On the metric dimension of two classes of convex polytopes*

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Abstract. A family $\mathcal{G}$ of connected graphs is a family with constant metric dimension if $dim(G)$ is finite and does not depend upon the choice of G in $\mathcal{G}$.
The metric dimension of some classes of plane graphs has been determined in [3], [4], [5], [10], [13] and [18] while metric dimension of some classes of convex polytopes has been determined in [8] and a question was raised as an open problem: Is it the case that the graph of every convex polytope has constant metric dimension?
In this paper, we study the metric dimension of two classes of convex polytopes. It is shown that these classes of convex polytopes have constant metric dimension and only three vertices chosen appropriately suffice to resolve all the vertices of these classes of convex polytopes. It is natural to ask for the characterization of classes of convex polytopes with constant metric dimension.

Keywords: *Metric dimension, basis, resolving set, plane graph, antiprism, convex polytopes*

1 Notation and preliminary results

If G is a connected graph, the *distance* $d(u, v)$ between two vertices $u, v \in V(G)$ is the length of a shortest path between them. Let $W = \{w_1, w_2, ..., w_k\}$ be an ordered set of vertices of G and let v be a vertex of G. The *representation* $r(v|W)$ of v with respect to W is the k-tuple

* This research is partially supported by Abdus Salam School of Mathematical Sciences, Lahore and Higher Education Commission of Pakistan.

$(d(v,w_1), d(v,w_2), ..., d(v,w_k))$. If distinct vertices of G have distinct representations with respect to W, then W is called a *resolving set* for G [3]. A resolving set of minimum cardinality is called a *basis* for G and this cardinality is the *metric dimension* of G, denoted by $dim(G)$. The concepts of resolving set and metric basis have previously appeared in the literature (see [3-6, 8-11, 13-18]).
For a given ordered set of vertices $W = \{w_1, w_2, ..., w_k\}$ of a graph G, the *ith* component of $r(v|W)$ is 0 if and only if $v = w_i$. Thus, to show that W is a resolving set it suffices to verify that $r(x|W) \neq r(y|W)$ for each pair of distinct vertices $x, y \in V(G) \backslash W$.
A useful property in finding $dim(G)$ is the following:

Lemma 1. *[17] Let W be a resolving set for a connected graph G and $u, v \in V(G)$. If $d(u,w) = d(v,w)$ for all vertices $w \in V(G) \setminus \{u,v\}$, then $\{u,v\} \cap W \neq \emptyset$.*

Slater refereed to the metric dimension of a graph as its location number and motivated the study of this invariant by its application to the placement of a minimum number of sonar/loran detecting devices in a network so that the position of every vertex in the network can be uniquely described in terms of its distances to the devices in the set ([15],[16]). These concepts have also some applications in chemistry for representing chemical compounds ([6],[11]) or to problems of pattern recognition and image processing, some of which involve the use of hierarchical data structures [14].
By denoting $G + H$, we mean the join of G and H. A *wheel* W_n is defined as $W_n = K_1 + C_n$, for $n \geq 3$, a *fan* is $f_n = K_1 + P_n$ for $n \geq 1$ and *Jahangir graph* $J_{2n}, (n \geq 2)$ (also known as *gear graph*) is obtained from the *wheel* W_{2n} by alternately deleting n spokes. Buczkowski *et al.* [3] determined the dimension of *wheel* W_n, Caceres *et al.* [5] the dimension of *fan* f_n and Tomescu and Javaid [18] the dimension of *Jahangir graph* J_{2n}.

Theorem 1. *([3], [5], [18]) Let W_n be a wheel of order $n \geq 3$, f_n be fan of order $n \geq 1$ and J_{2n} be a Jahangir graph. Then*
(i) For $n \geq 7$, $dim(W_n) = \lfloor \frac{2n+2}{5} \rfloor$;
(ii) For $n \geq 7$, $dim(f_n) = \lfloor \frac{2n+2}{5} \rfloor$;
(iii) For $n \geq 4$, $dim(J_{2n}) = \lfloor \frac{2n}{3} \rfloor$.

