay_1by_2$ or $x_0y_0ay_1x_1b$ is an induced P_6 in G. □

Now assume that $k > 1$. First, consider the case in which there is some i such that $y_i \sim x_{i+k+2}$.

Lemma 3.12. *If $y_{i+k+1} \sim x_{i+k+2}$ and $x_i \sim y_{i+1}$ then G is not $\mathcal{P}_b$-bisaturated.*

Proof. Let $\epsilon_1 = y_ix_{i+k+2}, \epsilon_2 = y_{i+1}x_{i+k+1}, \epsilon_3 = y_{i+k+1}x_{i+k+2}$. Note that ϵ_1, ϵ_3 are edges in G and ϵ_2 is not, (Fig. 6). The addition of ϵ_2, however, results in the asteroidal triple of edges $\{\epsilon_1, \epsilon_2, \epsilon_3\}$. By Lemma 3.9 G is not $\mathcal{P}_b$-bisaturated. □

Lemma 3.13. *If $y_{i+k+1} \not\sim x_{i+k+2}$ or $x_i \not\sim y_{i+1}$ then G is not $\mathcal{P}_b$-bisaturated.*

Proof. In the former case let $\epsilon = x_{i+1}y_{i+k+1}$, and in the latter let $\epsilon = y_{i+1}x_{i+k+1}$. Let $\epsilon_1 = y_ix_{i+k+1}, \epsilon_2 = e_{i+1}, \epsilon_3 = e_{i+k+1}$. The addition of edge ϵ results in the asteroidal triple of edges $\{\epsilon_1, \epsilon_2, \epsilon_3\}$. By Lemma 3.9, G is not $\mathcal{P}_b$-bisaturated. □

Now, consider the alternate case in which there is no i such that $y_i \sim x_{i+k+2}$.

Lemma 3.14. *Let $G = (X, Y)$ contain an edge asteroid $\{e_0, e_1, \ldots, e_{2k}\}$ of order greater than three with the property that for all i, $y_i \not\sim x_{i+k+2}$ and $x_i \not\sim y_{i+k+2}$. Then, G is not $\mathcal{P}_b$-bisaturated.*

Proof.

Case 1: Assume that there is an i such that a shortest path between e_i and e_{i+k+1} includes the vertex a_i, a shortest path between e_{i+1} and e_{i+k+2} includes the vertex $a_{i+1} \neq a_i$, e_i is not adjacent to a_{i+1}, and e_{i+1} is not adjacent to a_i. Then, we have one of the graphs in Fig. 7 as a subgraph of G. There is a path from the endpoints of e_i to the endpoints of e_{i+1} that avoids the neighbors of the endpoints of e_{i+k+1}, and thus both graphs in the figure contain an induced P_6 that includes as a subpath $y_ia_iy_{i+k+1}x_{i+k+1}$. By Lemma 3.8 G is not $\mathcal{P}_b$-bisaturated.

