A Four Colorings Theorem

Gary Chartrand
Department of Mathematics
Western Michigan University
Kalamazoo, MI 49008

Stephen T. Hedetniemi
Department of Computer Science
Clemson University
Clemson, SC 29634

Futaba Okamoto
Mathematics Department
University of Wisconsin - La Crosse
La Crosse, WI 54601

Ping Zhang
Department of Mathematics
Western Michigan University
Kalamazoo, MI 49008

ABSTRACT

A vertex u in a set S of vertices of a graph G has a private neighbor (with respect to S) if either (i) u has no neighbor in S or (ii) there exists some vertex $v \in V(G) - S$ that is a neighbor of u but not a neighbor of any other vertex in S. A set S is irredundant in a graph G if every vertex in S has a private neighbor. An irredundant k-coloring of a graph G is a partition of $V(G)$ into k irredundant sets. The minimum k for which G has an irredundant k-coloring is the irredundant chromatic number $\chi_{ir}(G)$ of G. A complete coloring of a graph G is a proper vertex coloring of G having the property that for every two distinct colors i and j used in the coloring, there exist adjacent vertices of G colored i and j. The maximum positive integer k for which G has a complete k-coloring is the achromatic number $\psi(G)$ of G. A Grundy coloring of a graph G is a proper vertex coloring of G having the property that for every two colors (positive integers) i and j with $i < j$, every vertex colored j has a neighbor colored i. The maximum positive integer k for which a graph G has a Grundy k-coloring is the Grundy number $\Gamma(G)$ of G. The chromatic number of a graph G is denoted by $\chi(G)$. For every graph G, these four coloring parameters satisfy the inequalities

$\chi_{ir}(G) \le \chi(G) \le \Gamma(G) \le \psi(G)$. It is shown that if a, b, c, and d are integers with $2 \le a \le b \le c \le d$, then there exists a nontrivial connected graph G with $\chi_{ir}(G) = a$, $\chi(G) = b$, $\Gamma(G) = c$, and $\psi(G) = d$ if and only if $d = 2$ or $c \ne 2$.

Key Words: irredundant coloring, proper coloring, Grundy coloring, complete coloring.
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1 Introduction

For a set S of vertices in a graph G, a vertex $u \in S$ has a *private neighbor* (with respect to S) if either (i) u has no neighbor in S or (ii) there exists some vertex $v \in V(G) - S$ that is a neighbor of u but not a neighbor of any other vertex in S. In the case of (i), u is called its *own private neighbor*; while in (ii), v is an *external private neighbor* of u. Therefore, a vertex $u \in S$ has a private neighbor (with respect to S) if either u is its own private neighbor or u has an external private neighbor. A set S is *irredundant* in a graph G if every vertex in S has a private neighbor with respect to S. An *irredundant k-coloring* of a graph G is a partition of $V(G)$ into k irredundant sets. The minimum k for which G has an irredundant k-coloring is the *irredundant chromatic number* $\chi_{ir}(G)$ of G. Obviously, if G is a nontrivial graph of order n, then $\chi_{ir}(G) = 1$ if and only if $G = \overline{K}_n$. This concept was introduced and studied in [4].

A (proper) *k-coloring* of a graph G is a function $c : V(G) \to \{1, 2, \ldots, k\}$ having the property that $c(u) \ne c(v)$ for every pair u, v of adjacent vertices of G. The *chromatic number* $\chi(G)$ of G is the minimum k for which G has a k-coloring. If S is an independent set in G, then every vertex of G is its own private neighbor and so S is an irredundant set as well. Hence every proper coloring of a graph is also an irredundant coloring. Therefore, $\chi_{ir}(G) \le \chi(G)$ for every graph G.

A *Grundy coloring* of a graph G is a proper vertex coloring of G having the property that for every two colors (positive integers) i and j with $i < j$, every vertex colored j has a neighbor colored i. The maximum k for which a graph G has a Grundy k-coloring is the *Grundy number* $\Gamma(G)$ of G. Thus

$$\chi_{ir}(G) \le \chi(G) \le \Gamma(G) \tag{1}$$

for every graph G. The Grundy number of a graph appears to have been introduced by Christen and Selkow in [1]. Since a vertex v assigned the color $\Gamma(G)$ in a Grundy coloring of G has a neighbor colored i for each positive integer i less than $\Gamma(G)$, it follows that $\Delta(G) \ge \deg v \ge \Gamma(G) - 1$ and so

$$\Gamma(G) \le \Delta(G) + 1. \tag{2}$$

A *complete coloring* of a graph G is a proper vertex coloring of G having the property that for every two distinct colors i and j used in the coloring, there exist adjacent vertices of G colored i and j. The maximum positive integer k for which G has a complete k-coloring is the *achromatic number* $\psi(G)$ of G. Thus $\Gamma(G) \leq \psi(G)$ and so

$$\chi_{ir}(G) \leq \chi(G) \leq \Gamma(G) \leq \psi(G). \tag{3}$$

Furthermore, the size of a graph G must be at least $\binom{\psi(G)}{2}$. Achromatic numbers were introduced by Harary, Hedetniemi, and Prins [6] and by Harary and Hedetniemi [5]. Among the results obtained on the achromatic number of a graph is the following by Geller and Kronk [3] concerning the deletion of a vertex from a graph.