The metric dimension of all these plane graphs depends upon the number of vertices in the graph.
On the other hand, we say that a family $\mathcal{G}$ of connected graphs is a family with constant metric dimension if dim(G) is finite and does not depend upon the choice of G in $\mathcal{G}$. In [6] it was shown that a graph has metric dimension 1 if and only if it is a *path*, hence paths on n vertices constitute

a family of graphs with constant metric dimension. Similarly, *cycles* with $n(\geq 3)$ vertices also constitute such a family of graphs as their metric dimension is 2 and does not depend upon on the number of vertices n. In [4] it was proved that

$$dim(P_m \times C_n)=\begin{cases} 2, & \text{if } n \text{ is odd;} \\ 3, & \text{if } n \text{ is even.} \end{cases}$$

Since *prisms* D_n are the trivalent plane graphs obtained by the cross product of path P_2 with a cycle C_n, this implies that

$$dim(D_n)=\begin{cases} 2, & \text{if } n \text{ is odd;} \\ 3, & \text{if } n \text{ is even.} \end{cases}$$

So, prisms constitute a family of 3-*regular graphs* with constant metric dimension. Javaid *et al.* proved in [10] that the plane graph *antiprism* A_n constitute a family of regular graphs with constant metric dimension as $dim(A_n) = 3$ for every $n \geq 5$. The prism and the antiprism are *Archimedean* convex polytopes defined e.g. in [12]. The metric dimension of some classes of convex polytopes has been determined in [8] and it is shown that these classes of convex polytopes have constant metric dimension 3 and an open problem was raised:

Open problem [8]: Is it the case that the graph of every convex polytope has constant metric dimension?

Note that the problem of determining whether $dim(G) < k$ is an NP-complete problem [7].

Let the graph of antiprism A_n [1] be given. We insert a vertex a_{n+1} inside the n-gone P and b_{n+1} inside the n-gone P'.

We join any vertex a_i of P with the vertex a_{n+1} and any vertex b_i of P' with the vertex b_{n+1} for $i = 1, 2, ..., n$. Thus we obtain the graph A'_n. The dual graph to A'_n with vertices $a_1, a_2, \ldots, a_n; b_1, b_2, ..., b_n; c_1, c_2, ..., c_n; d_1, d_2, ..., d_n$ is the graph of the convex polytope defined in [1] and is denoted by $\mathbb{D}_n$. It has 5-sided faces and n-sided faces.

The graph of convex polytope Q_n [2] (Fig. 1) can be obtained from the graph of convex polytope $\mathbb{D}_n$ [1] by adding new edges b_ib_{i+1}. i.e., $V(Q_n) = V(\mathbb{D}_n)$ and $E(Q_n) = E(\mathbb{D}_n) \cup \{b_ib_{i+1} : 1 \leq i \leq n\}$. It was shown in [8] that both graph of convex polytopes $\mathbb{D}_n$ and Q_n have same metric dimension 3.

In this paper, we extend this study by considering two classes of convex polytopes which can be obtained from the graph of convex polytope Q_n defined in [2] by adding new edges in it in specific ways . In the second section, we study the metric dimension of graph of convex polytope $\mathbb{L}_n$ consisting of 3-sided faces, 4-sided faces and n-sided faces. In the third section, we investigate the metric dimension of the graph of convex polytope $\mathbb{B}_n$ consisting of 3-sided faces and n-sided faces.

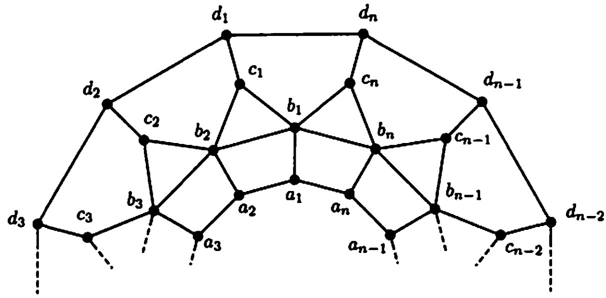

Fig. 1. The graph of convex polytope Q_n

2 The graph of convex polytope $\mathbb{L}_n$

The graph of convex polytope $\mathbb{L}_n$ is obtained from the graph of convex polytope Q_n [2] by adding new edges c_ic_{i+1} and $a_{i+1}b_i$ (i is taken modolu n). i.e., $V(\mathbb{L}_n) = V(Q_n)$ and $E(\mathbb{L}_n) = E(Q_n) \cup \{c_ic_{i+1} : 1 \leq i \leq n\} \cup \{a_{i+1}b_i : 1 \leq i \leq n\}$ (Fig 1). The graph of convex polytope $\mathbb{L}_n$ can