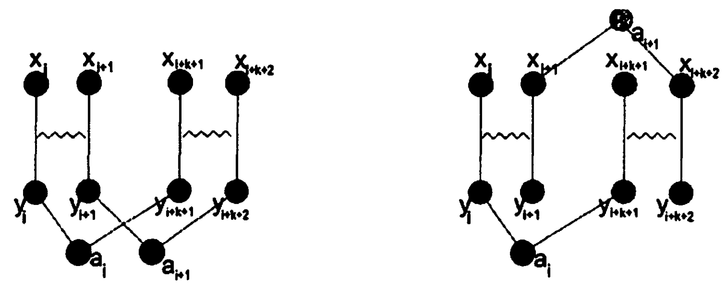

FIGURE 7. Possible configurations of the vertices a_i, a_{i+1}

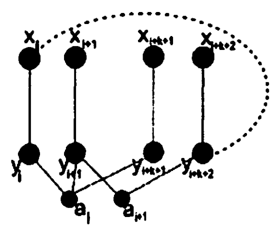

FIGURE 8. a_i, a_{i+1} in the same partite set

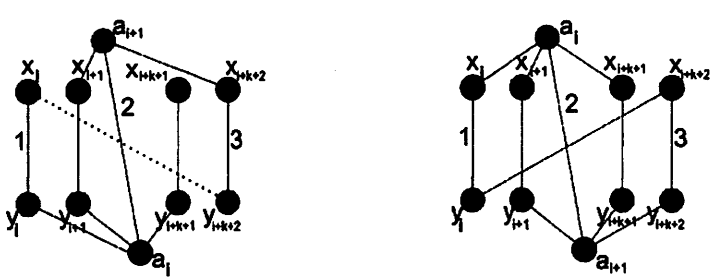

FIGURE 9. Possible configurations when a_i, a_{i+1} in different partite sets

Case 2: Now say that for all i there is a vertex a_i such that $e_i, e_{i+1}, e_{i+k+1} \sim a_i$, as depicted in Figure 8.

Case 2a: There is an integer i such that a_i and a_{i+1} are distinct vertices, (Fig. 9). If a_i, a_{i+1} are in the same partite set of G, set X, then let $\epsilon =$ x_iy_{i+k+2}. The addition of ϵ to G results in the asteroidal triple of edges $\{e_i, e_{i+1}, e_{i+k+2}\}$.
If $a_i \in X, a_{i+1} \in Y$, (Fig. 9), then either $a_i \nsim a_{i+1}$, in which case $a_iy_{i+1}x_{i+1}a_{i+1}x_{i+k+2}y_{i+k+2}$ forms an induced P_6, or $a_i \sim a_{i+1}$, in which case the addition of the edge x_iy_{i+k+2} completes an asteroidal triple of edges among $\{e_i, e_{i+k+2}, a_ia_{i+1}$.

Case 2b: There is a vertex a in G that satisfies $a_i = a$ for all $0 \le i \le 2k$. Let A be the collection of all such vertices, i.e. $A = \{a \in V(G) : a \sim e_i \forall i\}$.

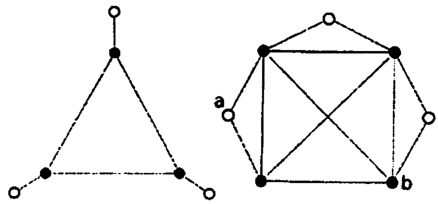

FIGURE 10. The forbidden subgraphs G_1 and G_3

$G' = G - A$ has the same edge asteroid as G since no edge asteroid path includes a vertex in A, and G' contains no vertex adjacent to all edges of the edge asteroid. So, G' falls into Case 1 or Case 2a above, and is an induced subgraph of G.

□

Lemmas 3.7 through 3.14 lead us to the following conjecture.

Conjecture 3.15. *G is $\mathcal{P}_b$-bisaturated if and only if $G \in C_6 * K_{l,m}$ for some integers l, m.*

Note that this family of maximally non-interval bigraphs is precisely the family of bicliques with 3 isolated edges removed.

Thus far we have assumed the forbidden subgraph characterization implied by Theorems 3.4 and 3.5. While there are current attempts at constructing examples of graphs that potentially invalidate this characterization, they are all quite large. Because complete bipartite graphs are interval bigraphs any non-interval bigraph that avoids the aforementioned forbidden subgraphs contains a high number of non-adjacencies. Therefore, we expect that any bipartite graphs that violate the characterization will contain rather long induced paths. We have examined one such graph, briefly mentioned at the top of page 323 in [8], and supplied to us [7] by the authors of [8], and we have found it to contain an induced P_7.

4. Edge-Maximal Split Non-Interval Graphs

Definition 4.1. *A graph G is a split graph if the vertices can be partitioned into sets A and B such that the induced subgraph on A is a complete graph and the the induced subgraph on B an independent set.*

See Fig. 10 for examples in which the white vertices represent independent sets and the black vertices cliques.

Let $\mathcal{P}_s$ denote the property of being an interval graph or a non-split graph. Let G_1 and G_3 be the graphs in Fig. 10, and let G_2 denote the graph in Fig. 1.