Theorem 1.1 *For each vertex v in a nontrivial graph G,*

$$\psi(G) - 1 \leq \psi(G - v) \leq \psi(G).$$

As a consequence of Theorem 1.1, it follows that if H is an induced subgraph of a graph G, then $\psi(H) \leq \psi(G)$. In particular, if an end-vertex is deleted from the path P_n of order $n \geq 2$, then it follows that $\psi(P_n) - 1 \leq \psi(P_{n-1}) \leq \psi(P_n)$. Hell and Miller [7], in fact, established the following.

Theorem 1.2 *For $n \geq 2$, $\psi(P_n) = \max\left\{k : \left(\lfloor \frac{k}{2} \rfloor + 1\right)(k-2) + 2 \leq n\right\}$.*

In this work, we investigate the relationship among the irredundant chromatic number, the chromatic number, the Grundy number, and the achromatic number of a connected graph.

2 Graphs With Four Prescribed Coloring Numbers

We have seen that if G is a nontrivial connected graph with $\chi_{ir}(G) = a$, $\chi(G) = b$, $\Gamma(G) = c$, and $\psi(G) = d$, then $2 \leq a \leq b \leq c \leq d$ by (3). We now determine all 4-tuples (a, b, c, d) of integers, where $2 \leq a \leq b \leq c \leq d$, for which there is a connected graph G such that $\chi_{ir}(G) = a$, $\chi(G) = b$, $\Gamma(G) = c$, and $\psi(G) = d$. We begin with those of the form (a, b, c, c).

Theorem 2.1 *For every triple (a, b, c) of integers with $2 \leq a \leq b \leq c$, there exists a connected graph G for which $\chi_{ir}(G) = a$, $\chi(G) = b$, and $\Gamma(G) = \psi(G) = c$.*

Proof. We construct a graph $G_{a,b,c,c}$ such that

$$\chi_{ir}(G_{a,b,c,c}) = a,\ \chi(G_{a,b,c,c}) = b, \text{ and } \Gamma(G_{a,b,c,c}) = \psi(G_{a,b,c,c}) = c.$$

We consider two cases.

Case 1. $b = c$. Let the vertex set of K_b be $\{v_1, v_2, \ldots, v_b\}$. We construct $G_{a,b,b,b}$ by adding $b-a+1$ new vertices $w_1, w_2, \ldots, w_{b-a+1}$ to K_b and joining w_i to v_i for each i $(1 \le i \le b-a+1)$. Then $\chi(G_{a,b,b,b}) = b$.

To show that $\chi_{ir}(G_{a,b,b,b}) = a$, define a (nonproper) vertex coloring $f : V(G_{a,b,b,b}) \to \{1, 2, \ldots, a\}$ by

$$f(x) = \begin{cases} 1 & \text{if } x = v_i \text{ for } 1 \le i \le b-a+1 \\ a-b+i & \text{if } x = v_i \text{ for } b-a+2 \le i \le b \\ a & \text{if } x = w_i \text{ for } 1 \le i \le b-a+1. \end{cases} \tag{4}$$

Since f is an irredundant a-coloring of $G_{a,b,b,b}$, it follows that

$$\chi_{ir}(G_{a,b,b,b}) \le a.$$

Assume, to the contrary, that $\chi_{ir}(G_{a,b,b,b}) < a$. Then there exists an irredundant $(a-1)$-coloring of $G_{a,b,b,b}$. Since $G_{a,b,b,b}$ is nonempty, $\chi_{ir}(G_{a,b,b,b}) \ge 2$ and so $a \ge 3$. Let

$$V_1 = \{v_1, v_2, \ldots, v_{b-a+1}\} \text{ and } V_2 = \{v_{b-a+2}, v_{b-a+3}, \ldots, v_b\}.$$

Since $N(u) - \{v\} = N(v) - \{u\}$ for every two distinct vertices u and v in V_2, no two vertices in V_2 belong to the same irredundant set. That is, the $a-1$ vertices in V_2 must be colored differently. Since there are $a-1$ colors available, this implies that there exist a vertex in V_1 and a vertex in V_2 that are colored the same, say v_1 and v_b. Therefore, there exists an irredundant set containing both v_1 and v_b. However, since $a \ge 3$ and every neighbor of v_b that is different from v_1 is also a neighbor of v_1, it follows that v_b has no external private neighbor, which is a contradiction. Therefore, $\chi_{ir}(G_{a,b,b,b}) \ge a$ and so $\chi_{ir}(G_{a,b,b,b}) = a$.

It remains to show that

$$\Gamma(G_{a,b,b,b}) = \psi(G_{a,b,b,b}) = b.$$

Since $\chi(G_{a,b,b,b}) = b$, we need only show that $\psi(G_{a,b,b,b}) \le b$. Observe that the size of $G_{a,b,b,b}$ is

$$\binom{b}{2} + (b-a+1) = \binom{b+1}{2} - a < \binom{b+1}{2}.$$

Therefore, $\psi(G_{a,b,b,b}) \le b$ and so $\Gamma(G_{a,b,b,b}) = \psi(G_{a,b,b,b}) = b$.