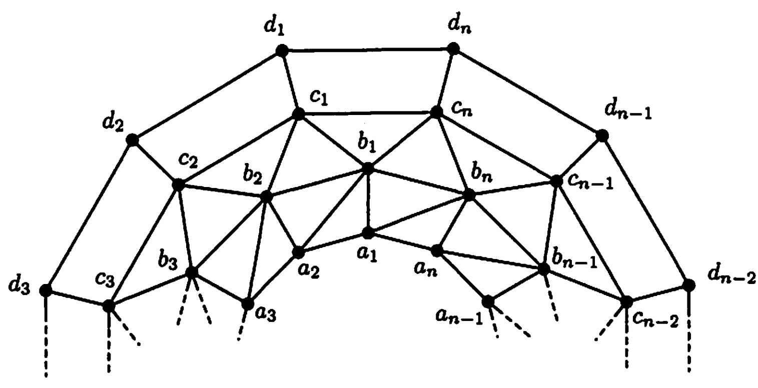

Fig. 2. The graph of convex polytope $\mathbb{L}_n$

also be obtained by taking cartesian product of path P_4 with a cycle C_n and then adding the new edges $a_{i+1}b_i; b_{i+1}c_i$. i.e. $V(\mathbb{L}_n) = V(P_4 \times C_n)$ and $E(\mathbb{L}_n) = E(P_4 \times C_n) \cup \{a_{i+1}b_i; b_{i+1}c_i : 1 \leq i \leq n\}$.
For our purpose, we call the cycle induced by $\{a_i : 1 \leq i \leq n\}$, the inner cycle, cycle induced by $\{b_i : 1 \leq i \leq n\}$, the interior cycle, cycle induced

by $\{c_i : 1 \leq i \leq n\}$, the exterior cycle, cycle induced by $\{d_i : 1 \leq i \leq n\}$, the outer cycle .
The metric dimension of graph of convex polytope Q_n has been determined in [8] and it is proved that $dim(Q_n) = 3$ for every $n \geq 6$. In the next theorem, we show that the metric dimension of the graph of convex polytope $\mathbb{L}_n$ is 3. Note that the choice of appropriate basis vertices (also referred to as landmarks in [13]) is core of the problem.

Theorem 2. *For $n \geq 6$, let the graph of convex polytopes be $\mathbb{L}_n$; then $dim(\mathbb{L}_n) = 3$.*

Proof. We will prove the above equality by double inequalities. We consider the two cases.
Case(i) When n is even.
In this case, we can write $n = 2k$, $k \geq 3$, $k \in \mathbf{Z}^+$. Let $W = \{a_1, a_2, a_{k+1}\} \subset V(\mathbb{L}_n)$, we show that W is a resolving set for $\mathbb{L}_n$ in this case. For this we give representations of any vertex of $V(\mathbb{L}_n) \setminus W$ with respect to W.
Representations of the vertices on inner cycle are

$$r(a_i|W) = \begin{cases} (i-1, i-2, k-i+1), & 3 \leq i \leq k; \\ (2k-i+1, 2k-i+2, i-k-1), & k+2 \leq i \leq 2k. \end{cases}$$

Representations of the vertices on interior cycle are

$$r(b_i|W) = \begin{cases} (1, 1, k), & i = 1; \\ (i, i-1, k-i+1), & 2 \leq i \leq k; \\ (k, k, 1), & i = k+1; \\ (2k-i+1, 2k-i+2, i-k), & k+2 \leq i \leq 2k. \end{cases}$$

Representations the vertices on exterior cycle are

$$r(c_i|W) = \begin{cases} (2, 2, k), & i = 1; \\ (i+1, i, k-i+1), & 2 \leq i \leq k-1; \\ (k+1, k, 2), & i = k; \\ (2k-i+1, 2k-i+2, i-k+1), & k+1 \leq i \leq 2k-1; \\ (2, 2, k+1), & i = 2k. \end{cases}$$

Representations of the vertices on outer cycle are

$$r(d_i|W) = \begin{cases} (3, 3, k+1), & i = 1; \\ (i+2, i+1, k-i+2), & 2 \leq i \leq k-1; \\ (k+2, k+1, 3), & i = k; \\ (2k-i+2, 2k-i+3, i-k+2), & k+1 \leq i \leq 2k-1; \\ (3, 3, k+2), & i = 2k. \end{cases}$$

We note that there are no two vertices having the same representations implying that $dim(\mathbb{L}_n) \leq 3$.

On the other hand, we show that $dim(\mathbb{L}_n) \geq 3$. Suppose on contrary that $dim(\mathbb{L}_n) = 2$, then there are following possibilities to be discussed.

(1) Both vertices are in the inner cycle. Without loss of generality, we can suppose that one resolving vertex is a_1. Suppose that the second resolving vertex is a_i $(2 \leq i \leq k+1)$. Then for $2 \leq i \leq k$, we have $r(a_n|\{a_1, a_i\}) = r(b_n|\{a_1, a_i\})$ and for $i = k+1$, we have $r(a_2|\{a_1, a_{k+1}\}) = r(a_n|\{a_1, a_{k+1}\})$, a contradiction.

