On Friendly Index Sets of $(p, p+1)$-Graphs

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Abstract

Let G be a simple graph. Any vertex labeling $f : V(G) \to \mathbb{Z}_2$ induces an edge labeling $f^* : E(G) \to \mathbb{Z}_2$ according to $f^*(xy) = f(x) + f(y)$. For each $i \in \mathbb{Z}_2$, define $v_f(i) = |\{v \in V(G) : f(v) = i\}|$, and $e_f(i) = |\{e \in E(G) : f^*(e) = i\}|$. The friendly index set of the graph G is defined as $\{|e_f(0) - e_f(1)| : |v_f(0) - v_f(1)| \leq 1\}$. We determine the friendly index sets of connected $(p, p+1)$-graphs with minimum degree 2. Many of them form arithmetic progressions. Those that are not miss only the second terms of the progressions.

1 Introduction

Let G be a simple graph with vertex set V and edge set E. A vertex labeling $f : V(G) \to \mathbb{Z}_2$ induces an edge labeling $f^* : E(G) \to \mathbb{Z}_2$ defined by $f^*(xy) = f(x) + f(y)$. For $i = 0, 1$, define

$$\begin{aligned} v_f(i) &= |\{v \in V : f(v) = i\}|, \\ e_f(i) &= |\{e \in E : f^*(e) = i\}|. \end{aligned}$$

When the context is clear, we will omit the subscript. We say f is ***friendly*** if $|v_f(1) - v_f(0)| \leq 1$. The ***friendly index set*** of G is defined as

$$\mathrm{FI}(G) = \{|e_f(1) - e_f(0)| : f \text{ is friendly}\}.$$

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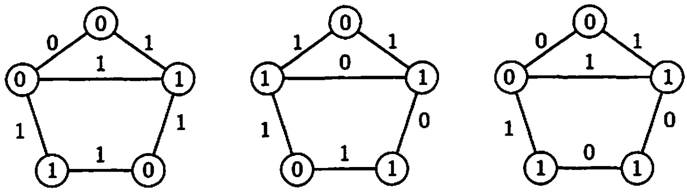
An example: $C_p(t)$ is the cycle on p vertices $x_1, x_2, \ldots, x_p$ with a chord joining x_1 to x_t. Using an exhaustive search, it can be shown that $\mathrm{FI}(C_5(3)) = \{4, 2, 0\}$.



Since its introduction by Lee and Ng in [4], the FI sets of many families of graphs had been completely determined. See, for examples, [1]–[10]. In this paper, we discuss how to determine the friendly index sets of $(p, p+1)$-graphs with minimum degree two. The degree sequence of such graphs is either $(4, 2, 2, \ldots, 2)$ or $(3, 3, 2, 2, \ldots, 2)$. Hence they are of the form $C(n_1, n_2, \ldots, n_k) \cup H$, where $C(n_1, n_2, \ldots, n_k)$ is the disjoint union of k cycles, and H is one of the following three connected graphs:

- The ***one-point union of two cycles*** of length ℓ_1 and ℓ_2, joined at the center c. We denote it $\mathrm{UC}(\ell_1, \ell_2)$. It has $p = \ell_1 + \ell_2 - 1$ vertices.

- The ***theta graph*** $\Theta(\ell_1, \ell_2, \ell_3)$, which consists of three paths of length ℓ_1, ℓ_2, ℓ_3 joined at their endpoints u and v. It has $p = \ell_1 + \ell_2 + \ell_3 - 1$ vertices.

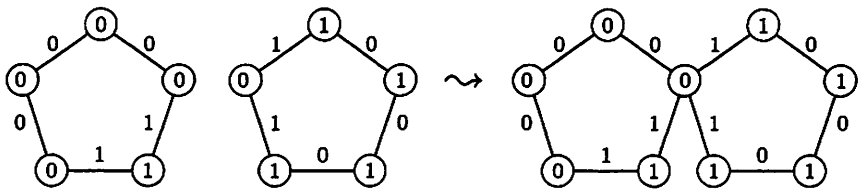
- The ***dumbbell graph*** $\mathrm{DB}(\ell_1, \ell_2; \ell_3)$. It consists of two cycles of length ℓ_1 and ℓ_2 connected by a path of length ℓ_3 at its endpoints u and v, and has $p = \ell_1 + \ell_2 + \ell_3 - 1$ vertices.