Case 2. $c \ge b+1$. We first construct a graph $H_{a,b,c}$ with

$$V(H_{a,b,c}) = U_1 \cup U_2 \cup V \cup W \cup X,$$

where U_1, U_2, V, W, and X are pairwise disjoint and

$$U_1 = \{u_1, u_2, \ldots, u_{b-a+1}\}, \quad U_2 = \{u_{b-a+2}, u_{b-a+3}, \ldots, u_b\},$$
$$V = \{v_1, v_2, \ldots, v_{c-b}\}, \quad W = \{w_1, w_2, \ldots, w_{c-b}\},$$
$$X = \{x_1, x_2, \ldots, x_{b-a+1}\},$$

such that (i) the subgraph of $H_{a,b,c}$ induced by $U_1 \cup U_2$ is K_b; (ii) the sets V, W, and X are independent in $H_{a,b,c}$; (iii) each vertex in U_1 is adjacent to every vertex in V and each vertex in U_2 is adjacent to every vertex in W; (iv) v_i $(1 \leq i \leq c-b)$ is adjacent to w_j $(1 \leq j \leq c-b)$ if and only if $i \neq j$; and (v) each x_i is joined to u_i for $1 \leq i \leq b-a+1$. Thus the subgraph of $H_{a,b,c}$ induced by $V \cup W$ is isomorphic to the graph obtained by deleting the matching $\{v_iw_i : 1 \leq i \leq c-b\}$ from $K_{c-b,c-b}$. The graph $H_{3,5,8}$ is shown in Figure 1. We now consider the three subcases: (1) $a = 2$, (2) $a \geq 3$ and $c = b+1$, and (3) $a \geq 3$ and $c \geq b+2$.

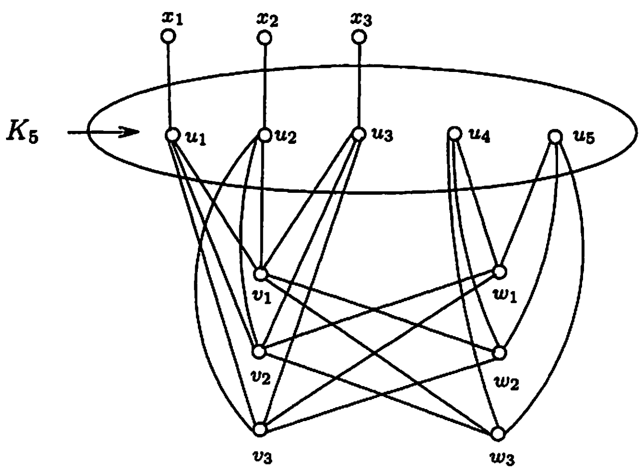

Figure 1: The graph $H_{3,5,8}$

Subcase 2.1. $a = 2$. Let $G_{2,b,c,c} = H_{2,b,c}$ and observe that $\chi(G_{2,b,c,c}) = b$. We first show that $\chi_{ir}(G_{2,b,c,c}) = 2$. Since $G_{2,b,c,c}$ is not empty, $\chi_{ir}(G_{2,b,c,c}) \geq 2$. To show that $\chi_{ir}(G_{2,b,c,c}) \leq 2$, observe that $U_2 = \{u_b\}$ and so the (nonproper) vertex coloring $f : V(G_{2,b,c,c}) \to \{1,2\}$ defined by

$$f(x) = \begin{cases} 1 & \text{if } x \in U_1 \cup W \\ 2 & \text{if } x \in U_2 \cup V \cup X \end{cases}$$

is an irredundant 2-coloring. Therefore, $\chi_{ir}(G_{2,b,c,c}) = 2$.

It remains to show that $\Gamma(G_{2,b,c,c}) = \psi(G_{2,b,c,c}) = c$. The (proper) vertex coloring $g : V(G_{2,b,c,c}) \to \{1, 2, \ldots, c\}$ given by

$$g(x) = \begin{cases} c+1-i & \text{if } x = u_i \text{ for } 1 \le i \le b \\ i & \text{if } x = v_i \text{ or } x = w_i \text{ for } 1 \le i \le c-b \\ 1 & \text{if } x \in X \end{cases}$$

is a Grundy c-coloring and so $\Gamma(G_{2,b,c,c}) \ge c$.

We claim that $\psi(G_{2,b,c,c}) = c$. Suppose that $\psi(G_{2,b,c,c}) \ge c+1$. Then there is a complete $(c+1)$-coloring h of $G_{2,b,c,c}$. Since the order of $G_{2,b,c,c}$ is $n = 2c-1 = 2(c+1)-3$, every complete $(c+1)$-coloring results in $c+1$ color classes at least three of which consist of a single vertex. Suppose that h produces the color classes $S_1, S_2, \ldots, S_{c+1}$, where $S_1, S_2, \ldots, S_\ell$ ($\ell \ge 3$) are singleton sets. For $1 \le i \le \ell$, suppose that $S_i = \{z_i\}$. Then z_i is adjacent to at least one vertex in each color class S_j with $j \ne i$, implying that $\deg z_i \ge c$. Thus, each z_i must belong to U_1 and so we may assume, without loss of generality, that $S_i = \{u_i\}$ for $1 \le i \le \ell$. Since each z_i ($1 \le i \le \ell$) must be adjacent to a vertex of each of the $c+1$ colors different from $h(z_i)$, it follows that the $c+1$ vertices in $U_1 \cup U_2 \cup V \cup \{x_i\}$ are colored differently for $1 \le i \le \ell$, implying that the vertices $x_1, x_2, \ldots, x_\ell$ are assigned the same color. Therefore, there exists a color class containing at least ℓ vertices. Since there are ℓ color classes containing exactly one vertex each, it follows that each of the remaining $c-\ell$ color classes contains at least two vertices. This implies that

