

GEODETIC DOMINATION IN GRAPHS

H. Escudro¹, R. Gera², A. Hansberg³, N. Jafari Rad⁴,
and L. Volkmann³

¹Department of Mathematics, Juniata College
Huntingdon, PA 16652; escudro@juniata.edu

²Department of Applied Mathematics, Naval Postgraduate School,
Monterey, CA 93943; rgera@nps.edu

³Lehrstuhl II für Mathematik, RWTH Aachen University,
52056 Aachen, Germany; hansberg, volkm@math2.rwth-aachen.de

⁴Department of Mathematics, Shahrood University of Technology
Shahrood, Iran; n.jafarirad@shahroodut.ac.ir

Abstract

A subset S of vertices in a graph G is called a *geodetic dominating set* if S is both a geodetic set and a (standard) dominating set. In this paper, we study geodetic domination on graphs.

Keywords: Domination; geodesic; geodetic

2000 Mathematical subject classification: 05C12, 05C69

1 Introduction

We consider finite graphs without loops and multiple edges. For any graph G the set of vertices is denoted by $V(G)$ and the edge set by $E(G)$. We define the *order* of G by $n = n(G) = |V(G)|$ and the *size* by $m = m(G) = |E(G)|$. For a vertex $v \in V(G)$, the *open neighborhood* $N(v)$ is the set of all vertices adjacent to v , and $N[v] = N(v) \cup \{v\}$ is the *closed neighborhood* of v . The *degree* $d(v)$ of a vertex v is defined by $d(v) = |N(v)|$. The *minimum* and *maximum degree* of a graph G are denoted by $\delta = \delta(G)$ and $\Delta = \Delta(G)$, respectively. For $X \subseteq V(G)$ let $G[X]$ the subgraph of

G induced by X , $N(X) = \bigcup_{x \in X} N(x)$ and $N[X] = \bigcup_{x \in X} N[x]$. If G is a connected graph, then the *distance* $d(x, y)$ is the length of a shortest $x - y$ path in G . The *diameter* $\text{diam}(G)$ of a connected graph is defined by $\text{diam}(G) = \max_{x, y \in V(G)} d(x, y)$. An $x - y$ path of length $d(x, y)$ is called an $x - y$ *geodesic*. A vertex v is said to *lie on* an $x - y$ geodesic P if v is an internal vertex of P . The closed interval $I[x, y]$ consists of x, y and all vertices lying on some $x - y$ geodesic of G , while for $S \subseteq V(G)$,

$$I[S] = \bigcup_{x, y \in S} I[x, y].$$

If G is a connected graph, then a set S of vertices is a *geodetic set* if $I[S] = V(G)$. The minimum cardinality of a geodetic set is the *geodetic number* of G , and is denoted by $g(G)$. The geodetic number of a disconnected graph is the sum of the geodetic numbers of its components. A geodetic set of cardinality $g(G)$ is called a *$g(G)$ -set*.

A vertex of G is *simplicial* if the subgraph induced by its neighborhood is complete. It is easily seen that every simplicial vertex belongs to every geodetic set. For references on geodetic sets see [1, 3, 4, 5, 10].

A vertex in a graph G *dominates* itself and its neighbors. A set of vertices S in a graph G is a *dominating set* if each vertex of G is dominated by some vertex of S . The *domination number* $\gamma(G)$ of G is the minimum cardinality of a dominating set of G . For references on domination parameters in graphs see [9].

If $e = \{u, v\}$ is an edge of a graph G with $d(u) = 1$ and $d(v) > 1$, then we call e a *pendant edge*, u a *leaf* and v a *support vertex*. Let $L(G)$ be the set of all leaves of a graph G . We denote by P_n , C_n , and $K_{r,s}$ the path on n vertices, the cycle on n vertices, and the complete bipartite graph in which one partite set has r vertices and the other partite set has s vertices, respectively. The *corona* $\text{cor}(G)$ of a graph G is constructed from G , where for each vertex $v \in V(G)$, a new vertex v' and a pendant edge vv' are added.