We will first determine the FI sets of *connected* $(p, p+1)$-graphs with minimum degree two. In other words, we will first determine the FI sets of H, where H is one of the three graphs described above. Next, we will discuss how to solve the general problem: the union of H with disjoint cycles.

2 $\mathrm{UC}(\ell_1, \ell_2)$

If all the 0-vertices in a friendly labeling are changed into 1-vertices, and 1-vertices into 0-vertices, the new labeling is still friendly, but the value of $|e(0) - e(1)|$ remains unchanged. Therefore we may assume the center c of the one-point union is an 0-vertex. Note that the restriction of the labeling on each cycle may not be a friendly labeling on that cycle.

Conversely, we could start with a vertex labeling (which needs not be friendly) on each cycle, and then pick an 0-vertex from each cycle, and "splice" them together to form an one-point union. We just have to choose the vertex labelings on the two cycles carefully so that the combined vertex labeling on the one-point union is friendly.



We need to analyze the vertex labeling (which needs not be friendly) of a cycle of length L. Any vertex labeling partitions the vertices of the cycles into B blocks, each block consists of consecutive 0-vertices followed by consecutive 1-vertices.

$$\underbrace{000\ldots0}_{c_1}\underbrace{111\ldots1}_{d_1}\underbrace{000\ldots0}_{c_2}\underbrace{111\ldots1}_{d_2}\ldots\ldots\underbrace{000\ldots0}_{c_B}\underbrace{111\ldots1}_{d_B}.$$

If all the vertices are labeled the same way, define $B=0$. In particular, if $B=0$, then either $(c_0,d_0)=(L,0)$, which means all vertices are 0-vertices, or $(c_0,d_0)=(0,L)$, which means all vertices are 1-vertices. It is easy to verify that

$$\begin{aligned} e(0) &= \sum_{j=1}^{B}(e_j-1)+\sum_{j=1}^{B}(d_j-1) &= L-2B, \\ e(1) &= \sum_{j=1}^{B}1+\sum_{j=1}^{B}1 &= 2B. \end{aligned}$$

Therefore

$$e(0)-e(1)=L-4B,$$

where $0\le B\le\lfloor L/2\rfloor$.

Recall that we assume the center is an 0-vertex. We may regard it the first 0-vertex in the first block of both cycles. Let $b=b_1+b_2$, where b_i denote the number of blocks in the ith cycle. We have

$$e(0)-e(1)=\ell_1+\ell_2-4(b_1+b_2)=p+1-4b,$$

where, *theoretically*, $0\le b\le\lfloor\ell_1/2\rfloor+\lfloor\ell_2/2\rfloor$.

We say theoretically because some of the b-values within the range may not lead to a friendly labeling. For example, $b=0$ requires $b_1=b_2=0$, which means the entire one-point union consists of 0-vertices. Therefore

$$1\le b\le\lfloor\ell_1/2\rfloor+\lfloor\ell_2/2\rfloor.$$

We now show that all these b-values are attainable. Given any b within this range, we can use a greedy approach to label the vertices:

- Label the first cycle with $b-1$ blocks of 01's if $b_1 < b$, or with as many blocks of 01's as it allows if $b_1 \geq b$.
- Continue with the second cycle, if necessary, until we have accumulated $b-1$ blocks of 01's.
- If there are unlabeled vertices in the first cycle, stretch out the last block with an equal (or almost equal) number of 0s and 1s to form a larger block.
- Label the last block in the second cycle with an equal (or almost equal) number of 0s and 1s. The exact numbers of 0s and 1s are selected so that the overall labeling is friendly.

For b less than the maximum allowable value $\lfloor \ell_1/2 \rfloor + \lfloor \ell_2/2 \rfloor$, we have enough unlabeled vertices to work around with. Hence it is always possible to find a friendly labeling with b blocks of vertices.