$$n \ge \ell + \ell + 2(c-\ell) = 2c,$$

which is a contradiction. Therefore,

$$c \le \Gamma(G_{2,b,c,c}) \le \psi(G_{2,b,c,c}) \le c$$

and so $\Gamma(G_{2,b,c,c}) = \psi(G_{2,b,c,c}) = c$.

Subcase 2.2. *$a \ge 3$ and $c = b+1$.* Let $G_{a,b,b+1,b+1} = H_{a,b,b+1}$ and again $\chi(G_{a,b,b+1,b+1}) = b$. We first show that $\chi_{ir}(G_{a,b,b+1,b+1}) = a$. Observe that every irredundant coloring of $G_{a,b,b+1,b+1}$ must assign distinct colors to the vertices in the set $U_2 \cup W$; for otherwise, if two vertices in this set are colored the same, then at least one of these vertices has no private neighbor. This implies that that

$$\chi_{ir}(G_{a,b,b+1,b+1}) \ge |U_2 \cup W| = a.$$

On the other hand, the (nonproper) vertex coloring $f : V(G_{a,b,b+1,b+1}) \to \{1, 2, \ldots, a\}$ of $G_{a,b,b+1,b+1}$ defined by

$$f(x) = \begin{cases} 1 & \text{if } x \in U_1 \\ a - b - 1 + i & \text{if } x = u_i \in U_2 \ (b - a + 2 \leq i \leq b) \\ a & \text{if } x \in V \cup W \\ 2 & \text{if } x \in X \end{cases}$$

is an irredundant a-coloring and so

$$\chi_{ir}(G_{a,b,b+1,b+1}) \leq a.$$

Hence, $\chi_{ir}(G_{a,b,b+1,b+1}) = a$.

It remains to verify that

$$\Gamma(G_{a,b,b+1,b+1}) = \psi(G_{a,b,b+1,b+1}) = b + 1.$$

Since the (proper) vertex coloring $g : V(G_{a,b,b+1,b+1}) \to \{1, 2, \ldots, b+1\}$ of $G_{a,b,b+1,b+1}$ defined by

$$g(x) = \begin{cases} b + 2 - i & \text{if } x = u_i \text{ for } 1 \leq i \leq b \\ 1 & \text{if } x \in V \cup W \cup X \end{cases}$$

is a Grundy $(b+1)$-coloring, it follows that $\Gamma(G_{a,b,b+1,b+1}) \geq b+1$. On the other hand, the size m of $G_{a,b,b+1,b+1}$ is

$$m = \binom{b}{2} + b + (b - a + 1) = \binom{b+2}{2} - a < \binom{b+2}{2},$$

implying that $\psi(G_{a,b,b+1,b+1}) \leq b + 1$. Hence,

$$b + 1 \leq \Gamma(G_{a,b,b+1,b+1}) \leq \psi(G_{a,b,b+1,b+1}) \leq b + 1$$

and so $\Gamma(G_{a,b,b+1,b+1}) = \psi(G_{a,b,b+1,b+1}) = b + 1$.

Subcase 2.3. $a \geq 3$ and $c \geq b + 2$. Then $G_{a,b,c,c}$ is constructed from $H_{a,b,c}$ and K_a with $V(K_a) = Y = \{y_1, y_2, \ldots, y_a\}$ by identifying x_1 and y_1. Observe that $\chi(G_{a,b,c,c}) = b$. We first show that $\chi_{ir}(G_{a,b,c,c}) = a$. Every irredundant coloring of $G_{a,b,c,c}$ must assign distinct colors to the vertices in the set Y; for otherwise, if two vertices in Y are colored the same, then at least one of these vertices has no private neighbor. This implies that $\chi_{ir}(G_{a,b,c,c}) \geq |Y| = a$. Since the (nonproper) vertex coloring $f : V(G_{a,b,c,c}) \to \{1, 2, \ldots, a\}$ defined by

$$f(x) = \begin{cases} 1 & \text{if } x \in U_1 \cup W - \{u_1\} \\ 2 & \text{if } x = u_1 \\ a - b + i & \text{if } x = u_i \in U_2 \ (b - a + 2 \leq i \leq b) \\ a & \text{if } x \in V \cup X \\ i - 1 & \text{if } x = y_i \text{ for } 2 \leq i \leq a \end{cases}$$

is an irredundant a-coloring of $G_{a,b,c,c}$,

$$\chi_{ir}(G_{a,b,c,c}) \le a.$$

Thus, $\chi_{ir}(G_{a,b,c,c}) = a$.