(2) Both vertices are in the interior cycle. Without loss of generality, we can suppose that one resolving vertex is b_1. Suppose that the second resolving vertex is b_i $(2 \leq i \leq k+1)$. Then for $2 \leq i \leq k$, we have $r(a_1|\{b_1, b_i\}) = r(c_n|\{b_1, b_i\})$ and for $i = k+1$, we have $r(b_2|\{b_1, b_{k+1}\}) = r(b_n|\{b_1, b_{k+1}\})$, a contradiction.

(3) Both vertices are in the exterior cycle. Due to the symmetry of the graph, this case is analogous to case **(2)**.

(4) Both vertices are in the outer cycle. Without loss of generality, we can suppose that one resolving vertex is d_1. Suppose that the second resolving vertex is d_i $(2 \leq i \leq k+1)$. Then for $2 \leq i \leq k$, we have $r(c_1|\{d_1, d_i\}) = r(d_n|\{d_1, d_i\})$ and for $i = k+1$, we have $r(d_2|\{d_1, d_{k+1}\}) = r(d_n|\{d_1, d_{k+1}\})$, a contradiction.

(5) One vertex is in the inner cycle and other in interior cycle. Without loss of generality, we can suppose that one resolving vertex is a_1 and the second resolving vertex is b_i $(1 \leq i \leq k+1)$. Then for $i = 1$, we have $r(a_2|\{a_1, b_1\}) = r(b_n|\{a_1, b_1\})$ and for $2 \leq i \leq k+1$, $r(a_2|\{a_1, b_i\}) = r(b_1|\{a_1, b_i\})$, a contradiction.

(6) One vertex is in the inner cycle and other in exterior cycle. Without loss of generality, we can suppose that one resolving vertex is a_1 and the second resolving vertex is c_i $(1 \leq i \leq k+1)$. Then for $i = 1$, we have $r(b_2|\{a_1, c_1\}) = r(c_n|\{a_1, c_1\})$ and for $2 \leq i \leq k$ $r(a_2|\{a_1, c_i\}) = r(b_1|\{a_1, c_i\})$. When $i = k+1$, $r(a_n|\{a_1, c_{k+1}\}) = r(b_1|\{a_1, c_{k+1}\})$, a contradiction.

(7) One vertex is in the inner cycle and other in the outer cycle. Again, we can suppose that one resolving vertex is a_1. Suppose that the second resolving vertex is d_i $(1 \leq i \leq k+1)$. Then for $i = 1$, $r(b_2|\{a_1, d_1\}) = r(c_n|\{a_1, d_1\})$ and for $2 \leq i \leq k+1$ we have $r(a_2|\{a_1, d_i\}) = r(b_1|\{a_1, d_i\})$, a contradiction.

(8) One vertex is in interior cycle and other in exterior cycle. This case is analogous to case **(5)**.

(9) One vertex is in interior cycle and other is in outer cycle. Without loss of generality, we can suppose that one resolving vertex is b_1 and the second resolving vertex is d_i $(1 \leq i \leq k+1)$. Then for $i = 1$, we have $r(b_2|\{b_1, d_1\}) = r(c_n|\{b_1, d_1\})$ and for $2 \leq i \leq k+1$, $r(b_2|\{b_1, d_i\}) = r(c_1|\{b_1, d_i\})$, a contradiction.

(10) One vertex is in exterior cycle and other is in outer cycle. Without

loss of generality, we can suppose that one resolving vertex is c_1 and the second resolving vertex is d_i $(1 \leq i \leq k+1)$. Then for $i=1$, we have $r(c_2|\{c_1,d_1\}) = r(c_n|\{c_1,d_1\})$, for $i=2$ $r(b_3|\{c_1,d_2\}) = r(d_n|\{c_1,d_2\})$ and for $3 \leq i \leq k+1$, $r(c_2|\{c_1,d_i\}) = r(d_1|\{c_1,d_i\})$, a contradiction.
Hence, from above it follows that there is no resolving set with two vertices for $V(\mathbb{L}_n)$ implying that $dim(\mathbb{L}_n) = 3$ in this case.
Case(ii) When n is odd.
In this case, we can write $n = 2k+1$, $k \geq 3$, $k \in \mathbf{Z}^+$. Again we show that $W = \{a_1, a_2, a_{k+1}\} \subset V(\mathbb{L}_n)$ is a resolving set for $\mathbb{L}_n$ in this case. For this we give representations of any vertex of $V(\mathbb{L}_n) \setminus W$ with respect to W.
Representations of the vertices on inner cycle are

$$r(a_i|W) = \begin{cases} (i-1, i-2, k-i+1), & 3 \leq i \leq k; \\ (k,k,1), & i = k+2; \\ (2k-i+2, 2k-i+3, i-k-1), & k+3 \leq i \leq 2k+1. \end{cases}$$

