Computing the Folkman Number $F_v(2,2,2,2,2;4)$

Joel Lathrop and Stanisław Radziszowski
Department of Computer Science
Rochester Institute of Technology
Rochester, NY 14623
joel.lathrop@alum.rit.edu, spr@cs.rit.edu

Abstract

For a graph G, the expression $G \xrightarrow{v} (a_1,\dots,a_r)$ means that for any r-coloring of the vertices of G there exists a monochromatic a_i-clique in G for some color $i \in \{1,\dots,r\}$. The vertex Folkman numbers are defined as $F_v(a_1,\dots,a_r;q) = \min\{|V(G)| : G \xrightarrow{v} (a_1,\dots,a_r) \text{ and } K_q \not\subseteq G\}$. Of these, the only Folkman number of the form $F(\underbrace{2,\dots,2}_{r};r-1)$ which has remained unknown up to this time is $F_v(2,2,2,2,2;4)$.

We show here that $F_v(2,2,2,2,2;4) = 16$, which is equivalent to saying that the smallest 6-chromatic K_4-free graph has 16 vertices. We also show that the sole witnesses of the upper bound $F_v(2,2,2,2,2;4) \leq 16$ are the two Ramsey (4,4)-graphs on 16 vertices.

1 Introduction

Let G be a finite, simple, undirected graph. We will denote the set of vertices of G as $V(G)$ and the set of edges as $E(G)$. The graphs obtained from G by addition and removal of an edge e will be written as $G+e$ and $G-e$, respectively. $\overline{G}$ stands for the complement of G, and $\chi(G)$ for the chromatic number of G. Finally, unless explicitly stated otherwise, it may be presumed that all integer variables we name are positive.

The two-color Ramsey number $R(k,l)$ is defined as the smallest number n such that for every graph G on n vertices, either G contains a K_k or $\overline{G}$

contains a K_l [4]. We say that a graph G is (k,l)-*good* if G does not contain a K_k and $\overline{G}$ does not contain a K_l. The set of all (k,l)-good graphs on n vertices is written as $\mathscr{R}(k,l;n)$.

The expression $G \overset{v}{\to} (a_1,\ldots,a_r)$ means that for any r-coloring of the vertices of G there exists a monochromatic a_i-clique in G for some color $i \in \{1,\ldots,r\}$. The vertex Folkman graphs $H_v(a_1,\ldots,a_r;q)$ are defined as

$$H_v(a_1,\ldots,a_r;q) = \{G : G \overset{v}{\to} (a_1,\ldots,a_r) \text{ and } K_q \not\subseteq G\}.$$

The vertex Folkman numbers $F_v(a_1,\ldots,a_r;q)$ are defined by

$$F_v(a_1,\ldots,a_r;q) = \min\{|V(G)| : G \in H_v(a_1,\ldots,a_r;q)\}.$$

Since the order of $a_1,\ldots,a_r$ is inconsequential to the definitions, we will assume that $a_1 \leq a_2 \leq \cdots \leq a_r$. Folkman [3] proved that $H_v(a_1,\ldots,a_r;q)$ is non-empty if and only if $q > \max\{a_1,\ldots,a_r\}$. Knowing that certain Folkman numbers exist, the natural next question is what bounds can be determined for those numbers. By the pigeonhole principle, we observe that $K_m \overset{v}{\to} (a_1,\ldots,a_r)$, where

$$m = 1 + \sum_{i=1}^{r}(a_i - 1).$$

This easily leads to $F_v(a_1,\ldots,a_r;m+1) = m$. Łuczak, Ruciński, and Urbański [7] obtained the next bound by proving that $F_v(a_1,\ldots,a_r;m) = a_r + m$. Nenov [12] proved certain bounds for a prohibited clique order of $m-1$, specifically

$$\begin{aligned} F_v(a_1,\ldots,a_r;m-1) &= m+6 \qquad \text{if } a_r = 3 \text{ and } m \geq 6, \text{ and} \\ F_v(a_1,\ldots,a_r;m-1) &= m+7 \qquad \text{if } a_r = 4 \text{ and } m \geq 6. \end{aligned}$$

Of particular interest are the vertex Folkman numbers $F_v(\underbrace{2,\ldots,2}_{r};q)$, also written as $F_v(2_r;q)$. Equivalently, these numbers can be defined as the order of the smallest $(r+1)$-chromatic graphs containing no K_q. Nenov [13] proved various bounds for Folkman numbers of this variety, however here we focus only on problems with q close to m. If $m = r+1$ then we consider only the case of $a_i = 2$ for all $1 \leq i \leq r$. From the proof of Łuczak et. al. [7] we know that $F_v(2_r;r+1) = r+3$.