It is easily seen that a dominating set is not in general a geodetic set in a graph G . Also the converse is not valid in general. This has motivated us to study the new domination conception of geodetic domination. We investigate those subsets of vertices of a graph that are both a geodetic set and a dominating set. We call these sets *geodetic dominating sets*. We call

the minimum cardinality of a geodetic dominating set of G , the *geodetic domination number* of G .

In section 2 we give some general results and sharp bounds for the geodetic domination number. In section 3 we focus on trees, by relating the new parameter to standard parameters in graph theory. In section 4 we present realization results on the geodetic domination number. In section 5 we study the effect on the geodetic domination number of a given graph by the removal of a vertex or an edge.

2 Geodetic Domination

In this section, we look closely at the concept of geodetic domination in a graph G , and obtain the geodetic domination number of some families of graphs. Further, we look at some relationships between the geodetic domination number and other parameters.

We call a set of vertices S in a graph G a *geodetic dominating set* if S is both a geodetic set and a dominating set. The minimum cardinality of a geodetic dominating set of G is its *geodetic domination number*, and is denoted by $\gamma_g(G)$. Since $V(G)$ is a geodetic dominating set for any graph G , the geodetic domination number of a graph is always defined. A geodetic dominating set of size $\gamma_g(G)$ is said to be a $\gamma_g(G)$ -set.

For example, if $G = K_{1,n-1}$ where $n \geq 3$ and v is the support vertex in G , then the set $\{v\}$ is a dominating set. However, $\{v\}$ is not a geodetic set of G . On the other hand, $S = V(G) \setminus \{v\}$ is a geodetic set of G . In fact, any geodetic set of G must contain every vertex in S , and hence S is a minimum geodetic set. Since S is also a dominating set, we deduce that S is a minimum geodetic dominating set of G and so $\gamma_g(G) = g(G) = \gamma_g(K_{1,n-1}) = n-1$. Chartand, Harary and Zhang [5] showed that $g(K_{r,s}) = \min\{r, s, 4\}$ for $r, s \geq 2$, and thus we obtain

$$\gamma_g(K_{r,s}) = g(K_{r,s}) = \min\{r, s, 4\}$$

for $r, s \geq 2$. The following bounds are immediate by the definitions.

Observation 2.1. *If G is a connected graph of order $n \geq 2$, then*

$$2 \leq \max\{g(G), \gamma(G)\} \leq \gamma_g(G) \leq n.$$

First we characterize all connected graphs of order $n \geq 2$ whose geodetic domination number is 2, n and $n - 1$.

Theorem 2.2. *Let G be a connected graph of order $n \geq 2$. Then:*

- (a) $\gamma_g(G) = 2$ if and only if there exists a geodetic set $S = \{u, v\}$ of G such that $d(u, v) \leq 3$,
- (b) $\gamma_g(G) = n$ if and only if G is the complete graph on n vertices.
- (c) $\gamma_g(G) = n - 1$ if and only if there is a vertex v in G such that v is adjacent to every other vertex of G and $G - v$ is the union of at least two complete graphs.

Proof. Let G be a connected graph of order $n \geq 2$.

(a) This part can be easily verified.

(b) Note that the result holds for $n = 2$. We now consider the case where $n \geq 3$. Assume first that $\gamma_g(G) = n$ and suppose to the contrary that there are two non-adjacent vertices x, y in G . Let P be an $x - y$ geodesic, and let v be a vertex on P which is adjacent to x . Then $V(G) \setminus \{v\}$ is a geodetic dominating set of G , contradicting the fact that $\gamma_g(G) = n$. Hence G is a complete graph. On the other hand, if $G = K_n$, then $\gamma_g(G) = n$.

