Some remarks on degree sets of multigraphs

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Abstract

The degree set of a graph G is the set S consisting of the distinct degrees of vertices in G. In 1977, Kapoor, Polimeni and Wall [2] have determined the least number of vertices among simple graphs with given degree set. In this note, we look at the analogue problem concerning the least order and the least size of a multigraph with a given degree set.

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We consider finite graphs G with vertex set $V(G)$ and edge set $E(G)$. A graph without loops is a multigraph, and a multigraph without parallel edges is a simple graph. We define the order of G by $n = n(G) = |V(G)|$ and the size by $|E(G)|$. For a vertex $v \in V(G)$ of a graph G let $d(v) = d_G(v)$ its degree. The degree sequence of a graph G is defined as the nonincreasing sequence of the degrees of the vertices of G. If $d_1, d_2, \ldots, d_n$ is a sequence of nonnegative intergers, then we say that this sequence can be realized by a graph G when the vertices of G can be labeled $v_1, v_2, \ldots, v_n$ so that $d_G(v_i) = d_i$ for all $1 \le i \le n$.

The degree set of a graph G is the set S consisting of the distinct degrees of vertices in G. In 1977, Kapoor, Polimeni and Wall [2] have determined the least number of vertices among simple graphs with given degree set. If $S = \{d_1, d_2, \ldots, d_k\}$ is a set of integers such that $d_1 > d_2 > \ldots > d_k \ge 1$, then Kapoor, Polimeni and Wall [2] have shown that there exists a simple graph G of order $1 + d_1$ with the degree set S. Recently, Tripathi and Vijay [5] have presented a short and constructive proof of this theorem, which immediately implies a fast algorithm that yields a degree sequence of a simple graph with $1 + d_1$ vertices for a given degree set. In addition, Tripathi and Vijay [4] also looked at the problem concerning the least size of a graph with a given degree set.

In this paper we investigate the analogue questions concerning the least order and the least size of a multigraph with a given degree set, and we shall answer these questions completely. The following well-known result on degree sequences of multigraphs is the main tool in our proofs.

Theorem 1 (Hakimi [1] 1962) Let $n \geq 2$ be an integer. A sequence $d_1 \geq d_2 \geq \ldots \geq d_n \geq 1$ of integers can be realized by a multigraph if and only if $\sum_{i=1}^{n} d_i$ is even and $d_1 \leq \sum_{i=2}^{n} d_i$.

In 1994, Takahashi, Imai and Asano [3] have given a short and constructive proof of Theorem 1.

Theorem 2 Let $S = \{d_1, d_2, \ldots, d_k\}$ be a set of integers such that

$$d_1 > d_2 > \ldots > d_k \geq 1.$$

(i) If $k = 1$, then S is the degree set of a multigraph of order two.

Assume now that $k \geq 2$ and $d_1 \leq \sum_{i=2}^{k} d_i$.

(ii) If $\sum_{i=1}^{k} d_i$ is even, then S is the degree set of a multigraph of order k.

(iii) If $\sum_{i=1}^{k} d_i$ is odd, then S is the degree set of a multigraph of order $k+1$.

Next assume that $k \geq 2$ and $d_1 > \sum_{i=2}^{k} d_i$.

(iv) If $d_1 + \sum_{i=1}^{k} d_i$ is even, then S is the degree set of a multigraph of order $k+1$.

In addition, assume in the following that $d_1 + \sum_{i=1}^{k} d_i$ is odd.

(v) Let $\sum_{i=1}^{k} d_i$ be even. If there exists an index $2 \leq j \leq k$ such that d_j is even and $d_1 \leq d_j + \sum_{i=2}^{k} d_i$, then S is the degree set of a multigraph of order $k+1$. If there is no such index, then S is the degree set of a multigraph of order $k+2$.