This leads us next to vertex Folkman numbers of the form $F_v(2_r;r)$. In the trivial case of $r = 2$ clearly $F_v(2,2;2)$ does not exist. Chvátal [1] proved $F_v(2_3;3) = 11$, and Nenov [11] proved $F_v(2_4;4) = 11$. The solution for the remainder of the cases of this form is complete with Nenov's proof that $F_v(2_r; r) = r + 5$ for $r \geq 5$ [11].

Finally, we consider vertex Folkman numbers of the form $F_v(2_r; r-1)$. Again, directly by definition $F_v(2_3; 2)$ does not exist. For $r = 4$, Jensen and Royle [6] showed that $F_v(2_4; 3) = 22$. For $r \geq 6$, Nenov [14] proved that $F_v(2_r; r-1) = r + 7$. This leaves only $F_v(2_5; 4)$, of which Nenov [14] proved the bounds $12 \leq F_v(2_5; 4) \leq 16$ and identified as "the only unknown number of the kind $F(2_r; r-1)$."

In the remainder of this paper, we will show that $F_v(2_5; 4) = 16$ by computationally proving that $F_v(2_5; 4) > 15$. Computationally proving Folkman number lower bound such as this is not easy. To do so requires showing that *every* graph on 15 vertices is not in $H_v(2_4; 4)$. Even with isomorph rejection it is computationally intractable to generate all graphs on 15 vertices: There are 31,426,485,969,804,308,768 such non-isomorphic graphs [10]. Therefore, we must use certain theoretical properties to prove that only a subset of all graphs on 15 vertices can possibly be in $H_v(2_5; 4)$, and then enumerate and test that subset. Thus, the proof is part theoretical and part computational: We theoretically show that some graphs on 15 vertices cannot be in $H_v(2_5; 4)$ and then computationally enumerate the rest and show that they are also not in $H_v(2_5; 4)$. Our method for doing this is based on that used by Coles and Radziszowski [2] to prove $F_v(2, 2, 3; 4) = 14$.

In addition to proving the lower bound, we also find all graphs on 16 vertices in $H_v(2_5; 4)$ which are witnesses to the bound $F_v(2_5; 4) \leq 16$. This is done using the same process of theoretical elimination, computational enumeration, and testing used to prove the lower bound.

2 Algorithms

In order to determine if the graphs we computationally enumerate are in $H_v(2_5; 4)$, we must test to see if they meet the Folkman property of $F_v(2_5; 4)$. Given some graph G to test, this means that $G \overset{v}{\rightarrow} (2_5)$ and $K_4 \not\subseteq G$ must hold. Since $G \overset{v}{\rightarrow} (2_5)$ if and only if $\chi(G) > 5$, we can test $G \in H_v(2_5; 4)$ by simply verifying $\chi(G) > 5$ and $K_4 \not\subseteq G$. As Nenov has already proven that $F_v(2_5; 4) \leq 16$, it is sufficient to computationally prove that $F_v(2_5; 4) > 15$.

2.1 Theoretical constraints

We first define a *maximal-Folkman graph.*

Definition 2.1. For Folkman number $F(a_1, \ldots, a_i; q)$, a graph G is a *maximal-Folkman graph* if and only if $G \in H(a_1, \ldots, a_i; q)$ and for all $u, v \in V(G), u \neq v, \{u, v\} \notin E(G)$ it holds that $G+\{u, v\} \notin H(a_1, \ldots, a_i; q)$. The set of all maximal-Folkman graphs is written as $H^{max}(a_1, \ldots, a_i; q)$.

Consider any K_q-free graph G. Observe that any supergraph H of G on the same set of vertices, such that addition of any edge to H creates K_q, also satisfies $\chi(G) \leq \chi(H)$. Any such H is a maximal-Folkman graph with the same parameters as G, and every G has at least one such maximal-Folkman supergraph. Hence, in our case, it is sufficient to find all graphs in $H_v^{max}(2_5; 4)$ from which we can derive $H_v(2_5; 4)$ via the ReduceSize algorithm of [2] (shown later as Algorithm 1).