Representations of the vertices on interior cycle are

$$r(b_i|W) = \begin{cases} (1,1,k), & i = 1; \\ (i, i-1, k-i+1), & 2 \leq i \leq k; \\ (k+1, k, 1), & i = k+1; \\ (2k-i+2, 2k-i+3, i-k), & k+2 \leq i \leq 2k+1. \end{cases}$$

Representations of the vertices on exterior cycle are

$$r(c_i|W) = \begin{cases} (2,2,k), & i = 1; \\ (i+1, i, k-i+1), & 2 \leq i \leq k-1; \\ (k+1, k, 2), & i = k; \\ (k+1, k+1, 2), & i = k+1; \\ (2k-i+2, 2k-i+3, i-k+1), & k+2 \leq i \leq 2k; \\ (2,2,k+1), & i = 2k+1. \end{cases}$$

Representations of the vertices on outer cycle are

$$r(d_i|W) = \begin{cases} (3,3,k+1), & i = 1; \\ (i+2, i+1, k-i+2), & 2 \leq i \leq k-1; \\ (k+2, k+1, 3), & i = k; \\ (k+2, k+2, 3), & i = k+1; \\ (2k-i+3, 2k-i+4, i-k+2), & k+2 \leq i \leq 2k; \\ (3,3,k+2), & i = 2k+1. \end{cases}$$

Again we see that there are no two vertices having the same representations which implies that $dim(\mathbb{L}_n) \leq 3$.
On the other hand, suppose that $dim(\mathbb{L}_n) = 2$, then there are the same possibilities as in case (i) and contradiction can be deduced analogously. This implies that $dim(\mathbb{L}_n) = 3$ in this case, which completes the proof.

3 The graph of convex polytope $\mathbb{B}_n$

The graph of convex polytope $\mathbb{B}_n$ is obtained from the graph of convex polytope Q_n [2] by adding new edges c_ic_{i+1}, $a_{i+1}b_i$ and $c_{i+1}d_i$ (i is taken modolu n). i.e., $V(\mathbb{B}_n) = V(Q_n)$ and $E(\mathbb{B}_n) = E(Q_n) \cup \{c_ic_{i+1} : 1 \le i \le n\} \cup \{a_{i+1}b_i : 1 \le i \le n\} \cup \{c_{i+1}d_i : 1 \le i \le n\}$ (Fig 2).
The graph of convex polytope $\mathbb{B}_n$ can also be obtained by taking carte-

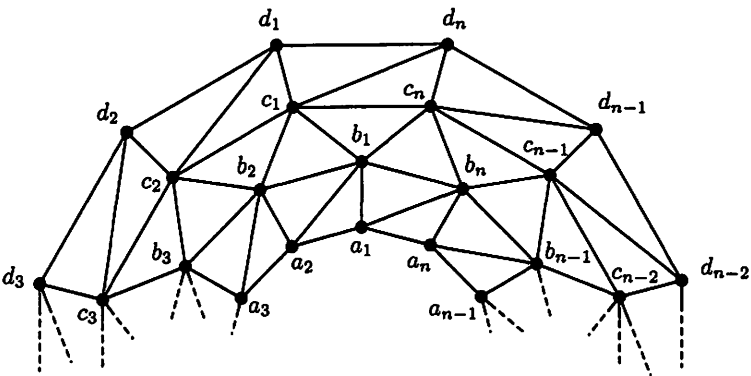

Fig. 3. The graph of convex polytope $\mathbb{B}_n$

sian product of path P_4 with a cycle C_n and then adding the new edges $a_{i+1}b_i; b_{i+1}c_i; c_{i+1}d_i$. i.e. $V(\mathbb{B}_n) = V(P_4 \times C_n)$ and $E(\mathbb{B}_n) = E(P_4 \times C_n) \cup \{a_{i+1}b_i; b_{i+1}c_i; c_{i+1}d_i : 1 \le i \le n\}$.
For our purpose, we call the cycle induced by $\{a_i : 1 \le i \le n\}$, the inner cycle, cycle induced by $\{b_i : 1 \le i \le n\}$, the interior cycle, cycle induced by $\{c_i : 1 \le i \le n\}$, the exterior cycle, cycle induced by $\{d_i : 1 \le i \le n\}$, the outer cycle .
In the next theorem, we show that with only three vertices, we can resolve all the vertices of the graph of convex polytope $\mathbb{B}_n$. Once gain, choice of appropriate landmarks is crucial.