(vi) Let $\sum_{i=1}^{k} d_i$ be odd. If there exists an index $2 \leq j \leq k$ such that d_j is odd and $d_1 \leq d_j + \sum_{i=2}^{k} d_i$, then S is the degree set of a multigraph of order $k+1$. If there is no such index, then S is the degree set of a multigraph of order $k+2$.

In all cases the given order of the multigraph is the least possible one.

Proof. (i) If $k = 1$, then let G consists of two vertices, which are joined by d_1 parallel edges. This multigraph G of order two has the degree set $S = \{d_1\}$. Since order one is impossible, order two is the least one.

Next assume that $k \geq 2$ and $d_1 \leq \sum_{i=2}^{k} d_i$.

(ii) Let $\sum_{i=1}^{k} d_i$ be even. Applying Theorem 1, we see that the sequence $d_1, d_2, \ldots, d_k$ can be realized by a multigraph G, and thus S is the degree set of the multigraph G of order k.

(iii) Let $\sum_{i=1}^{k} d_i$ be odd. In view of Theorem 1, there does not exist a multigraph of order k with degree set S. Since $\sum_{i=1}^{k} d_i$ is odd, there exists an index j such that d_j is odd. According to Theorem 1, the degree sequence $d_1, d_2, \ldots, d_{j-1}, d_j, d_j, d_{j+1}, \ldots, d_k$ can be realized by a multigraph G, and thus S is the degree set of the multigraph G of order $k + 1$.

Now assume that $k \geq 2$ and $d_1 > \sum_{i=2}^{k} d_i$. In view of Theorem 1, there does not exist a multigraph of order k with the degree set S.

(iv) Let $d_1 + \sum_{i=1}^{k} d_i$ be even. By Theorem 1, the degree sequence $d_1, d_1, d_2, \ldots, d_k$ can be realized by a multigraph G, and thus S is the degree set of the multigraph G of order $k + 1$.

Finally, assume that $d_1 + \sum_{i=1}^{k} d_i$ is odd.

(v) Let $\sum_{i=1}^{k} d_i$ be even. This assumption implies that d_1 is odd. If there exists an index $2 \leq j \leq k$ such that d_j is even and $d_1 \leq d_j + \sum_{i=2}^{k} d_i$, then Theorem 1 shows that the degree sequence

$$d_1, d_2, \ldots, d_{j-1}, d_j, d_j, d_{j+1}, \ldots, d_k$$

can be realized by a multigraph G, and thus S is the degree set of the multigraph G of order $k + 1$. If there is no such index, then S is not the degree set of a multigraph of order less or equal $k+1$. However, by Theorem 1, the degree sequence $d_1, d_1, d_1, d_2, \ldots, d_k$ can be realized by a multigraph G, and thus S is the degree set of the multigraph G of order $k + 2$.

(vi) Let $\sum_{i=1}^{k} d_i$ be odd. This assumption implies that d_1 is even. If there exists an index $2 \leq j \leq k$ such that d_j is odd and $d_1 \leq d_j + \sum_{i=2}^{k} d_i$, then Theorem 1 shows that the degree sequence

$$d_1, d_2, \ldots, d_{j-1}, d_j, d_j, d_{j+1}, \ldots, d_k$$

can be realized by a multigraph G, and thus S is the degree set of the multigraph G of order $k + 1$. If there is no such index, then S is a not the degree set of a multigraph of order less or equal $k + 1$. However, since $\sum_{i=1}^{k} d_i$ is odd, there exists an index $2 \leq j \leq k$ such that d_j is odd. From Theorem 1 we deduce that the degree sequence

$$d_1, d_1, d_2, \ldots, d_{j-1}, d_j, d_j, d_{j+1}, \ldots, d_k$$

can be realized by a multigraph G, and thus S is the degree set of the multigraph G of order $k+2$. □

The proof of Theorem 2 and the proof of Theorem 1 in [3] lead to an efficient algorithm that yields a multigraph of least order with given degree set.