Finally, we show that $\Gamma(G_{a,b,c,c}) = \psi(G_{a,b,c,c}) = c$. The (proper) vertex coloring $g : V(G_{a,b,c,c}) \to \{1, 2, \ldots, c\}$ of $G_{a,b,c,c}$ defined by

$$g(x) = \begin{cases} c+1-i & \text{if } x = u_i \text{ for } 1 \le i \le b \\ i & \text{if } x = v_i \text{ or } x = w_i \text{ for } 1 \le i \le c-b \\ 1 & \text{if } x \in X \\ i & \text{if } x = y_i \text{ for } 2 \le i \le a \end{cases}$$

is a Grundy c-coloring and so $\Gamma(G_{a,b,c,c}) \ge c$.

We claim that $\psi(G_{a,b,c,c}) = c$, for assume, to the contrary, that

$$\psi(G_{a,b,c,c}) \ge c+1.$$

Then there exists a complete $(c+1)$-coloring $h : V(G_{a,b,c,c}) \to \{1, 2, \ldots, c+1\}$ of $G_{a,b,c,c}$. Since the order of $G_{a,b,c,c}$ is $n = 2c = 2(c+1) - 2$, every complete $(c+1)$-coloring results in $c+1$ color classes at least two of which consist of a single vertex. Suppose that h partitions $V(G_{a,b,c,c})$ into the color classes $S_1, S_2, \ldots, S_{c+1}$, where $S_1, S_2, \ldots, S_\ell$ ($\ell \ge 2$) are singleton sets. For $1 \le i \le \ell$, let $S_i = \{z_i\}$. Then z_i is adjacent to at least one vertex in each color class S_j with $j \ne i$, implying that $\deg z_i \ge c$. Since there are $b - a + 1$ vertices of degree $c = \Delta(G_{a,b,c,c})$, namely those vertices in U_1, it follows that $\ell \le b - a + 1$ and, for each $1 \le i \le \ell$, there exists a unique integer j_i with $1 \le j_i \le b - a + 1$ such that $z_i = u_{j_i} \in U_1$. Since each u_{j_i} ($1 \le i \le \ell$) must be adjacent to a vertex of each of the $c+1$ colors different from $h(u_{j_i})$, it follows that the $c+1$ vertices in $U_1 \cup U_2 \cup V \cup \{x_{j_i}\}$ are colored differently for $1 \le i \le \ell$. This implies that the ℓ vertices $x_{j_1}, x_{j_2}, \ldots, x_{j_\ell}$ must be assigned the same color, say 1. Therefore, there exists a color class, say S_{c+1}, containing at least ℓ vertices. Because there are ℓ color classes containing a single vertex, each of the remaining $c - \ell$ color classes contains at least two vertices, implying that

$$n \ge \ell + \ell + 2(c - \ell) = 2c = n$$

and so $|S_{c+1}| = \ell$ and $S_{c+1} \subseteq X$. However, this implies that

$$\sum_{x \in S_{c+1}} \deg x \le a + \ell - 1 \le a + (b - a + 1) - 1 = b < c.$$

That is, there are fewer than c vertices that are adjacent to at least one vertex colored 1, which contradicts the fact that h is a complete coloring. Therefore, $\psi(G_{a,b,c,c}) \le c$. Then

$$c \le \Gamma(G_{a,b,c,c}) \le \psi(G_{a,b,c,c}) \le c$$

and so $\Gamma(G_{a,b,c,c}) = \psi(G_{a,b,c,c}) = c$. ■

We now consider all 4-tuples (a, b, c, d) of integers with $2 \le a \le b \le c \le d$. First, we state two useful lemmas, the first of which appeared in [2], while the second one appeared in [1].

Lemma 2.2 *If G is a connected graph with $\Gamma(G) = 2$, then $\psi(G) = 2$.*

Lemma 2.3 *For a graph G and an integer k with $\chi(G) \le k \le \Gamma(G)$, there is a Grundy k-coloring of G.*

Theorem 2.4 *Let a, b, c, and d be integers with $2 \le a \le b \le c \le d$. Then there exists a connected graph G with $\chi_{ir}(G) = a$, $\chi(G) = b$, $\Gamma(G) = c$, and $\psi(G) = d$ if and only if $d = 2$ or $c \ne 2$.*

Proof. First, let G be a connected graph such that $\chi_{ir}(G) = a$, $\chi(G) = b$, $\Gamma(G) = c$, and $\psi(G) = d$. Thus $c = 2$ or $c \ne 2$. If $c = 2$, then $d = 2$ by Lemma 2.2.

For the converse, let a, b, c, and d be integers with $2 \le a \le b \le c \le d$. If $c = d$, then the graph $G_{a,b,c,c}$ constructed in the proof of Theorem 2.1 satisfies the condition. Thus we may assume that $3 \le c < d$. We consider two cases, according to whether $b = c$ or $b \ne c$.

Case 1. $b = c$. We consider two subcases, according to whether $a = 2$ or $a \ge 3$.