The only possible exception is the largest allowable value of b. This happens when b_i equals $\lfloor \ell_i/2 \rfloor$ for each i. Let $v_i(0) - v_i(1)$ represent the restriction of $v(0) - v(1)$ on the ith cycle, we find

$$v_i(0) - v_i(1) = \begin{cases} 0 & \text{if } \ell_i \text{ is even,} \\ 1 \text{ or } -1 & \text{if } \ell_i \text{ is odd.} \end{cases}$$

Over the entire one-point union, since c is an 0-vertex, we have

$$v(0) - v(1) = -1 + \sum_{i=1}^{2} (v_i(0) - v_i(1)).$$

It is easy to verify that we can select $v_i(0) - v_i(1)$ and mix them together so that $|v(0) - v(1)| \leq 1$. Therefore a friendly labeling exists with $b = \lfloor \ell_1/2 \rfloor + \lfloor \ell_2/2 \rfloor$.

We have just proved that the full range of values $1 \leq b \leq \lfloor \ell_1/2 \rfloor + \lfloor \ell_2/2 \rfloor$ is attainable. Since $e(0) - e(1) = p + 1 - 4b$, we obtain the following data.

parities of ℓ_i	range of values of $e(0) - e(1)$
ℓ_1 and ℓ_2 are odd	$p-3, p-7, p-11, \ldots, -(p-7), -(p-3)$
$\ell_1 + \ell_2$ is odd	$p-3, p-7, p-11, \ldots, -(p-5), -(p-1)$
ℓ_1 and ℓ_2 are even	$p-3, p-7, p-11, \ldots, -(p-3), -(p+1)$

Taking absolute values produce the desired FI set.

Theorem 2.1 *Let $G = UC(\ell_1, \ell_2)$, where $p = \ell_1 + \ell_2 - 1$. Then*

$$FI(G) = \begin{cases} \{p-3, p-7, p-11, \ldots, 2 \text{ or } 0\} & \text{if } \ell_1 \text{ and } \ell_2 \text{ are odd,} \\ \{p-1, p-3, p-5, \ldots, 1\} & \text{if } \ell_1 + \ell_2 \text{ is odd,} \\ \{p+1, p-3, p-7, \ldots, 2 \text{ or } 0\} & \text{if } \ell_1 \text{ and } \ell_2 \text{ are even.} \end{cases}$$

Example 1. The vertex labelings that affirm $\text{FI}(\text{UC}(3,5)) = \{4,0\}$ are tabulated below. For each cycle, we start with $f(c)$, move along the vertices on the cycle, until we return to $f(c)$.

C_3	C_5	$e(0) - e(1)$
0010	010110	−4
0010	001110	0

In a similar manner, the following vertex labelings

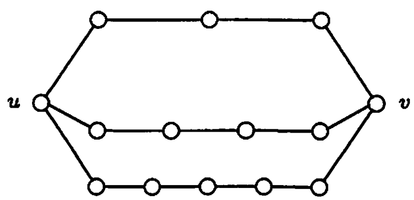
C_4	C_5	$e(0) - e(1)$
01010	010010	−7
00000	011110	5
01010	000110	−3
00110	000110	1

C_4	C_6	$e(0) - e(1)$
01010	0101010	−10
00000	0111110	6
00110	0001110	2

confirm that $\text{FI}(\text{UC}(4,5)) = \{7,5,3,1\}$ and $\text{FI}(\text{UC}(4,6)) = \{10,6,2\}$. □

3 The Theta Graph $\Theta(\ell_1, \ell_2, \ell_3)$

The theta graph $\Theta(\ell_1, \ell_2, \ell_3)$ can be considered as three paths of length ℓ_1, ℓ_2 and ℓ_3 joined together at their endpoints u and v. The theta graph $\Theta(4,5,6)$ is displayed below.



