On Balance Index Sets of Halin graphs of Stars and Double Stars

Alexander Nien-Tsu Lee
Department of Bioengineering
University of California at San Diego
La Jolla, California 92092

Sin-Min Lee
Department of Computer Science
San Jose State University
San Jose, CA 95192

Sheng-Ping Bill Lo
Cisco Systems, Inc.
170, West Tasman Drive
San Jose, CA 95134

Ho Kuen Ng
Department of Mathematics
San Jose State University
San Jose, CA 95192

Abstract Let G be a graph with vertex set V(G) and edge set E(G), and let A = {0, 1}. A labeling $f : V(G) \to A$ induces a partial edge labeling $f^* : E(G) \to A$ defined by $f^*((u, v)) = f(u)$ if and only if $f(u) = f(v)$ for each edge $(u, v) \in E(G)$. For $i \in A$, let $v_f(i) = \text{card}\{v \in V(G) : f(v) = i\}$ and $e_f(i) = \text{card}\{e \in E(G) : f^*(e) = i\}$. A labeling f of G is said to be friendly if $| v_f(0) - v_f(1) | \leq 1$. The ***balance index set*** of the graph G, BI(G), is defined as $\{| e_f(0) - e_f(1) | :$ the vertex labeling f is friendly$\}$. We determine the balance index sets of Halin graphs of stars and double stars.

1. Introduction

A. Liu , S.K. Tan and the second author [7] considered a new labeling problem of graph theory that was similar to Cahit's cordial graph labeling. Let G be a graph with vertex set V(G) and edge set E(G). A vertex labeling of G is a mapping f from V(G) into the set {0, 1}. For each vertex labeling f of G, we can define a partial edge labeling f* of G in the following way. For each edge (u, v) in E(G), where u, v are in V(G), we have

$$f^*((u, v)) = \begin{cases} 0 & \text{if } f(u) = f(v) = 0 \\ 1 & \text{if } f(u) = f(v) = 1 \end{cases}.$$

For those edges with two end vertices labeled by different values by f, f* doesn't label on them. Thus f* is a partial function from E(G) into the set {0, 1}, and we shall refer to f* as the induced partial function of f. Let $v_f(0)$ and $v_f(1)$ denote the number of vertices of G that are labeled by 0 and 1 under the mapping f respectively. Likewise, let $e_f(0)$ and $e_f(1)$ denote the number of edges of G that are labeled by 0 and 1 under the induced partial function f* respectively. With these notations, we now introduce the notion of the balance index set of a graph.

Definition 1.1. A labeling f of a graph G is said to be ***friendly*** if $| v_f(0) - v_f(1) | \le 1$. If, in addition, $| e_f(0) - e_f(1) | \le 1$ then G is said to be ***balanced***. See [2, 3, 7, 9].

Definition 1.2. The ***balance index set*** of the graph G, BI(G), is defined as $\{| e_f(0) - e_f(1) | :$ the vertex labeling f is friendly$\}$.

Balance index sets have been studied in [3, 4, 5, 6, 7,8,9,10].

Example 1. Figure 1 shows that the BI(G) = {0, 1, 2}.

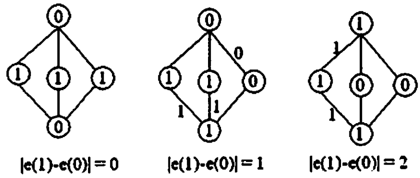


Figure 1.

Example 2. $BI(K_{3,3}) = \{0\}$.

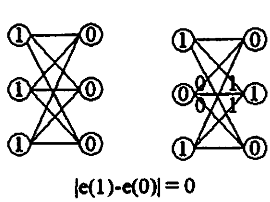


Figure 2.

We note that when determining the balance index set of a graph, we may fix an arbitrary vertex in the graph and label it 0. If another vertex labeling gives it the label 1, simply replace each vertex label by its complement. Then v(0) and v(1) are interchanged, and e(0) and e(1) are interchanged. Since we are only concerned with absolute values, interchanging v(0) and v(1), e(0) and e(1) would not make any difference.

The double star D(m, n) is a tree of diameter three such that there are m appended edges on one end of P_2 and n appended edges on the other end (Figure 3). Without loss of generality, we assume that $m \leq n$.

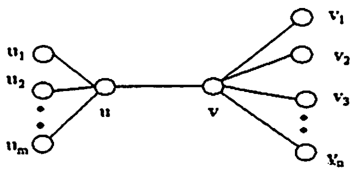


Figure 3.