Subcase 1.1. $a = 2$. In the graph $G_{2,b,b,b}$ of Theorem 2.1, $\deg v_1 = b$. By Theorem 1.2, there is a positive integer k such that $\psi(P_k) = d$. Let F be the graph obtained by joining v_1 of $G_{2,b,b,b}$ and an end-vertex of P_k. Thus $\psi(F) \ge d$. Since $\psi(G_{2,b,b,b}) = b \le d$, it follows by Theorem 1.1 that there exists an integer $k' \le k$ such that joining v_1 of $G_{2,b,b,b}$ and an end-vertex of $P_{k'}$ results in a graph G with $\psi(G) = d$. Suppose that G is obtained from $G_{2,b,b,b}$ and the path $P_{k'} : z_1, z_2, \ldots, z_{k'}$ by joining v_1 and z_1. Then $\chi(G) = b$.

We first verify that $\chi_{ir}(G) = 2$. Since G is nonempty, $\chi_{ir}(G) \ge 2$. On the other hand, the (nonproper) vertex coloring $f^* : V(G) \to \{1, 2\}$ of G defined by

$$f^*(x) = \begin{cases} f(x) & \text{if } x \in V(G_{2,b,b,b}) \\ 1 & x = z_i \text{ for } 2 \le i \le k' \text{ and } i \text{ is even} \\ 2 & x = z_i \text{ for } 1 \le i \le k' \text{ and } i \text{ is odd,} \end{cases}$$

where f is the irredundant 2-coloring of $G_{2,b,b,b}$ defined by (4) in the proof of Theorem 2.1, is an irredundant 2-coloring of G. Therefore, $\chi_{ir}(G) = 2$ as claimed.

We now show that $\Gamma(G) = b$. Since $\chi(G) = b$ and $\Delta(G) = b+1$, it follows by (1) and (2) that $b \leq \Gamma(G) \leq b+2$. Suppose that $\Gamma(G) \geq b+1$. By Lemma 2.3, there is a Grundy $(b+1)$-coloring g^* of G. Then $g^*(v_{i_1}) = b+1$ and $g^*(v_{i_2}) = b$ for some i_1 and i_2 with $1 \leq i_1, i_2 \leq b-1$. Since $b \geq 3$, it follows that $g^*(w_{i_1}) = g^*(w_{i_2}) = 1$. If $2 \leq i_1 \leq b-1$, say $i_1 = 2$, then each of the vertices in $N(v_2) - \{w_2\}$ must be colored uniquely with the colors $2, 3, \ldots, b$ by g^*. This implies that no vertex in $\{v_1, v_2, \ldots, v_b\}$ is assigned the color 1. However, this is a contradiction since $g^*(v_b) \neq 1$ and v_b is adjacent to no vertex colored 1 while g^* is a Grundy coloring. Therefore, $g^*(v_1) = b+1$ and assume, without loss of generality, that $g^*(v_2) = b$. Then since $g^*(w_2) = 1$, each of the vertices $v_3, v_4, \ldots, v_b$ must be colored uniquely with the colors $2, 3, \ldots, b-1$ and so again no vertex in $\{v_1, v_2, \ldots, v_b\}$ is assigned the color 1. This is impossible since $g^*(v_b) \neq 1$ and v_b is adjacent to no vertex colored 1.

Subcase 1.2. $a \geq 3$. In the graph $G_{a,b,b,b}$ of Theorem 2.1, $\deg w_1 = 1$. As described in Subcase 1.1, since $\psi(G_{a,b,b,b}) = b \leq d$, it follows by Theorems 1.1 and 1.2 that there exists an integer k' such that joining w_1 of $G_{a,b,b,b}$ and an end-vertex of the path $P_{k'}$ of order k' results in a graph G with $\psi(G) = d$. Suppose that G is obtained from $G_{a,b,b,b}$ and the path $P_{k'} : z_1, z_2, \ldots, z_{k'}$ by joining w_1 and z_1. Then $\chi(G) = b$.

We first verify that $\chi_{ir}(G) = a$. As discussed in the proof of Theorem 2.1, every irredundant coloring assigns distinct colors to the vertices in $V_2 \cup \{v_i\}$ for each i with $1 \leq i \leq b-a+1$; for otherwise, if two vertices of $V_2 \cup \{v_i\}$ are colored the same, then at least one of these vertices has no private neighbor. This implies that $\chi_{ir}(G) \geq |V_2| + 1 = a$. Observe that the (nonproper) vertex coloring $f^* : V(G) \to \{1, 2, \ldots, a\}$ of G defined by

$$f^*(x) = \begin{cases} f(x) & \text{if } x \in V(G_{a,b,b,b}) \\ 1 & \text{if } x = z_i \text{ for } 2 \leq i \leq k' \text{ and } i \text{ is even} \\ 2 & \text{if } x = z_i \text{ for } 1 \leq i \leq k' \text{ and } i \text{ is odd} \end{cases}$$

is an irredundant a-coloring of G, where f is the irredundant a-coloring of $G_{a,b,b,b}$ defined by (4) in the proof of Theorem 2.1. Therefore, $\chi_{ir}(G) = a$.