Due to symmetry, we may assume $1 \leq \ell_1 \leq \ell_2 \leq \ell_3$, where $\ell_2 \geq 2$, and that the two vertices u and v are labeled 00 or 01. We first study what happens to the labeling (which needs not be friendly) of a path of length L with B blocks of vertices.

Case 1: $f(u) = f(v) = 0$. We need $c_B > 0$ and $d_B = 0$:

$$\underbrace{000\ldots0}_{c_1}\underbrace{111\ldots1}_{d_1}\underbrace{000\ldots0}_{c_2}\underbrace{111\ldots1}_{d_2}\cdots\cdots\underbrace{000\ldots0}_{c_B}.$$

Notice that this requires $B \geq 1$. In particular, $B = 1$ means all the vertices on the path are labeled 0. We find

$$\begin{aligned} e(0) &= \sum_{j=1}^{B}(e_j - 1) + \sum_{j=1}^{B-1}(d_j - 1) &&= (L+1) - (2B-1), \\ e(1) &= \sum_{j=1}^{B-1} 1 + \sum_{j=1}^{B-1} 1 &&= 2B - 2. \end{aligned}$$

Therefore, on a single path,

$$e(0) - e(1) = L + 4 - 4B,$$

where $1 \leq B \leq \lceil (L+1)/2 \rceil$. Consequently, over $\Theta(\ell_1, \ell_2, \ell_3)$, we find, if we let $b = b_1 + b_2 + b_3$,

$$e(0) - e(1) = \ell_1 + \ell_2 + \ell_3 + 12 - 4(b_1 + b_2 + b_3) = p + 13 - 4b,$$

where $3 \leq b \leq \lceil (\ell_1 + 1)/2 \rceil + \lceil (\ell_2 + 1)/2 \rceil + \lceil (\ell_3 + 1)/2 \rceil$.

However, $b = 3$ implies that $b_1 = b_2 = b_3 = 1$. This in turn implies that all the vertices in each path are labeled 0, thus all the vertices in $\Theta(\ell_1, \ell_2, \ell_3)$ are 0-vertices. If $b = 4$, we may assume $b_i = b_j = 1$ and $b_k = 2$. Then all the vertices on the two paths of length ℓ_i and ℓ_j are 0-vertices. For the labeling to be friendly, we need $\ell_i + \ell_j \leq \ell_k$. Since we assume $\ell_1 \leq \ell_2 \leq \ell_3$, this is same as requiring $\ell_1 + \ell_2 \leq \ell_3$. Therefore

$$\left\lceil \frac{\ell_1 + 1}{2} \right\rceil + \left\lceil \frac{\ell_2 + 1}{2} \right\rceil + \left\lceil \frac{\ell_3 + 1}{2} \right\rceil \geq b \geq \begin{cases} 4 & \text{if } \ell_1 + \ell_2 \leq \ell_3, \\ 5 & \text{otherwise.} \end{cases}.$$

Using a greedy algorithm, it is clear that these values of b can be attained, with perhaps the exception of the largest possible value for b.

Denote the restriction of $v(0) - v(1)$ on the ith path as $v_i(0) - v_i(1)$. The maximum value of b is reached when $b_i = \lceil (\ell_i + 1)/2 \rceil$ for each i. When this happens, if ℓ_i is even, the path will be labeled $0101\ldots010$, but if ℓ_i is odd, the path will be labeled $0101\ldots0110$ or $0101\ldots0100$. Hence

$$v_i(0) - v_i(1) = \begin{cases} 1 & \text{if } \ell_i \text{ is even,} \\ 0 \text{ or } 2 & \text{if } \ell_i \text{ is odd.} \end{cases}$$

Since u and v are shared by all three paths, and they are both labeled 0, over the entire theta graph,

$$v(0) - v(1) = -4 + \sum_{i=1}^{3} (v_i(0) - v_i(1)).$$

It is easy to verify we can always find a friendly labeling with b reaching its maximum possible value. The ranges of attainable values of $e(0) - e(1)$ are listed in the following table, in which ℓ denotes the number of odd values among ℓ_1, ℓ_2, ℓ_3.