(c) Let G be a graph with $\gamma_g(G) = n - 1$, and let S be a $\gamma_g(G)$ -set such that $V(G) \setminus S = \{v\}$. Let H be a component of $G - v$, and suppose that H contains two non-adjacent neighbors u and w of v . Let $x_1 x_2 \dots x_t$ be a shortest $u - w$ path in H with $x_1 = u$ and $x_t = w$. Then $t \geq 3$, and we obtain the contradiction that $V(G) \setminus \{v, x_2\}$ is a geodetic dominating set. Thus $N(v) \cap V(H)$ induces a complete graph. If $G - v$ consists of only one component, then v is a simplicial vertex, again a contradiction.

Hence $G - v$ is the disjoint union of $p \geq 2$ graphs H_1, H_2, \dots, H_p . We now show that v is adjacent to every other vertex in G . Suppose to the contrary that v is not adjacent to some vertex in H_i , say in H_1 . This implies that there is a $v - u$ path vwu with $w, u \in V(H_1)$ such that $uv \notin E(G)$. Since G is connected, H_2 contains a neighbor y of v . Now $d(u, y) = 3$, and we arrive at the contradiction that $V(G) \setminus \{v, w\}$ is a geodetic dominating set.

Obviously, if G has a vertex v such that $d(v) = n - 1$ and $G - v$ is the union of (at least two) complete graphs, then $\gamma_g(G) = n - 1$. \square

Theorem 2.2 (b), (c) and the inequality $g(G) \leq \gamma_g(G)$ imply the next well-known result.

Corollary 2.3. (Buckley, Harary, Quintas [1] 1988) *Let G be a connected graph of order $n \geq 2$. Then $g(G) = n - 1$ if and only if there is a vertex v in G such that v is adjacent to every other vertex of G and $G - v$ is the union of at least two complete graphs.*

Lemma 2.4. *If G is a connected graph with $\gamma(G) = 1$, then $\gamma_g(G) = g(G)$.*

Proof. If $G = K_n$, then $\gamma(G) = 1$ and $\gamma_g(G) = n = g(G)$. So we only have to consider the case $G \neq K_n$. Since $\gamma(G) = 1$, it follows that $\Delta(G) = n - 1$ and $\text{diam}(G) \leq 2$. The assumption $G \neq K_n$ shows that G has at least two non-adjacent vertices and so $\text{diam}(G) = 2$. Let S be a minimum geodetic set of G , and let $x \notin S$ (such a vertex exists as $G \neq K_n$). Since S is a geodetic set, there exist vertices $x_1, x_2 \in S$ such that x belongs to an $x_1 - x_2$ geodesic. But $\text{diam}(G) = 2$ implies that the $x_1 - x_2$ geodesic containing x must be the path $x_1 x x_2$. Thus x_1 dominates x , and so S is a dominating set of G . It follows that S is a geodetic dominating set of G . Hence $\gamma_g(G) \leq |S| = g(G)$ and $g(G) \leq \gamma_g(G)$ leads to $\gamma_g(G) = g(G)$ as desired. \square

Next we present two sharp upper bounds of the geodetic domination number in terms of diameter and girth.

Proposition 2.5. *If G is a connected graph of order $n \geq 2$, then*

$$\gamma_g(G) \leq n - \left\lfloor \frac{2\text{diam}(G)}{3} \right\rfloor. \quad (1)$$

Proof. Define $\text{diam}(G) = d = 3t + r$ with integers r, t such that $0 \leq r \leq 2$, and select two vertices u_0 and u_d in G such that $d(u_0, u_d) = d$. Let $P = u_0 u_1 \dots u_d$ be a shortest path from u_0 to u_d , and let $A = \{u_0, u_3, \dots, u_{3t}, u_{3t+r}\}$. It is a simple matter to verify that $D = V(G) \setminus (V(P) \setminus A)$ is a geodetic dominating set of G . If we note that $|A| = t + 1$ when $r = 0$ and $|A| = t + 2$ when $1 \leq r \leq 2$, then we find that

$$|V(P) \setminus A| = \left\lfloor \frac{6t + 2r}{3} \right\rfloor = \left\lfloor \frac{2\text{diam}(G)}{3} \right\rfloor,$$

and this leads to the desired bound (1). \square

If P_n is the path of order n , then

$$\gamma_g(P_n) = \left\lfloor \frac{n+2}{3} \right\rfloor = n - \left\lfloor \frac{2(n-1)}{3} \right\rfloor = n - \left\lfloor \frac{2\text{diam}(P_n)}{3} \right\rfloor.$$

This shows that we have equality in inequality (1) if G is the path of order n and consequently, the bound (1) is sharp.