Theorem 3. *Let the graph of convex polytopes be $\mathbb{B}_n$; then $dim(\mathbb{B}_n) = 3$ for every $n \ge 6$.*

Proof. We will prove the above equality by double inequalities. We consider the two cases.
Case(i) When n is even.
In this case, we can write $n = 2k$, $k \ge 3$, $k \in \mathbb{Z}^+$. Let $W = \{a_1, a_3, a_{k+1}\} \subset V(\mathbb{B}_n)$, we show that W is a resolving set for $\mathbb{B}_n$ in this case. For this we

give representations of any vertex of $V(\mathbb{B}_n) \setminus W$ with respect to W. Representations of the vertices on inner cycle are

$$r(a_i|W) = \begin{cases} (1,1,k-1), & i=2; \\ (i-1,i-3,k-i+1), & 4 \le i \le k; \\ (k-1,k-1,1), & i=k+2; \\ (2k-i+1,2k-i+3,i-k-1), & k+3 \le i \le 2k. \end{cases}$$

Representations of the vertices on interior cycle are

$$r(b_i|W) = \begin{cases} (1,2,k), & i=1; \\ (2,1,k-1), & i=2; \\ (i,i-2,k-i+1), & 3 \le i \le k; \\ (k,k-1,1), & i=k+1; \\ (k-1,k,2), & i=k+2; \\ (2k-i+1,2k-i+3,i-k), & k+3 \le i \le 2k. \end{cases}$$

Representations of the vertices on exterior cycle are

$$r(c_i|W) = \begin{cases} (2,2,k), & i=1; \\ (3,2,k-1), & i=2; \\ (i+1,i-1,k-i+1), & 3 \le i \le k-1; \\ (k+1,k-1,2), & i=k; \\ (k,k,2), & i=k+1; \\ (2k-i+1,2k-i+3,i-k+1), & k+2 \le i \le 2k-1; \\ (2,3,k+1), & i=2k. \end{cases}$$

Representations of vertices on outer cycle are

$$r(d_i|W) = \begin{cases} (3,3,k), & i=1; \\ (4,3,k-1), & i=2; \\ (i+2,i,k-i+1), & 3 \le i \le k-2; \\ (k+1,k-1,3), & i=k-1; \\ (k+1,k,3), & i=k; \\ (k,k+1,3), & i=k+1; \\ (2k-i+1,2k-i+3,i-k+2), & k+2 \le i \le 2k-2; \\ (3,4,k+1), & i=2k-1; \\ (3,3,k+1), & i=2k. \end{cases}$$

We see that there are no two vertices having the same representations implying that $dim(\mathbb{B}_n) \le 3$.
On the other hand, we show that $dim(\mathbb{B}_n) \ge 3$. Suppose on contrary that $dim(\mathbb{B}_n) = 2$, then there are following possibilities to be discussed.
(1) Both vertices are in the inner cycle. Without loss of generality, we can suppose that one resolving vertex is a_1. Suppose that the second resolving vertex is a_i $(2 \le i \le k+1)$. Then for $2 \le i \le k$, we have