Foldes and Hammer showed [5] that a split graph is interval if and only if it does not contain an induced subgraph isomorphic to one of the graphs G_1, G_2, G_3.

Lemma 4.2. *$G_1, G_2 \in Sat(n, \mathcal{P}_s)$, but G_3 is not.*

Proof. G_1 and G_2 are both split graphs that contain asteroidal triples denoted by white vertices in Figures 1 and 10. Any edge that can be added to either graph without resulting in an interval graph is an edge that eliminates the split property. However, the addition of edge ab to G_3, (see Fig. 10), does not destroy the asteroidal triple nor result in a non-split graph. □

Theorem 4.3. $Sat(n, \mathcal{P}_s) = \{G_1 \vee K_{n-6}, G_2 \vee K_{n-6}\}, n \geq 6$

Proof. Say $G \in Sat(n, \mathcal{P}_s)$. One of $\{G_1, G_2\}$ must therefore be a subgraph of G. If $n > 6$ then G also contains least one other vertex v. The smaller graph $G - v$ is a split graph, with vertices appropriately partitioned into sets with induced subgraphs A, a clique, and B, an independent set. Let u be a neighbor of v. If u is in B then in order for G to be a split graph $A \cup \{v\}$ must be a complete graph. We can also join v to every other vertex in B without destroying either condition. So, v is is adjacent to every vertex in $G - \{v\}$. If u is in A then either $< V(A) \cup \{v\} >$ is a clique, which again implies that v is adjacent to every vertex in $G - \{v\}$, or there is a vertex $a \in A$ such that $v \nsim a$ and $V(B) \cup \{v\}$ is an independent set. In this case the edge va can be added to G without violating the split property, and therefore $A \cup \{v\}$ must be a clique. Hence v is adjacent to every vertex in B in order for G to be $\mathcal{P}$-saturated.

Therefore, $Sat(n, \mathcal{P}_s) = \{G_1 \vee K_{n-6}, G_2 \vee K_{n-6}\}$ when $n \geq 6$ □

5. Other Examples

Definition 5.1. *A graph G is a unit interval graph if it has an interval representation in which every vertex is associated with an interval of length* 1.

The following theorem of Roberts [11] will be useful.

Theorem 5.2. *An interval graph G is a unit interval graph if and only if it does not contain $K_{1,3}$ as an induced subgraph.*

Let $\mathcal{P}_u$ denote the family of unit interval graphs.

Theorem 5.3. $Sat(n, \mathcal{P}_u) = \{K_{1,3} \vee K_{n-4}, C_4 \vee K_{n-4}\}, n \geq 4$

Proof. Let $n \geq 4$. We have already seen in Theorem 2.2 that $C_4 \vee K_{n-4}$ is an edge maximal non-interval graph. Since the addition of an edge results in a clique with a single edge removed, which does not contain an induced $K_{1,3}$, it is also in $Sat(n, P_u)$. $K_{1,3} \vee K_{n-4}$ is an interval graph, but by Theorem 5.2 is not unit interval. However, any edge added to $K_{1,3} \vee K_{n-4}$, (in fact, there is only one such edge without loss of generality), eliminates the induced $K_{1,3}$ without creating either an asteroidal triple or a large induced cycle. So, this graph is also in $Sat(n, P_u)$.

Now say that G is a P_u-saturated graph. Then G must contain either a large cycle, an asteroidal triple, or a $K_{1,3}$ as an induced subgraph. Denote this induced subgraph J. By Lemma 2.1 we know G cannot contain an asteroidal triple, and by Theorem 2.2 there is no induced cycle with length greater than 4. So, G contains either an induced $K_{1,3}$ or an induced C_4. If G is not precisely $K_{1,3} \vee K_{n-4}$ or $C_4 \vee K_{n-4}$, then it is a proper subgraph of one of them. There is an edge that can