Now, let us consider other attributes of potential $G \in H_v(2_5; 4)$ on 15 vertices. Because $K_4 \not\subseteq G$ and $R(4, 3) = 9$ [15], it follows that G has a $\overline{K_3}$. Thus, G can be seen as a 12-vertex graph G' with an added $\overline{K_3}$ and corresponding edges. For each vertex in the $\overline{K_3}$ we will add all possible edges to a corresponding triangle-free subset in G'. This is illustrated in Figure 1.

Obviously $K_4 \not\subseteq G'$. Also, since $\chi(G) \geq 6$ and the addition of an independent set can increase chromatic number by at most one, we know that $\chi(G') \geq 5$. Finally, since we are only trying to obtain $G \in H_v^{max}(2_5; 4)$ we can restrict ourselves to only those G' which are connected graphs (since G must be K_4-free maximal).

2.2 The extension algorithm

The above constraints allow us to tractably enumerate a set of 15-vertex graphs containing all 15-vertex graphs in $H_v^{max}(2_5; 4)$, using the following algorithm called Extend:

1. For every G' which (a) has 12 vertices, (b) is connected, (c) has no 4-clique, and (d) has $\chi(G') \geq 5$, perform steps 2–4 below. All 12-vertex, connected graphs can be generated using the geng utility of the nauty software package [8] and then filtered for properties (c) and (d).

2. Extend G' by adding $\overline{K_3}$ and incident edges to it. Each vertex in the added $\overline{K_3}$ is made incident to all vertices of a maximal triangle-free

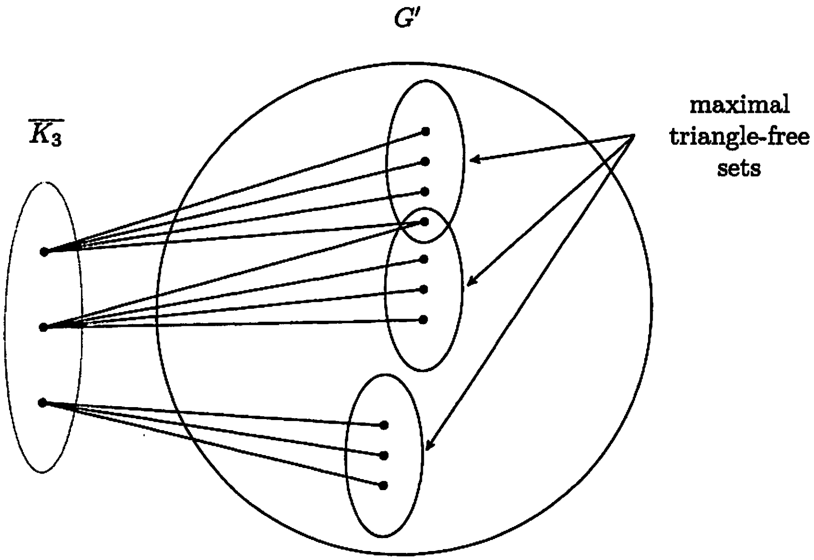

Figure 1: G as a $\overline{K_3}$-extension to triangle-free subsets in G'

subset[1]. This is done in all possible ways for all maximal triangle-free subsets of G', skipping obvious isomorphisms (e.g., permutations of the vertices in $\overline{K_3}$). The output will contain all the maximal-Folkman graphs containing G', as well as other Folkman and non-Folkman graphs.

3. Eliminate isomorphs using nauty's canonization functionality.

4. Filter out graphs with $\chi(G) \leq 5$. Since we started with graphs that had no 4-clique and our extension algorithm does not allow the creation of a 4-clique, our final output will be graphs G such that $K_4 \not\subseteq G$ and $\chi(G) \geq 6$. This implies that $G \in H_v(2_5; 4)$.

2.3 The reduction algorithm

The EXTEND algorithm will generate all 15-vertex graphs in $H_v^{max}(2_5; 4)$. However, if we want all graphs of that order in $H_v(2_5; 4)$, we must reduce the maximal-Folkman graphs to produce all their non-maximal-Folkman

[1] Just to be clear, a "maximal triangle-free" subset S of G' is such that S contains no triangles and the addition of any new vertex from $V(G')$ to S induces a triangle in S.

subgraphs. This can be done with the REDUCESIZE algorithm of [2], given here as Algorithm 1 for convenience.