ℓ	$e(0) - e(1)$	condition
3	$p-3, p-7, p-11, \ldots, -(p-9), -(p-5)$	if $\ell_1 + \ell_2 \leq \ell_3$
	$p-7, p-11, p-15, \ldots, -(p-9), -(p-5)$	otherwise
2	$p-3, p-7, p-11, \ldots, -(p-7), -(p-3)$	if $\ell_1 + \ell_2 \leq \ell_3$
	$p-7, p-11, p-15, \ldots, -(p-7), -(p-3)$	otherwise
1	$p-3, p-7, p-11, \ldots, -(p-5), -(p-1)$	if $\ell_1 + \ell_2 \leq \ell_3$
	$p-7, p-11, p-15, \ldots, -(p-5), -(p-1)$	otherwise
0	$p-3, p-7, p-11, \ldots, -(p-3), -(p+1)$	if $\ell_1 + \ell_2 \leq \ell_3$
	$p-7, p-11, p-15, \ldots, -(p-3), -(p+1)$	otherwise

Case 2: $f(u) = 0$ and $f(v) = 1$. For a path of length L with B blocks whose endpoints are labeled 0 and 1,

$$\begin{aligned} e(0) &= \sum_{j=1}^{B}(e_j - 1) + \sum_{j=1}^{B}(d_j - 1) &= (L+1) - 2B, \\ e(1) &= \sum_{j=1}^{B} 1 + \sum_{j=1}^{B-1} 1 &= 2B - 1. \end{aligned}$$

Therefore, on a single path,

$$e(0) - e(1) = L + 2 - 4B,$$

where $1 \leq B \leq \lceil L/2 \rceil$. Over the entire theta graph, $\Theta(\ell_1, \ell_2, \ell_3)$, we find

$$e(0) - e(1) = \ell_1 + \ell_2 + \ell_3 + 6 - 4(b_1 + b_2 + b_3) = p + 7 - 4b,$$

where $3 \leq b \leq \lceil \ell/2 \rceil + \lceil \ell_2/2 \rceil + \lceil \ell_3/2 \rceil$. Again, a greedy approach yields friendly labeling for each value of b within this range, with the last maximum value as the only possible exception. When that happens,

$$v_i(0) - v_i(1) = \begin{cases} 1 \text{ or } -1 & \text{if } \ell_i \text{ is even}, \\ 0 & \text{if } \ell_i \text{ is odd}. \end{cases}$$

Along with

$$v(0) - v(1) = \sum_{i=1}^{3} (v_i(0) - v_i(1))$$

over the entire graph $\Theta(\ell_1, \ell_2, \ell_3)$, it is easy to see that we can find a friendly labeling in this case. Therefore b can attain its full range of possible values, from which we could compute the values of $e(0) - e(1)$:

ℓ	$e(0) - e(1)$
3	$p-5, p-9, p-13, \ldots -(p-3), -(p+1)$
2	$p-5, p-9, p-13, \ldots -(p-5), -(p-1)$
1	$p-5, p-9, p-13, \ldots -(p-7), -(p-3)$
0	$p-5, p-9, p-13, \ldots -(p-9), -(p-5)$

Combining the data from the two cases, we obtain a surprisingly simple result.

Theorem 3.1 *The FI set of the theta graph* $\Theta(\ell_1, \ell_1, \ell_3)$ *is*

$$\begin{cases} \{p+1\} \cup \{p-3, p-5, p-7, \ldots, 1 \text{ or } 0\} & \text{if } \ell_1 \equiv \ell_2 \equiv \ell_3 \pmod 2, \\ \{p-1\} \cup \{p-3, p-5, p-7, \ldots, 1 \text{ or } 0\} & \text{otherwise,} \end{cases}$$

where $p = \ell_1 + \ell_2 + \ell_3 - 1$.