$r(a_n|\{a_1, a_i\}) = r(b_n|\{a_1, a_i\})$ and for $i = k+1$, we have $r(a_2|\{a_1, a_{k+1}\}) = r(a_n|\{a_1, a_{k+1}\})$, a contradiction.
(2) Both vertices are in the interior cycle. Without loss of generality, we can suppose that one resolving vertex is b_1. Suppose that the second resolving vertex is b_i $(2 \leq i \leq k+1)$. Then for $2 \leq i \leq k$, we have $r(b_n|\{b_1, b_i\}) = r(c_n|\{b_1, b_i\})$ and for $i = k+1$, we have $r(b_2|\{b_1, b_{k+1}\}) = r(b_n|\{b_1, b_{k+1}\})$, a contradiction.
(3) Both vertices are in the exterior cycle. Due to the symmetry of the graph, this case is analogous to case **(2)**.
(4) Both vertices are in the outer cycle. This case is analogous to case **(1)**.
(5) One vertex is in the inner cycle and other in interior cycle. Without loss of generality, we can suppose that one resolving vertex is a_1 and the second resolving vertex is b_i $(1 \leq i \leq k+1)$. Then for $i = 1$, we have $r(a_2|\{a_1, b_1\}) = r(b_1|\{a_1, b_1\})$ and for $2 \leq i \leq k+1$, $r(a_2|\{a_1, b_i\}) = r(b_1|\{a_1, b_i\})$, a contradiction.
(6) One vertex is in the inner cycle and other in exterior cycle. Without loss of generality, we can suppose that one resolving vertex is a_1 and the second resolving vertex is c_i $(1 \leq i \leq k+1)$. Then for $i = 1$, we have $r(a_2|\{a_1, c_1\}) = r(b_2|\{a_1, c_1\})$ and for $2 \leq i \leq k$ $r(a_2|\{a_1, c_i\}) = r(b_1|\{a_1, c_i\})$. When $i = k+1$, $r(a_n|\{a_1, c_{k+1}\}) = r(b_n|\{a_1, c_{k+1}\})$, a contradiction.
(7) One vertex is in the inner cycle and other in the outer cycle. Again, we can suppose that one resolving vertex is a_1. Suppose that the second resolving vertex is d_i $(1 \leq i \leq k+1)$. Then for $i = 1$, $r(b_2|\{a_1, d_1\}) = r(c_n|\{a_1, d_1\})$ and for $2 \leq i \leq k+1$ we have $r(a_n|\{a_1, d_i\}) = r(b_1|\{a_1, d_i\})$, a contradiction.
(8) One vertex is in interior cycle and other in exterior cycle. Without loss of generality, we can suppose that one resolving vertex is b_1. Suppose that the second resolving vertex is c_i $(1 \leq i \leq k+1)$. Then for $i = 1$, $r(b_2|\{b_1, c_1\}) = r(c_n|\{b_1, c_1\})$ and for $2 \leq i \leq k+1$ we have $r(a_2|\{b_1, c_i\}) = r(b_1|\{b_1, c_i\})$, a contradiction.
(9) One vertex is in interior cycle and other is in outer cycle. This case is analogous to case **(5)** due to the symmetry of the graph.
(10) One vertex is in exterior cycle and other is in outer cycle. This case is symmetric to case **(6)**.
So from above, we conclude that there is no resolving set with two vertices for $V(\mathbb{B}_n)$ implying that $dim(\mathbb{B}_n) = 3$ in this case.
Case(ii) When n is odd.
In this case, we can write $n = 2k+1$, $k \geq 3$, $k \in \mathbf{Z}^+$. Let $W = \{a_1, a_3, a_{k+1}\} \subset V(\mathbb{B}_n)$, we show that W is a resolving set for $\mathbb{B}_n$ in this case. For this we give representations of any vertex of $V(\mathbb{B}_n) \setminus W$ with respect to W.

Representations of the vertices on inner cycle are

$$r(a_i|W) = \begin{cases} (1,1,k-1), & i=2; \\ (i-1,i-3,k-i+1), & 4 \le i \le k; \\ (k,k-1,1), & i=k+2; \\ (k-1,k,2), & i=k+3; \\ (2k-i+2,2k-i+4,i-k-1), & k+4 \le i \le 2k+1. \end{cases}$$

Representations of the vertices on interior cycle are

$$r(b_i|W) = \begin{cases} (1,2,k), & i=1; \\ (2,1,k-1), & i=2; \\ (i,i-2,k-i+1), & 3 \le i \le k; \\ (k+1,k-1,1), & i=k+1; \\ (k,k,2), & i=k+2; \\ (2k-i+2,2k-i+4,i-k), & k+3 \le i \le 2k+1. \end{cases}$$

Representations of the vertices on exterior cycle are

$$r(c_i|W) = \begin{cases} (2,2,k), & i=1; \\ (3,2,k-1), & i=2; \\ (i+1,i-1,k-i+1), & 3 \le i \le k-1; \\ (k+1,k-1,2), & i=k; \\ (k+1,k,2), & i=k+1; \\ (k,k+1,3), & i=k+2; \\ (2k-i+2,2k-i+4,i-k+1), & k+3 \le i \le 2k; \\ (2,3,k+1), & i=2k+1. \end{cases}$$

Representations of vertices on outer cycle are

$$r(d_i|W) = \begin{cases} (3,3,k), & i=1; \\ (4,3,k-1), & i=2; \\ (i+2,i,k-i+1), & 3 \le i \le k-2; \\ (k+1,k-1,3), & i=k-1; \\ (k+2,k,3), & i=k; \\ (k+1,k+1,3), & i=k+1; \\ (2k-i+2,2k-i+4,i-k+2), & k+2 \le i \le 2k-1; \\ (3,4,k+2), & i=2k: \\ (3,3,k+1), & i=2k+1. \end{cases}$$

Again we see that there are no two vertices having the same representations implying that $dim(\mathbb{B}_n) \le 3$.
On the other hand, we show that $dim(\mathbb{B}_n) \ge 3$. Suppose on contrary that $dim(\mathbb{B}_n) = 2$, then there are the same possibilities as in case (i) and contradiction can be obtained analogously. It follows that $dim(\mathbb{B}_n) = 3$, which completes the proof.