Proposition 2.6. *If G is a connected graph of girth $c(G) \geq 6$, then*

$$\gamma_g(G) \leq n - \left\lfloor \frac{2c(G)}{3} \right\rfloor. \quad (2)$$

Proof. Let $c = c(G) = 3t + r$ with integers r, t such that $0 \leq r \leq 2$, and let $C = u_1 u_2 \dots u_c u_1$ be an induced cycle of length c . In addition, let $A = \{u_1, u_4, \dots, u_{3t-2}\}$ when $r = 0$ and $A = \{u_1, u_4, \dots, u_{3t-2}, u_{3t+1}\}$ when $1 \leq r \leq 2$. Then $D = V(G) \setminus (V(C) \setminus A)$ is a geodetic dominating set of G . If we note that $|A| = t$ when $r = 0$ and $|A| = t + 1$ when $1 \leq r \leq 2$, then we find that

$$|V(C) \setminus A| = \left\lfloor \frac{6t + 2r}{3} \right\rfloor = \left\lfloor \frac{2c(G)}{3} \right\rfloor,$$

and this yields the desired bound (2). \square

If C_n is the cycle of order $n \geq 6$, then

$$\gamma_g(C_n) = \left\lfloor \frac{n}{3} \right\rfloor = n - \left\lfloor \frac{2n}{3} \right\rfloor = n - \left\lfloor \frac{2c(C_n)}{3} \right\rfloor.$$

This shows that we have equality in (2) if G is the cycle of order $n \geq 6$, and thus (2) is also sharp.

Notice that Proposition 2.6 remains true if $c(G) = 4$. However, since $\gamma_g(C_5) = 3$, we only arrive to the bound $\gamma_g(G) \leq n - 2$ if $c(G) = 5$.

Finally, we give upper bounds of the geodetic domination number for triangle-free graphs.

Proposition 2.7. *Let G be a triangle-free graph with minimum degree $\delta \geq 2$. If M is a maximal matching of G , then $\gamma_g(G) \leq 2|M|$.*

Proof. Let S be the set of all vertices incident with an edge of M . The maximality of M shows that $V(G) \setminus S$ is independent. Because of $\delta \geq 2$, each vertex $v \in V(G) \setminus S$ has at least two neighbors x and y in S . Since G is triangle-free, the path xvy is an $x - y$ geodesic. Hence S is a geodetic dominating set of cardinality $2|M|$, and the proof is complete. \square

Let $H_1 = K_{p,q}$ be the complete bipartite graph with the partite sets $\{u_1, u_2, \dots, u_q\}$ and $\{x_1, x_2, \dots, x_p\}$ such that $q > p \geq 2$, and let $H_2 = K_{p,r}$ be the complete bipartite graph with the partite sets $\{v_1, v_2, \dots, v_r\}$ and $\{y_1, y_2, \dots, y_p\}$ such that $r > p \geq 2$. Define the graph H as the disjoint union of H_1 and H_2 together with the edge set $M' = \{x_1y_1, x_2y_2, \dots, x_py_p\}$. Then $S = \{x_1, x_2, \dots, x_p\} \cup \{y_1, y_2, \dots, y_p\}$ is a minimum geodetic dominating set of the triangle-free graph H with the maximal matching M' . Thus $\gamma_g(H) = 2|M'|$, and therefore Proposition 2.7 is sharp.