Algorithm 1 REDUCESIZE(G) for some $H_v(a_1, \ldots, a_i; q)$

```
if G ∈ H_v(a_1, ..., a_i; q) then
  output G
  for all e ∈ E(G) do
    G ← G − e
    REDUCESIZE(G)
  end for
end if
```

3 Results

3.1 Computing the lower bound $F_v(2_5; 4) > 15$

Through theoretical constraints we applied we were able to substantially reduce our search space. There are only 41,364 connected graphs with 12 vertices, with no K_4 and chromatic number at least 5. While the $\overline{K_3}$-extension substantially expanded that set, the computation remained quite tractable.

We implemented the EXTEND algorithm described above and executed it for this case. The computation took place on a modern dual-core desktop and was completed in a matter of hours. It produced no maximal-Folkman graphs for $F_v(2_5; 4)$. We verified this computation by performing a 4-extension yielding a 3-independent-set[2] starting from a set of 11 vertex graphs and received the same result. This shows that $H_v^{max}(2_5; 4)$ contains no 15-vertex graphs. Therefore, we have computationally determined that $F_v(2_5; 4) > 15$.

3.2 Witnesses to the upper bound $F_v(2_5; 4) \leq 16$

As noted previously, Nenov [14] proved that $F_v(2_5; 4) \leq 16$. He did so by showing that $\mathscr{R}(4, 4; 16) \subseteq H_v(2_5; 4)$. We performed another extension and reduction process similar to the one we used to show $F_v(2_5; 4) > 15$ in order to determine if there were any other witnesses of $F_v(2_5; 4) \leq 16$. This time

[2]Specifically, the 4-extension was performed by a regular 3-extension using a 3-independent-set followed by extending by one more vertex which could have edges to vertices in that 3-independent-set.

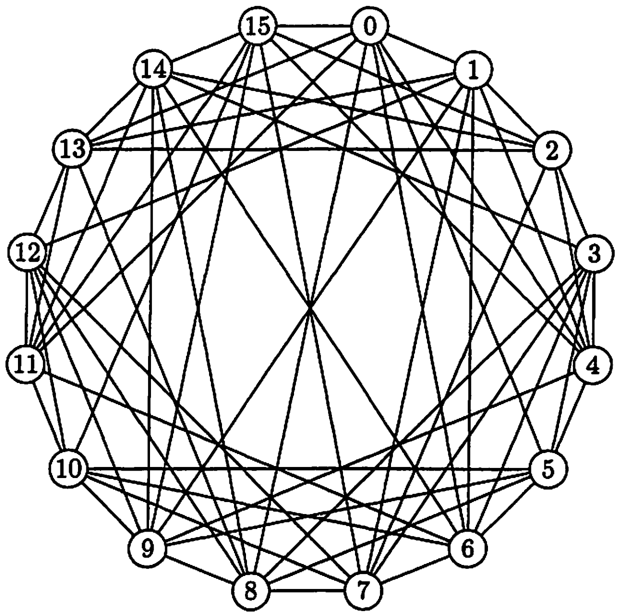

Figure 2: The 16-vertex witness $W_2 \in H_v(2_5; 4)$

however, we extended 12-vertex graphs by a $\overline{K_4}$. This extension produced no Folkman witnesses. Since all of these extended graphs had a $\overline{K_4}$, the only remaining graphs on 16 vertices to test were the graphs of $\mathcal{R}(4,4;16)$, and Nenov had already proved they were such witnesses. Therefore, 16-vertex graphs in $H_v(2_5;4)$ are exactly those in $\mathcal{R}(4,4;16)$. From [9] we know that $|\mathcal{R}(4,4;16)| = 2$. We will call these two graphs W_1 and W_2 and describe their properties.

3.2.1 Witness $W_1 \in H_v(2_5; 4)$

The first witness graph W_1 was described by Greenwood and Gleason [5] and is derived from the Paley graph of order 17. A Paley graph P_q for some prime q, $q \equiv 1 \mod 4$, is a graph on q vertices $\{0, \ldots, q-1\}$, in which two distinct vertices u and v are adjacent if and only if $|u-v| \equiv x^2 \mod q$, for some x. The witness W_1 is formed by removing any single vertex from P_{17}.

3.2.2 Witness $W_2 \in H_v(2_5; 4)$

The second witness graph W_2 is less well known. It is shown in Figure 2 with its vertices labelled for the sake of discussion. Its symmetrical properties are captured by eight graph automorphisms derivable from three automorphism generators.