4 Conclusion

In this paper, we have studied the metric dimension of two classes of convex polytopes by giving answer to an open problem proposed in [8]. We show that the metric dimension of these two classes of convex polytopes is finite and does not depend upon the number of vertices in these graphs and only three vertices appropriately chosen suffice to resolve all the vertices of these classes of plane graphs. It is natural to ask for the characterizations of classes of convex polytopes with constant metric dimension.

Note that in [14] Melter and Tomescu gave an example of infinite regular graphs (namely the digital plane endowed with city-block and chessboard distances, respectively) having no finite metric basis. We close this section by raising a question that naturally arises form the text.

Open Problem: Let G be the graph of some convex polytope which is obtained from cartesian product of path $P_m (m \geq 3)$ with a cycle C_n and then adding (or deleting) some families of edges. Is it the case that G always will have constant metric dimension?

References

1. M. Bača, Labellings of two classes of convex polytopes, *Utilitas Math.* 34(1988), 24 – 31.
2. M. Bača, On magic labellings of convex polytopes, *Annals Disc. Math.* 51(1992), 13 – 16.
3. P. S. Buczkowski, G. Chartrand, C. Poisson, P. Zhang, On k-dimensional graphs and their bases, *Periodica Math. Hung.*, 46(1)(2003), 9 – 15.
4. J. Caceres, C. Hernando, M. Mora, I. M. Pelayo, M. L. Puertas, C. Seara, D. R. Wood, On the metric dimension of cartesian product of graphs, *SIAM J. Disc. Math.*, 2(21), (2007), 423 – 441.
5. J. Caceres, C. Hernando, M. Mora, I. M. Pelayo, M. L. Puertas, C. Seara, D. R. Wood, On the metric dimension of some families of graphs. *Electronic Notes in Disc. Math.*, 22(2005), 129 – 133.
6. G. Chartrand, L. Eroh, M. A. Johnson, O. R. Oellermann, Resolvability in graphs and metric dimension of a graph, *Disc. Appl. Math.*, 105(2000), 99 – 113.
7. M. R. Garey, D. S. Johnson, *Computers and Intractability: A Guide to the Theory of NP- Completeness*, Freeman, New York, 1979.
8. M. Imran, A. Q. Baig, A. Ahmad, Families of plane graphs with constant metric dimension, *to appear in Utilitas Math.*
9. M. Imran, A. Q. Baig, M. K. Shafiq, I. Tomescu, On metric dimension of generalized Petersen graphs $P(n, 3)$, *to appear in Ars Combin.*
10. I. Javaid, M. T. Rahim, K. Ali, Families of regular graphs with constant metric dimension, *Utilitas Math.*, 75(2008), 21 – 33.

11. M. A. Johnson, Structure-activity maps for visualizing the graph variables arising in drug design, *J. Biopharm. Statist.*, 3(1993), 203 – 236.
12. E. Jucovič, Convex polyhedra, Veda, Bratislava, 1981. (Slovak)
13. S. Khuller, B. Raghavachari, A. Rosenfeld, Landmarks in graphs, *Disc. Appl. Math.*, 70(1996), 217 – 229.
14. R. A. Melter, I. Tomescu, Metric bases in digital geometry, *Computer Vision, Graphics, and Image Processing,* 25(1984), 113 – 121.
15. P. J. Slater, Leaves of trees, *Congress. Numer.*, 14(1975), 549 – 559.
16. P. J. Slater, Dominating and reference sets in graphs, *J. Math. Phys. Sci.*, 22(1998), 445 – 455.
17. I. Tomescu, M. Imran, On metric and partition dimensions of some infinite regular graphs, *Bull. Math. Soc. Sci. Math. Roumanie,* 52(100), 4(2009), 461 – 472.
18. I. Tomescu, I. Javaid, On the metric dimension of the Jahangir graph, *Bull. Math. Soc. Sci. Math. Roumanie,* 50(98), 4(2007), 371 – 376.