Example 2. $\mathrm{FI}(\Theta(2,3,4)) = \{7,5,3,1\}$, $\mathrm{FI}(\Theta(2,4,6)) = \{12,8,6,4,2,0\}$ and $\mathrm{FI}(\Theta(3,5,5))) = \{13,9,7,5,3,1\}$:

P_3	P_4	P_5	index
010	0100	01010	−7
001	0101	01011	−5
010	0100	01100	−3
000	0100	01110	1

P_3	P_5	P_7	index
010	01010	0101010	−12
000	00000	0111110	8
001	00111	0001111	6
000	01110	0001110	4
001	00111	0100111	2
000	01010	0011110	0

P_4	P_6	P_6	index
0101	010101	010101	−13
0101	010101	010011	−9
0011	000111	000111	7
0101	010101	000111	−5
0101	000111	000111	3
0101	010011	000111	−1

Notice that 10 and 11, respectively, are missing in the last two arithmetic progressions. □

Example 3. When $\ell_1 = 1$, $p = \ell_1 + \ell_2$, and $\Theta(1, \ell_2, \ell_3)$ reduces to a cycle on p vertices $x_1, x_2, \ldots, x_p$, with a chord joining x_1 to x_t. We denote such a graph $C_p(t)$. Obviously, $C_p(t) = \Theta(1, t-1, p-t+1)$. Therefore the FI set is

$$\begin{cases} \{p+1\} \cup \{p-3, p-5, p-7, \ldots, 1 \text{ or } 0\} & \text{if } t \text{ and } p-t \text{ are even,} \\ \{p-1\} \cup \{p-3, p-5, p-7, \ldots, 1 \text{ or } 0\} & \text{otherwise,} \end{cases}$$

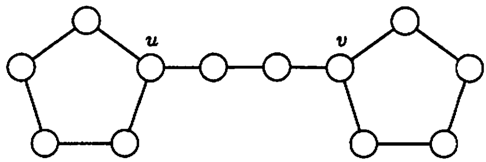
Further, if either t or $p-t$ is odd, the chord is called a ***parallel chord***. Lee and Ng had studied the FI sets of cycles with multiple parallel chords in [5]. They found

$$\mathrm{FI}(C_p(2s+1)) = \{p-1, p-3, p-5, \ldots, 1 \text{ or } 0\},$$

which agrees with the result given above. □

4 The Dumbbell Graph $\mathrm{DB}(\ell_1, \ell_2; \ell_3)$

The dumbbell graph $\mathrm{DB}(\ell_1, \ell_2; \ell_3)$ consists of two cycles of length ℓ_1 and ℓ_2, which are connected via a path of length ℓ_3 at its endpoints u and v. The dumbbell graph $\mathrm{DB}(5, 5; 3)$ is displayed below.



Case 1: $f(u) = f(v) = 0$. We may assume u and v are both the first 0-vertex in the first block in their respective cycle. We find

$$e(0) - e(1) = (\ell_1 - 4b_1) + (\ell_2 - 4b_2) + (\ell_3 + 4 - 4b_3) = p + 5 - 4b.$$

It is impossible for $b = 1$, because it requires $(b_1, b_2, b_3) = (0, 0, 1)$, which in turn implies all the vertices are labeled 0. For $b = 2$, we need (b_1, b_2, b_3) equals to either $(0, 1, 1)$ or $(1, 0, 1)$, which in turn requires $\ell_2 \geq \ell_1 + \ell_3$ or $\ell_1 \geq \ell_2 + \ell_3$ respectively. Therefore

$$\left\lfloor \frac{\ell_1}{2} \right\rfloor + \left\lfloor \frac{\ell_2}{2} \right\rfloor + \left\lceil \frac{\ell_3 + 1}{2} \right\rceil \geq b \geq \begin{cases} 2 & \text{if } |\ell_1 - \ell_2| \geq \ell_3, \\ 3 & \text{otherwise.} \end{cases}$$

The same technique we employed earlier affirms that b can assume its full range of possible values. Due to symmetry, we need to inspect 6 cases of various values of ℓ_i (mod 2). They lead to 4 different sets of values of $e(0) - e(1)$.