The same arguments as in the proof of Proposition 2.7 lead to the next two upper bounds. A subset $D \subseteq V(G)$ is a *2-dominating set* of G if every vertex of $V(G) \setminus D$ has at least two neighbors in D . The cardinality of a minimum 2-dominating set is called the *2-domination number* $\gamma_2(G)$ of G .

Proposition 2.8. *If G is a triangle-free graph, then $\gamma_g(G) \leq \gamma_2(G)$.*

Using Proposition 2.8 and known upper bounds on $\gamma_2(G)$ (see for example [2, 7, 8]), we obtain upper bounds of $\gamma_g(G)$ for triangle-free graphs.

Proposition 2.9. *If G is a triangle-free graph of order n with minimum degree $\delta \geq 2$, then $\gamma_g(G) \leq n - \alpha(G)$, where $\alpha(G)$ is the independence number of G .*

3 Geodetic Domination in Trees

If G is a graph and X a subset of $V(G)$, then, following Cockayne, Goodman and Hedetniemi [6], we call a set $D \subseteq V(G)$ an *X -dominating set* of G if $X \subseteq N[D]$. The *X -domination number* $\gamma_X(G)$ is the cardinality of a minimum X -dominating set of G .

Proposition 3.1. *If T is a tree of order $n \geq 2$ and $X = V(T) \setminus N[L(T)]$, then*

$$\gamma_g(T) = |L(T)| + \gamma_X(T).$$

Proof. Let S be a $\gamma_g(T)$ -set. As every geodetic set of T contains $L(T)$, we observe that $L(T) \subseteq S$. Since S is a dominating set of T , and $L(T)$

only dominates the vertices of $N[L(T)]$, the set $S \setminus L(T)$ is a minimum $(V(T) \setminus N[L(T)])$ -dominating set of T . This implies that

$$\gamma_g(T) - |L(T)| = |S| - |L(T)| = |S \setminus L(T)| = \gamma_X(T),$$

and the proof is complete. \square

Cockayne, Goodman and Hedetniemi [6] presented an $O(n)$ algorithm for determining $\gamma_X(T)$, and finding a corresponding minimum X -dominating set, for any tree T of order n . Applying this algorithm and Theorem 3.1, we see that we can find $\gamma_g(T)$ in linear time.

We now present conditions that force $\gamma_g(G) = g(G)$ and also $\gamma_g(G) = \gamma(G)$, parameters that have already been studied on trees.

Theorem 3.2. *If T is a tree of order $n \geq 3$, then the following conditions are equivalent.*

- (a) $\gamma_g(T) = g(T) = \gamma(T)$,
- (b) $L(T)$ is a minimum dominating set of T ,
- (c) $T = \text{cor}(T')$, where T' is an arbitrary tree of order at least 2.

Proof. Since the set of leaves $L(T)$ is a minimum geodetic set of a tree T , (a) and (b) are equivalent. Furthermore, if T' is a tree of order at least 2 and $T = \text{cor}(T')$, then $\gamma(T) = n/2 = |L(T)|$.

Finally, assume that $L(T)$ is a minimum dominating set of T . It follows that each non-leaf of T is adjacent to at least one leaf of T (note that $n \geq 3$ implies the existence of non-leaves). Now we show that each non-leaf of T is adjacent to at most one leaf of T . Suppose, to the contrary, that a support vertex u is adjacent to $k \geq 2$ leaves v_1, v_2, \dots, v_k . Then $D = (L(T) - \{v_1, v_2, \dots, v_k\}) \cup \{u\}$ is also a dominating set of T with $|D| < |L(T)|$. This is a contradiction to the minimality of $L(T)$. Altogether we see that each non-leaf of T is adjacent to exactly one leaf of T , and so $T = \text{cor}(T')$ with an arbitrary tree T' of order at least 2. \square

Finally, we notice the following proposition. The proof is similar to this one of Theorem 3.2 and is therefore omitted.