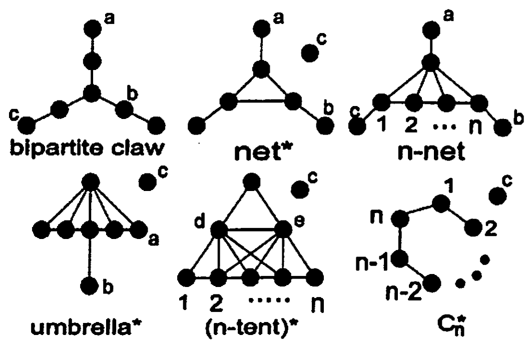

FIGURE 11. The basic minimally non-circular arc graphs

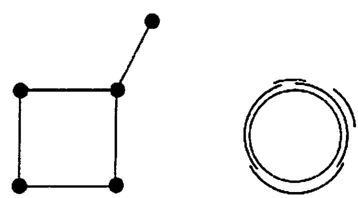

FIGURE 12. A graph and its circular arc representation

be added to G without eliminating J as an induced subgraph. Hence, $Sat(n, \mathcal{P}_u) = \{K_{1,3} \vee K_{n-4}, C_4 \vee K_{n-4}\}$ □

We now introduce a type of graph often studied alongside interval graphs.

Definition 5.4. *A graph G is called a* <u>*circular arc graph*</u> *if it is an interval graph of a family of arcs of a circle.*

In other words, instead of modeling intervals in the real line, the vertices of a circular arc graph can be represented by arcs on a circle. Note that all interval graphs are easily seen to be circular arc graphs by applying an isomorphism from the real line to the unit circle minus a point.

A graph G is *minimally* $\mathcal{P}$ if it has property $\mathcal{P}$ but no proper subgraph has property $\mathcal{P}$. Denote by G^* the graph obtained from G by adding an isolated vertex. We will use the following Lemma from [12].

Lemma 5.5. *The following graphs are minimally non-circular arc graphs: bipartite claw, net***, n-net for all* $n \geq 3$*, umbrella***, (n-tent)** *for all* $n \geq 3$*, and* C_n^* *for every* $n \geq 4$*. Any other minimally non-circular arc graph is connected.*

Bonomo et al. refers to the graphs above (see Fig. 11) as basic minimally non-circular arc graphs in [1]. Let $\mathcal{P}_c$ denote the family of circular arc graphs.

Theorem 5.6. *The only* $\mathcal{P}_c$*-saturated basic minimally non-circular arc graph is* C_4^*.

Proof. It is easy to show that C_4^* is $\mathcal{P}_c$-saturated, since the addition of any edge results in either a C_4 with a pendant edge, realizable by the arcs

$$(0, 2.1), (2, 4.1), (4, 6.1), (6, 0.1), (1, 1.5)$$

in radians (Fig. 12), or $(K_4 - e) \cup K_1$, which is an interval graph and therefore a circular arc graph. Now let G be a bipartite claw, net*, n-net or umbrella*. The addition of the edge ab as labeled in figure 11 creates an induced C_4^* from the ab-path, edge ab, and vertex c. In the n-tent if we add the edge joining vertices 1 and n, then we get an induced C_4^* from the vertices: $1, d, e, n$, and c. If $n \geq 5$ in the C_n^*, then the addition of the edge between vertices 1 and $n-2$ creates an induced C_4^* from the vertices: $1, n, n-1, n-2$, and c. Therefore, the only basic minimally non-circular arc graph in $Sat(n, \mathcal{P}_c)$ is C_4^*. □

6. Conclusion

Given a family $\mathcal{G}$ of graphs characterized by a family $\mathcal{F}$ of forbidden induced subgraphs it is not a surprise that any graph that is edge-maximal non-$\mathcal{G}$ is the join of a graph in $\mathcal{F}$ with a complete graph, as we saw in Theorem 2.2. What is interesting, however, is how similar the families in Theorem 2.2 and Conjecture 3.15 are in light of the differences between the forbidden subgraphs associated with interval graphs and interval bigraphs. It is of course our hope that any new developments in the identification of interval bigraphs will support Conjecture 3.15.

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