Corollary 3 Let $S = \{d_1, d_2, \ldots, d_k\}$ be a set of k positive and distinct integers. Then S is a degree set of a multigraph G of order $k \leq n(G) \leq k+2$.

Using the proof of Theorem 2, we now determine the least size of a multigraph with given degree set.

Remark 4 The multigraphs given in the proof of Theorem 2 (i) and (ii) are of least size.

If we choose in the proof of Theorem 2 (iii) the largest index $1 \leq j \leq k$ such that d_j is odd, then we arrive at a multigraph of least size.

If $k \geq 2$ and $d_1 > \sum_{i=2}^{k} d_i$, then we can split the prolem into two cases.

Case 1. Determine the least size among all multigraphs of least order with a given degree set.

Subcase 1.1. Choose the largest index $1 \leq j \leq k$ such that $d_1 \leq d_j + \sum_{i=2}^{k} d_i$ and $d_j + \sum_{i=1}^{k} d_i$ is even. Then the sequence

$$d_1, d_2, \ldots, d_{j-1}, d_j, d_j, d_{j+1}, \ldots, d_k$$

can be realized by a multigraph G of order $k+1$, and G is of least size among all multigraphs of least order with a given degree set.

Subcase 1.2. If there does not exist such an index, then choose two indices $1 \leq p \leq q \leq k$ as large as possible ($p = q$ is admissible) such that $d_1 \leq d_p + d_q + \sum_{i=2}^{k} d_i$ and $d_p + d_q + \sum_{i=1}^{k} d_i$ is even. Then the sequence

$$d_1, d_2, \ldots, d_{p-1}, d_p, d_{p+1}, \ldots, d_{q-1}, d_q, d_{q+1}, \ldots, d_k$$

can be realized by a multigraph G of order $k+2$, and G is of least size among all multigraphs of least order with a given degree set.

Case 2. Determine only the least size of a multigraph with a given degree set S. Choose a sequence $d_{j_1}, d_{j_2}, \ldots, d_{j_t}$ of degrees ($j_p = j_q$ is admissible) such that $d_1 \leq d_{j_1} + d_{j_2} + \ldots + d_{j_t} + \sum_{i=2}^{k} d_i$ and $d_{j_1} + d_{j_2} + \ldots + d_{j_t} + \sum_{i=1}^{k} d_i$ is even such that $d_{j_1} + d_{j_2} + \ldots + d_{j_t}$ is minimal with respect to these properties. This leads to a multigraph of least size with a given degree set S.

Demonstrating the difference between Case 1 and Case 2 in Remark 4, we present two examples.

Example 5 Let $S = \{11, 2, 1\}$. Then the degree sequence 11,11,2,1,1 leads to a multigraph G of least size with degree set S under the assumption that G is of least order. However, the degree sequence 11,2,2,2,2,2,1 leads to a multigraph of least size with degree set S.

Let $S = \{12, 2\}$. Then the degree sequence 12,12,2 leads to a multigraph G of least size with degree set S under the assumption that G is of least order. However, the degree sequence 12,2,2,2,2,2,2 leads to a multigraph of least size with degree set S.

Remark 6 Let $S = \{d_1, d_2, \ldots, d_k\}$ be a degree set such that $d_1 > d_2 > \ldots > d_k \geq 1$. If loops are allowed, then it is sraightforward to verify the following properties.

If $\sum_{i=1}^{k} d_i$ is even, then S is the degree set of a graph of order k, and we arrive at the least size when we choose the maximum number of loops.

Assume that $\sum_{i=1}^{k} d_i$ is odd. If we choose the largest index d_j such that $d_j + \sum_{i=1}^{k} d_i$ is even, then the degree sequence

$$d_1, d_2, \ldots, d_{j-1}, d_j, d_j, d_{j+1}, \ldots, d_k$$

can be realized by a graph G, and thus S is the degree set of G of order $k+1$. In addition, we arrive at the least size when we choose the maximum number of loops in G.

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