Proposition 3.3. *If T is a tree of order $n \geq 2$, then the following conditions are equivalent.*

- (a) $\gamma_g(T) = g(T)$,
- (b) $L(T)$ is a dominating set of T ,
- (c) Every vertex is either a leaf or a support vertex.

4 Realization Results

In this section we give realization results concerning the geodetic domination number. We first establish the existence of a connected graph G with $\gamma_g(G) = a$ and $|V(G)| = n$ for any two positive integers a, n with $2 \leq a \leq n$.

Proposition 4.1. *For any two positive integers a and n with $2 \leq a \leq n$ there exists a connected graph G with $\gamma_g(G) = a$ and $|V(G)| = n$.*

Proof. It can be verified that the result is true for $2 \leq n \leq 3$ since if $n = 2$, then $G = P_2$ while if $n = 3$, then $G \in \{P_3, K_3\}$. Let us now consider the case that $n \geq 4$. If $a = n$, let $G = K_n$ and if $a = n - 1$, let $G = K_{1, n-1}$. For $a \leq n - 2$, let G be the graph obtained from the star $K_{1, n-2}$ with leaves x_1, x_2, \dots, x_{n-2} , by adding a new vertex y and joining y to the vertices x_i ($a \leq i \leq n - 2$). Then the set $S = \{x_1, x_2, \dots, x_{a-1}, y\}$ is a minimum geodetic dominating set of G . \square

Since the union of a dominating set and a geodetic set gives us a geodetic dominating set, it follows that $\max\{\gamma(G), g(G)\} \leq \gamma_g(G) \leq \gamma(G) + g(G)$. We now consider triples $a, b, c \in \mathbb{Z}^+$, where $a, b \geq 2$ and $\max\{a, b\} \leq c \leq a + b$ for which there is a graph G such that $\gamma(G) = a$, $g(G) = b$ and $\gamma_g(G) = c$. Note that we only consider the cases where $a, b \geq 2$ since if $a = 1$, then Lemma 2.4 tells us that $g(G) = b = c = \gamma_g(G)$ for which we can take $G = K_{1, b} = K_{1, c}$ while if $b = 1$, then G has to be K_1 .

Lemma 4.2. *For any two integers $a, b \geq 2$, there is a connected graph G such that $\gamma(G) = a$, $g(G) = b$ and $\gamma_g(G) = a + b$.*

Proof. Let $a, b \geq 2$ be two integers. Consider the graph H obtained as follows.

1. Take a copy of C_6 and let x and y be antipodal vertices.
2. Add new vertices x_1, x_2, \dots, x_{b-1} and join each to the vertex x .

Let G be the graph obtained from H by taking a copy of the path on $3(a-2) + 1$ vertices $y_0 y_1 \dots y_{3(a-2)}$ and joining y_0 to the vertex y . Observe that the sets $\{x, y, y_2, y_5, \dots, y_{3(a-2)-1}\}$ and $\{x_1, x_2, \dots, x_{b-1}, y_{3(a-2)}\}$ are a minimum dominating set and a minimum geodetic set of G , respectively.

Thus $\gamma(G) = a$ and $g(G) = b$. Moreover, the union of the two sets given above is a minimum geodetic dominating set of G . It follows that $\gamma_g(G) = a + b$. \square

Theorem 4.3. *Let $a, b, c \in \mathbb{Z}^+$ with $a, b \geq 2$. Then there is a connected graph G such that $\gamma(G) = a$, $g(G) = b$ and $\gamma_g(G) = c$, where $\max\{a, b\} \leq c \leq a + b$.*

Proof. We consider four cases depending on whether some of a, b and c are equal or not.

Case 1 : $a = b = c$:

Take $G = \text{cor}(K_a)$, the corona of the complete graph on a vertices. Then $\gamma(G) = g(G) = \gamma_g(G) = a$.