$\ell_1\ell_2\ell_3$	$e(0)-e(1)$	condition
1 1 1	$p-3, p-7, p-11, \ldots, -(p-9), -(p-5)$	if $\lvert\ell_1-\ell_2\rvert \geq \ell_3$
	$p-7, p-11, p-15, \ldots, -(p-9), -(p-5)$	otherwise
1 0 1	$p-3, p-7, p-11, \ldots, -(p-7), -(p-3)$	if $\lvert\ell_1-\ell_2\rvert \geq \ell_3$
1 1 0	$p-7, p-11, p-15, \ldots, -(p-7), -(p-3)$	otherwise
1 0 0	$p-3, p-7, p-11, \ldots, -(p-5), -(p-1)$	if $\lvert\ell_1-\ell_2\rvert \geq \ell_3$
0 0 1	$p-7, p-11, p-15, \ldots, -(p-5), -(p-1)$	otherwise
0 0 0	$p-3, p-7, p-11, \ldots, -(p-3), -(p+1)$	if $\lvert\ell_1-\ell_2\rvert \geq \ell_3$
	$p-7, p-11, p-15, \ldots, -(p-3), -(p+1)$	otherwise

Case 2: $f(u) = 0$ and $f(v) = 1$. We may assume u is the first 0-vertex in the first block of the first cycle, and v is the last 1-vertex in the last block of the second cycle. We have $0 \leq b_1 \leq \lfloor \ell_1/2 \rfloor$, $0 \leq b_1 \leq \lfloor \ell_2/2 \rfloor$, and $1 \leq b_3 \leq \lceil \ell_3/2 \rceil$, and

$$e(0) - e(1) = (\ell_1 - 4b_1) + (\ell_2 - 4b_2) + (\ell_3 + 2 - 4b_3) = p + 3 - 4b.$$

Notice that $b = 1$ requires $(b_1, b_2, b_3) = (0, 0, 1)$, which implies $|\ell_1 - \ell_2| \leq \ell_3$. Thus

$$\left\lfloor \frac{\ell_1}{2} \right\rfloor + \left\lfloor \frac{\ell_2}{2} \right\rfloor + \left\lceil \frac{\ell_3}{2} \right\rceil \geq b \geq \begin{cases} 1 & \text{if } |\ell_1 - \ell_2| \leq \ell_3, \\ 2 & \text{otherwise.} \end{cases}$$

Again, using the same technique we used earlier, we find that b assumes its full range of possible values, hence the following values for $e(0) - e(1)$.

$\ell_1\ell_2\ell_3$	$e(0)-e(1)$	condition
1 1 1	$p-1, p-5, p-9, \ldots, -(p-7), -(p-3)$	if $\lvert\ell_1-\ell_2\rvert \leq \ell_3$
1 0 0	$p-5, p-9, p-13, \ldots, -(p-7), -(p-3)$	otherwise
1 0 1	$p-1, p-5, p-9, \ldots, -(p-5), -(p-1)$	if $\lvert\ell_1-\ell_2\rvert \leq \ell_3$
0 0 0	$p-5, p-9, p-13, \ldots, -(p-5), -(p-1)$	otherwise
0 0 1	$p-1, p-5, p-9, \ldots, -(p-3), -(p+1)$	if $\lvert\ell_1-\ell_2\rvert \leq \ell_3$
	$p-5, p-9, p-13, \ldots, -(p-3), -(p+1)$	otherwise
1 1 0	$p-1, p-5, p-9, \ldots, -(p-9), -(p-5)$	if $\lvert\ell_1-\ell_2\rvert \leq \ell_3$
	$p-5, p-9, p-13, \ldots, -(p-9), -(p-5)$	otherwise

We now need to combine the sets of values of $e(0) - e(1)$ from the two cases. There are 6 combinations of ℓ_1, ℓ_2 and ℓ_3. In each combination, we have three possibilities, depending on whether $|\ell_1 - \ell_2|$ equals, larger than, or smaller than ℓ_3. Notice that some cases are impossible. For example, we cannot have $|\ell_1 - \ell_2| = \ell_3$ when ℓ_1, ℓ_2, ℓ_3 are all odd. We obtain the following table.