1. `(0 7)(1 6)(2 5)(3 4)(8 15)(9 14)(10 13)(11 12)`
2. `(0 15)(1 14)(2 13)(3 12)(4 11)(5 10)(6 9)(7 8)`
3. `(1 6)(2 10)(3 12)(4 11)(5 13)(9 14)`

The first two generators are fairly straightforward: They describe graph symmetry about the horizontal and vertical axes.

The third graph automorphism generator of W_2 is more subtle and first requires an examination of the *orbits* of W_2, i.e. groups of vertices such that each vertex can be swapped with any of the other vertices in the group through one of the graph automorphims of W_2. The four orbits of W_2 are $O_1 = \{0, 7, 8, 15\}$, $O_2 = \{1, 6, 9, 14\}$, $O_3 = \{2, 5, 10, 13\}$, and $O_4 = \{3, 4, 11, 12\}$. The third automorphism generator operates on each of the orbits seperately: It fixes O_1 in place, flips O_2 about the horizontal axis, flips O_4 about the vertical axis, and flips O_3 about both the horizonal and vertical axes. It is worth noting that by composing these three generators, an automorphism of W_2 can be produced to fix any one of the four orbits while performing symmetrical flips on the rest.

4 Conclusion

By computationally proving $F_v(2_5; 4) > 15$ and using Nenov's upper bound of $F_v(2_5; 4) \leq 16$, we have proven that $F_v(2, 2, 2, 2, 2; 4) = 16$. We have also shown that the two graphs of $\mathcal{R}(4, 4; 16)$ are the sole 16-vertex witnesses of the upper bound.

References

[1] Vašek Chvátal. *Graphs and Combinatorics*, chapter The minimality of the Mycielski graph, pages 243–246. Lecture Notes in Mathematics. Springer, 1974.

[2] Jonathan P. Coles and Stanisław P. Radziszowski. Computing the Folkman number $F_v(2,2,3;4)$. *Journal of Combinatorial Mathematics and Combinatorial Computing*, 58:13–22, 2006.

[3] Jon Folkman. Graphs with monochromatic complete subgraphs in every edge coloring. *SIAM Journal on Applied Mathematics*, 18:19–24, 1970.

[4] Ronald L. Graham, Bruce L. Rothschild, and Joel H. Spencer. *Ramsey Theory*. Jon Wiley & Sons Inc., New York, 1990.

[5] Robert E. Greenwood and Andy M. Gleason. Combinatorial relations and chromatic graphs. *Canadian Journal of Mathematics*, 7(1):1–7, 1955.

[6] Tommy R. Jensen and Gordon F. Royle. Small graphs with chromatic number 5: A computer search. *Journal of Graph Theory*, 19(1):107–116, 1995.

[7] Tomasz Łuczak, Andrzej Ruciński, and Sebastian Urbański. On minimal Folkman graphs. *Discrete Math.*, 236(1-3):245–262, 2001.

[8] Brendan D. McKay. *nauty User's Guide (version 2.2)*. Computer Science Department of Australian National University, 2004. Software available at `http://cs.anu.edu.au/~bdm/nauty`.

[9] Brendan D. McKay and Stanisław P. Radziszowski. A new upper bound for the Ramsey number $R(5,5)$. *Australasian Journal of Combinatorics*, 5:13–20, 1992.

[10] Brendan D. McKay and Gordon Royle. Counts of unlabelled simple graphs. `http://cs.anu.edu.au/people/bdm/data/graphcounts.txt`, 2009.

[11] Nedyalko Nenov. *Ramsey graphs and some constants related to them.* PhD thesis, University of Sofia, 1980.

[12] Nedyalko Nenov. On a class of vertex Folkman numbers. *Serdica Mathematical Journal*, 28:219–232, November 2002.

[13] Nedyalko Nenov. On the vertex Folkman numbers $F_v(2,\ldots,2;q)$. *Serdica Mathematical Journal*, 35:251–272, November 2009.

[14] Nedyalko Nenov. On the vertex Folkman numbers $F_v(\underbrace{2,\ldots,2}_{r};r-1)$ and $F_v(\underbrace{2,\ldots,2}_{r};r-2)$. *eprint arXiv:0903.3151*, 2009.

[15] Stanisław P. Radziszowski. Small Ramsey numbers. *Electronic Journal of Combinatorics*, Dynamic Survey DS1:1–72, 1994–2009. Revision #12, http://www.combinatorics.org/Surveys/index.html.