Case 2 : $a < b = c$:

Take a copy of $K_{1,b}$ with the leaves x_1, x_2, \dots, x_b and the support vertex x . For $a < b$, subdivide each of the edges $xx_1, xx_2, \dots, xx_{a-1}$ to obtain a new graph G . Then the set $\{x, x_1, x_2, \dots, x_{a-1}\}$ is a minimum dominating set for G while $\{x_1, x_2, \dots, x_b\}$ is both a minimum geodetic set and a minimum geodetic dominating set for G . Thus $\gamma(G) = a$ and $g(G) = b = c = \gamma_g(G)$.

Case 3 : $b < a = c$:

Take a copy of $K_{1,b-1}$ with the leaves x_1, x_2, \dots, x_{b-1} and the support vertex x . Subdivide the edges $xx_i, i = 1, 2, \dots, b-1$ calling the new vertices y_1, y_2, \dots, y_{b-1} where x_i is adjacent to y_i for $i = 1, 2, \dots, b-1$. Obtain the graph G by taking a copy of the path of length $3(a-b)$, say $w_0w_1 \dots w_{3(a-b)}$, and joining w_0 to x . Then the set $\{x_1, x_2, \dots, x_{b-1}, w_0, w_3, \dots, w_{3(a-b)}\}$ is both a minimum dominating set and a minimum geodetic dominating set of G . Moreover, observe that the set $\{x_1, x_2, \dots, x_{b-1}, w_{3(a-b)}\}$ is a minimum geodetic set of G . Thus $\gamma(G) = a = c = \gamma_g(G)$ and $g(G) = b$.

Case 4 : $\max\{a, b\} < c < a + b$:

Let H be the graph obtained from $c - b$ copies of P_5 by identifying the corresponding leaves, and denoting them by x and y . Obtain a new graph G as follows.

1. Add new vertices x_1, x_2, \dots, x_{c-a} and joining each one to x .
2. Take $a+b-c-1$ copies of K_2 , say v_i, w_i where $i = 1, 2, \dots, a+b-c-1$ and joining each v_i to x .

Observe that the set $\{x, w_1, w_2, \dots, w_{a+b-c-1}\} \cup N(y)$ is a minimum dominating set of G while the set $\{y, x_1, x_2, \dots, x_{c-a}, w_1, w_2, \dots, w_{a+b-c-1}\}$ is a minimum geodetic set of G . On the other hand, the set

$$\{y, x_1, x_2, \dots, x_{c-a}, w_1, w_2, \dots, w_{a+b-c-1}\} \cup (N(N(y)) \setminus N(y))$$

is a minimum geodetic dominating set of G . Thus $\gamma(G) = a$, $g(G) = b$ and $\gamma_g(G) = c$. □

5 How the Geodetic Domination Number Changes When a Small Change is Made to the Graph

For many graph parameters, it is fundamental to ask how much the given parameter changes when a small change is done to a given graph. In this section, we study the effect on the geodetic domination number of a given graph by the removal of a vertex or an edge.

Proposition 5.1. *If G is a connected graph of order $n \geq 2$, then for every vertex $v \in V(G)$,*

$$\gamma_g(G - v) \leq d(v) + \gamma_g(G) - 1.$$

This bound is sharp.

Proof. Let S be a $\gamma_g(G)$ -set, and let $v \in V(G)$ be an arbitrary vertex. Then it is easy to see that $(S \cup N(v)) \setminus \{v\}$ is a geodetic dominating set of $G - v$. To obtain the desired bound, we distinguish two cases.

Case 1: $v \in S$. Then $|(S \cup N(v)) \setminus \{v\}| \leq d(v) + \gamma_g(G) - 1$.

Case 2: $v \notin S$. Then, since S is a dominating set of G , it follows that $S \cap N(v) \neq \emptyset$. Thus $|(S \cup N(v)) \setminus \{v\}| = |S \cup N(v)| \leq d(v) + \gamma_g(G) - 1$, and the inequality is proved.