It remains to show that $\Gamma(G) = b$. Since $\chi(G) = \Delta(G) = b$, it follows by (1) and (2) that $\Gamma(G) \in \{b, b+1\}$. Assume, to the contrary, that $\Gamma(G) = b+1$ and let there be given a Grundy $(b+1)$-coloring g^* of G. Since $\Delta(G) = b$, each vertex that is assigned the color $b+1$ must be of degree b. Thus $g^*(v_i) = b+1$ for some i with $1 \leq i \leq b-a+1$.

If $g^*(w_i) = 1$, then each of the $b-1$ vertices in $\{v_1, v_2, \ldots, v_b\} - \{v_i\}$ must be colored differently with the colors $2, 3, \ldots, b$. However, this implies that $g^*(v_b) \neq 1$ and v_b is adjacent to no vertex colored 1, contradicting the fact that g^* is a Grundy coloring. Hence $g^*(w_i) \neq 1$. If $2 \leq i \leq b-a+1$, then this implies that $g^*(w_i) = 2$ and $g^*(v_i) = 1 < b+1$, which is a contradiction. Therefore, $i = 1$ and in particular, $g^*(w_1) = 2$. Then the vertices in $\{v_2, v_3, \ldots, v_b\}$ must be colored differently with the colors $1, 3, 4, \ldots, b$; that is, no vertex in $\{v_1, v_2, \ldots, v_b\}$ is colored 2. Observe that at least one of the vertices v_{b-1} and v_b must be assigned the color greater than 2, say $g^*(v_b) \geq 3$. However then, v_b is adjacent to no vertex colored 2 by g^*, which is a contradiction.

Case 2. $b \neq c$. In the graph $G_{a,b,c,c}$ of Theorem 2.1, $\deg v_1 = c - a \leq c - 2$. By an argument similar to that given in Case 1, there is a path $P_{k'} : z_1, z_2, \ldots, z_{k'}$ of order k' for some positive integer k' such that the graph G obtained from $G_{a,b,c,c}$ and $P_{k'}$ by joining v_1 and z_1 has $\psi(G) = d$. Again, $\chi(G) = b$.

We first verify that $\chi_{ir}(G) = a$. Since G is nonempty, $\chi_{ir}(G) \geq 2$. In addition, as discussed in the proof of Theorem 2.1, if $c = b + 1$, then every irredundant coloring assigns distinct colors to the vertices in $U_2 \cup W$; while if $c \geq b + 2$, then every irredundant coloring assigns distinct colors to the vertices in Y. Therefore, $\chi_{ir}(G) \geq a$. On the other hand, observe that the (nonproper) vertex coloring $f^* : V(G) \to \{1, 2, \ldots, a\}$ of G defined by

$$f^*(x) = \begin{cases} f(x) & \text{if } x \in V(G_{a,b,c,c}) \\ 1 & \text{if } x = z_i \text{ for } 1 \leq i \leq k' \text{ and } i \text{ is odd} \\ 2 & \text{if } x = z_i \text{ for } 2 \leq i \leq k' \text{ and } i \text{ is even} \end{cases}$$

is an irredundant a-coloring, where f is the irredundant a-coloring of $G_{a,b,c,c}$ defined in the proof of Theorem 2.1. Therefore, $\chi_{ir}(G) = a$.

It remains to show that $\Gamma(G) = c$. Observe that the (proper) vertex coloring $g^* : V(G) \to \{1, 2, \ldots, c\}$ of G defined by

$$g^*(x) = \begin{cases} g(x) & \text{if } x \in V(G_{a,b,c,c}) \\ 1 & \text{if } x = z_i \text{ for } 2 \leq i \leq k' \text{ and } i \text{ is even} \\ 2 & \text{if } x = z_i \text{ for } 1 \leq i \leq k' \text{ and } i \text{ is odd} \end{cases}$$

is a Grundy c-coloring, where g is the Grundy c-coloring of $G_{a,b,c,c}$ defined in the proof of Theorem 2.1. Therefore, $\Gamma(G) \geq c$. Since $\Delta(G) = c$, it follows by (2) that $\Gamma(G) \in \{c, c+1\}$. Now assume, to the contrary, that there is a Grundy $(c+1)$-coloring $g^{**} : V(G) \to \{1, 2, \ldots, c+1\}$. Then there exists a vertex in U_1 that is assigned the color $c+1$. We consider three subcases.

Subcase 2.1. $c = b+1$. Then we may assume, without loss of generality, that $g^{**}(u_1) = c+1$. Since $g^{**}(x_1) = 1$, each vertex in $[U_1 \cup U_2 \cup V] - \{u_1\}$ is colored uniquely with one of the $c-1$ colors $2, 3, \ldots, c$. Since $g^{**}(v_1) \neq 1$ and no vertex in $U_1 \cup U_2$ is colored 1, it follows that $g^{**}(z_1) = 1$. Furthermore, since $g^{**}(u_b) \neq 1$ and no vertex in $U_1 \cup U_2 - \{u_b\}$ is colored 1, it follows that $g^{**}(w_1) = 1$. If $g^{**}(v_1) < g^{**}(u_b)$, then no neighbor of u_b is colored $g^{**}(v_1)$, contradicting the fact that g^{**} is a Grundy coloring. Similarly, if $g^{**}(u_b) < g^{**}(v_1)$, then no neighbor of v_1 is colored $g^{**}(u_b)$, which is again impossible.