	FI set	condition
ℓ_1, ℓ_2 odd, ℓ_3 odd	$\{p-3, p-5, p-7, \ldots, 1\}$ $\{p-1, p-3, p-5, \ldots, 1\}$	if $\|\ell_1 - \ell_2\| > \ell_3$ if $\|\ell_1 - \ell_2\| < \ell_3$
$\ell_1 + \ell_2$ odd, ℓ_3 odd	$\{p-1, p-3, p-5, \ldots, 0\}$	always
ℓ_1, ℓ_2 even, ℓ_3 odd	$\{p+1, p-1, p-3, \ldots, 1\}$ $\{p+1, p-1, p-5, \ldots, 1\}$	if $\|\ell_1 - \ell_2\| > \ell_3$ if $\|\ell_1 - \ell_2\| < \ell_3$
ℓ_1, ℓ_2 odd, ℓ_3 even	$\{p-1, p-3, p-5, \ldots, 0\}$ $\{p-3, p-5, p-7, \ldots, 0\}$ $\{p-1, p-5, p-7, \ldots, 0\}$	if $\|\ell_1 - \ell_2\| = \ell_3$ if $\|\ell_1 - \ell_2\| > \ell_3$ if $\|\ell_1 - \ell_2\| < \ell_3$
$\ell_1 + \ell_2$ odd, ℓ_3 even	$\{p-1, p-3, p-5, \ldots, 1\}$ $\{p-1, p-3, p-5, \ldots, 1\}$	if $\|\ell_1 - \ell_2\| > \ell_3$ if $\|\ell_1 - \ell_2\| < \ell_3$
ℓ_1, ℓ_2 even, ℓ_3 even	$\{p+1, p-1, p-3, \ldots, 0\}$	always

Theorem 4.1 *Let $p = \ell_1 + \ell_2 + \ell_3 - 1$, then the FI set of DB($\ell_1, \ell_2; \ell_3$) is*

$$\begin{cases} \{p-3, p-5, p-7, \ldots, 1 \text{ or } 0\} & \text{if } \ell_1, \ell_2 \text{ odd, and } |\ell_1 - \ell_2| > \ell_3, \\ \{p+1, p-1, p-3, \ldots, 1 \text{ or } 0\} & \text{if } \ell_1, \ell_2 \text{ odd,} \\ \{p-1, p-3, p-5, \ldots, 1 \text{ or } 0\} & \text{otherwise.} \end{cases}$$

Example 4. We find FI(DB(3,5;1)) = $\{5, 3, 1\}$, as shown below.

C_3	P_2	C_5	$e(0) - e(1)$
0010	01	101011	−5
0000	01	101111	3
0010	01	100111	−1

The following labelings demonstrate that FI(DB(3,4;3)) = $\{8, 6, 4, 2, 0\}$ and FI(DB(4,6;2)) = $\{12, 10, 8, 6, 4, 2, 0\}$. □

C_3	P_4	C_4	$e(0) - e(1)$
0000	0011	11111	8
0010	0110	01010	−6
0000	0101	11111	4
0110	0000	01110	2
0010	0011	10111	0

C_4	P_3	C_6	$e(0) - e(1)$
01010	010	0101010	−12
00000	001	1111111	10
00000	000	0111110	8
00000	001	1011111	6
00110	000	0011110	4
00110	001	1001111	2
01010	000	0011110	0

5 The General Case

In general, a $(p, p+1)$-graph G with minimum degree two is the union of disjoint cycles and H, where H is one of the three connected graphs we have studied above. To find its FI set, we follow these steps:

1. First find a formula for $e(0) - e(1)$. This is easy because we already know the formula for individual cycle, and for each of the three types of graphs described by H.
2. The formula poses a range of possible values of b, where b is the total number of blocks on the entire graph.
3. Decide which b-values are attainable. A greedy approach can be used for those b-values in the middle of the range. But caution must be exercised for those b-values close to the two extreme ends. This is the most difficult part of the solution.
4. Compute the corresponding values of $e(0) - e(1)$, take absolute values, and gather them together to form the FI set.

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