To see that equality is attained, let H_1 be the graph obtained from $k \geq 3$ copies of a P_4 , by identifying the corresponding leaves. Let H_1, H_2, \dots, H_s

be $s \geq 1$ copies of the graph H_1 , where x_i and y_i , for $1 \leq i \leq s$ are the two vertices of H_i that are of maximum degree. Let G be the graph obtained from H_1, H_2, \dots, H_s by identifying all vertices x_i ($1 \leq i \leq s$). For convenience, we call this new vertex x . Then a minimum geodetic dominating set of G is $\{x, y_1, y_2, \dots, y_s\}$, and for any vertex $v \in V(G)$, $\gamma_g(G - v) = d(v) + \gamma_g(G) - 1$. \square

The following is an immediate consequence of Proposition 5.1, and it is sharp for the same graph given in the proof of Proposition 5.1.

Corollary 5.2. *For every non-trivial connected graph G and for every vertex $v \in V(G)$,*

$$\gamma_g(G - v) \leq \Delta(G) + \gamma_g(G) - 1.$$

We now show how the geodetic domination number of a connected graph G changes when an edge of G is removed.

Proposition 5.3. *If G is a connected graph of order $n \geq 2$, then for every edge $e \in E(G)$ we have that*

$$2 \leq \gamma_g(G - e) \leq \gamma_g(G) + 2.$$

The bounds are sharp.

Proof. Let G be a connected graph of order $n \geq 2$ and let $e = xy \in E(G)$. We first prove the left inequality.

Since K_1 is the only graph whose geodetic domination number is 1, it follows that $\gamma_g(G) \geq 2$ and $\gamma_g(G - e) \geq 2$ for every edge e in G .

If we take $G = K_n$, where $n \geq 2$, then $\gamma_g(G) = n$ while for every $e \in E(G)$, $\gamma_g(G - e) = 2$ and the equality is attained in the left inequality.

For the right inequality, let S be a $\gamma_g(G)$ -set and $e = uv \in E(G)$. Then $S \cup \{u, v\}$ is a geodetic dominating set of $G - e$, which proves the result.

To see that equality is attained, consider the graph obtained from $k \geq 3$ copies of a P_4 , by identifying the corresponding leaves. If e is an edge incident to two vertices of degree 2, then $\gamma_g(G) = 2$, while $\gamma_g(G - e) = 4 = \gamma_g(G) + 2$. \square

References

- [1] F. Buckley, F. Harary, and L.V. Quintas, *Extremal results on the geodesic number of a graph*, *Scientia 2A* (1988), 17-26.
- [2] Y. Caro and R. Yuster, *Dominating a family of graphs with small connected subgraphs*, *Combin. Probab. Comput.* 9 (2000), 309-313.
- [3] G. Chartrand, F. Harary, H. C. Swart and P. Zhang, *Geodomination in graphs*, *Bull. ICA* 31 (2001), 51-59.
- [4] G. Chartrand, F. Harary and P. Zhang, *Geodetic sets in graphs*, *Discussiones Mathematicae Graph Theory* 20 (2000), 129-138.
- [5] G. Chartrand, F. Harary and P. Zhang, *On the geodetic number of a graph*, *Networks* 39 (2002), 1-6.
- [6] E.J. Cockayne, S. Goodman and S.T. Hedetniemi, *A linear algorithm for the domination number of a tree*, *Inform. Process. Lett.* 4 (1975), 41-44.
- [7] O. Favaron, A. Hansberg and L. Volkmann, *On k -domination and minimum degree*, *J. Graph Theory* 57 (2008), 33-40.
- [8] A. Hansberg and L. Volkmann, *Upper bounds on the k -domination number and the Roman k -domination number*, *Discrete Appl. Math.* 157 (2009), 1634-1639.
- [9] T. W. Haynes, S. T. Hedetniemi and P. J. Slater, editors. *Fundamentals of domination in graphs*. Marcel Dekker, Inc, New York, NY, 1998.
- [10] R. Muntean and P. Zhang, *On geodomination in graphs*, *Congressus Numerantium* 143 (2000), 161-174.