Subcase 2.2. $a = 2$ and $c \geq b + 2$. Then we may assume, without loss of generality, that $g^{**}(u_1) = c + 1$. Since $g^{**}(x_1) = 1$, each vertex in $[U_1 \cup U_2 \cup V] - \{u_1\}$ is colored uniquely with one of the $c-1$ colors $2, 3, \ldots, c$. Thus, every vertex in W is colored 1. If $g^{**}(v_2) < g^{**}(u_b)$, then there exists no neighbor of u_b that is colored $g^{**}(v_2)$; while if $g^{**}(u_b) < g^{**}(v_2)$, then v_2 is adjacent to no vertex colored $g^{**}(u_b)$. In either case, we obtain a contradiction.

Subcase 2.3. $a \geq 3$ *and* $c \geq b + 2$. Then $g^{**}(u_i) = c + 1$ for some i with $1 \leq i \leq b - a + 1$. If $2 \leq i \leq b - a + 1$, then $g^{**}(x_i) = 1$ and consequently, every vertex in W is colored 1. Then applying an argument similar to that given in Subcase 2.2 to the vertices u_b and v_2, a contradiction is produced. Therefore, $g^{**}(u_1) = c + 1$.

We claim that $g^{**}(x_1) \neq 1$ and $g^{**}(x_i) = 1$ for $2 \leq i \leq b - a + 1$. If $g^{**}(x_1) = 1$, then no vertex in $U_2 \cup V$ is colored 1. Since no vertex in W is adjacent to a vertex colored 1, every vertex in W must be colored 1. Considering the vertices u_b and v_2, we see (as in the argument given in Subcase 2.2) that a contradiction is produced. Therefore, as claimed, $g^{**}(x_1) \neq 1$. Next, suppose that $g^{**}(x_i) \neq 1$ for some i with $2 \leq i \leq b-a+1$, say $g^{**}(x_2) \neq 1$. Then $g^{**}(x_2) = 2$ and $g^{**}(u_2) = 1$. This again implies that no vertex in $U_2 \cup V$ is colored 1 and so every vertex in W must be colored 1, which is impossible. Therefore, as claimed, $g^{**}(x_1) \neq 1$ and $g^{**}(x_i) = 1$ for $2 \leq i \leq b - a + 1$.

Since $g^{**}(x_i) = 1$ for $2 \leq i \leq b - a + 1$, it follows that $g^{**}(u) \neq 1$ for every $u \in U_1$. Since some vertex in $U_1 \cup U_2 \cup V$ is colored 1, there exists a vertex in $U_2 \cup V$ that is colored 1. For each $u \in (U_1 \cup U_2) - \{u_1\}$ such that $g^{**}(u) \neq 1$, let S_u be the set of vertices that are adjacent to u in G and that are not colored $c + 1$ by g^{**}. Then

$$|S_u| \leq (b - 2) + (c - b) = c - 2$$

and so $g^{**}(u) \leq c - 1$. Thus $g^{**}(u) \leq c - 1$ for all $u \in (U_1 \cup U_2) - \{u_1\}$. For each $v \in V$, let S_v be the set of vertices that are adjacent to v in G and that are not colored $c + 1$ by g^{**}. Then

$$|S_v| \leq (b - a) + (c - b - 1) + 1 = c - a \leq c - 3$$

and so $g^{**}(v) \leq c - 2$. Furthermore, $g^{**}(x_1) \leq a \leq c - 2$. Therefore,

$$g^{**}(x) \leq \begin{cases} c - 1 & \text{if } x \in (U_1 \cup U_2) - \{u_1\} \\ c - 2 & \text{if } x \in V \cup \{x_1\}. \end{cases}$$

That is, there is no vertex that is adjacent to u_1 and colored c, which is a contradiction.

Therefore, $\Gamma(G) \leq c$ and so $\Gamma(G) = c$. ∎

References

[1] C. A. Christen and S. M. Selkow, Some perfect coloring properties of graphs. *J. Combin. Theory Ser. B* **27** (1979) 49-59.

[2] G. Chartrand, F. Okamoto, Z. Tuza, and P. Zhang, A note on graphs with prescribed complete coloring numbers. *J. Combin. Math. Combin. Comput.* **73** (2010) 77-84.

[3] D. P. Geller and H. Kronk, Further results on the achromatic number. *Fund. Math.* **85** (1974) 285-290.

[4] T. W. Haynes, S. M. Hedetniemi, S. T. Hedetniemi, A. A. McRae, and P. J. Slater, Irredundant colorings of graphs. *Bull. Inst. Combin. Appl.* **54** (2008) 103-121.

[5] F. Harary and S. T. Hedetniemi, The achromatic number of a graph. *J. Combin. Theory* **8** (1970) 154-161.

[6] F. Harary, S. T. Hedetniemi, and G. Prins, An interpolation theorem for graphical homomorphisms. *Portugal. Math.* **26** (1967) 453-462.

[7] P. Hell and D. J. Miller, Graphs with given achromatic number. *Discrete Math.* **16** (1976) 195-207.