In [6], it has been shown that

Theorem 1.1. The balance index set of the star St(n) is
(a) $\{k\}$, if $n = 2k + 1$ is odd, and
(b) $\{k - 1, k\}$, if $n = 2k$ is even.

Theorem 1.2. The balance index set of the double star D(m, n), where $m \leq n$, is
(a) $\{(n - m)/2, (n + m)/2\}$ if $m + n$ is even, and
(b) $\{(n - m - 1)/2, (n - m + 1)/2, (n + m - 1)/2, (n + m + 1)/2\}$ if $m + n$ is odd.

A Halin graph is a planar 3-connected graph that consists of a tree and a cycle connecting the end vertices of the tree. This class of graphs has been studied extensively in the literature. The purpose of this paper is to determine the balance index sets of two classes of Halin graphs, those of stars and double stars.

We questioned whether the numbers in BI(H(T)) for a tree T would form an arithmetic progression. Our results in this paper show that the conjecture is valid for Halin graphs of stars but not for Halin graphs of double stars.

2. On balance index sets of wheel graphs

For $n \geq 3$, the *wheel* on n vertices, W_n, is a graph with (n + 1) vertices x_0, x_1, x_2, ... , x_n, with x_0 having degree n and all the other vertices having degree 3. It is the cycle C_n with an additional vertex x_0 connected to each of its vertices. The vertex x_0 is called the hub, and the edges connecting the hub to the other vertices are called the spokes. Clearly, wheels are Halin graphs of stars.

We first cite a result in [6].

Lemma 2.1. For any (not necessarily friendly) vertex labeling of C_n, the differences $e(0) - e(1)$ and $v(0) - v(1)$ are the same.

Theorem 2.2. The balance index set of W_{2k-1} is $\{k - 2\}$ for all $k \geq 2$.
Proof. Without loss of generality, assume that the hub x_0 is labeled 0. Among the $(2k - 1)$ vertices on C_{2k-1}, $(k - 1)$ of them must have label 0, and the other k of them must have label 1. By Lemma 2.1, $e(0) - e(1) = v(0) - v(1) = -1$ on the cycle C_{2k-1}. Of the $(2k - 1)$ spokes, $(k - 1)$ of them have label 0, and the other k have no label. Thus $BI(W_{2k-1}) = \{k - 2\}$.□

Example 3. Figure 4 shows $BI(W_{2k-1})$ for k = 2, 3, and 4.

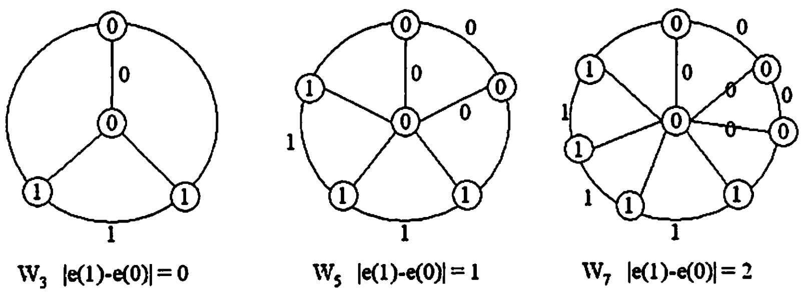


Figure 4.

Theorem 2.3. The balance index set of W_{2k} is
(1) $\{1, 2\}$ for k = 2, and
(2) $\{k - 3, k\}$ for $k \geq 3$.
Proof. Without loss of generality, assume that the hub x_0 is labeled 0.
Case 1: Among the 2k vertices on C_{2k}, k of them have label 0, and the other k of them have label 1. By Lemma 2.1, $e(0) - e(1) = v(0) - v(1) = 0$ on the cycle C_{2k}. Of the 2k spokes, k of them have label 0, and the other k have no label. This gives $e(0) - e(1) = k$.
Case 2: Among the 2k vertices on C_{2k}, $(k - 1)$ of them have label 0, and the other $(k + 1)$ of them have label 1. By Lemma 2.1, $e(0) - e(1) = v(0) - v(1) = -2$ on the cycle C_{2k}. Of the 2k spokes, $(k - 1)$ of them have label 0, and the other $(k + 1)$ have no label. This gives $e(0) - e(1) = k - 3$.□

Example 4. Figure 5 shows $BI(W_{2k})$ for k = 2, 3, and 4.

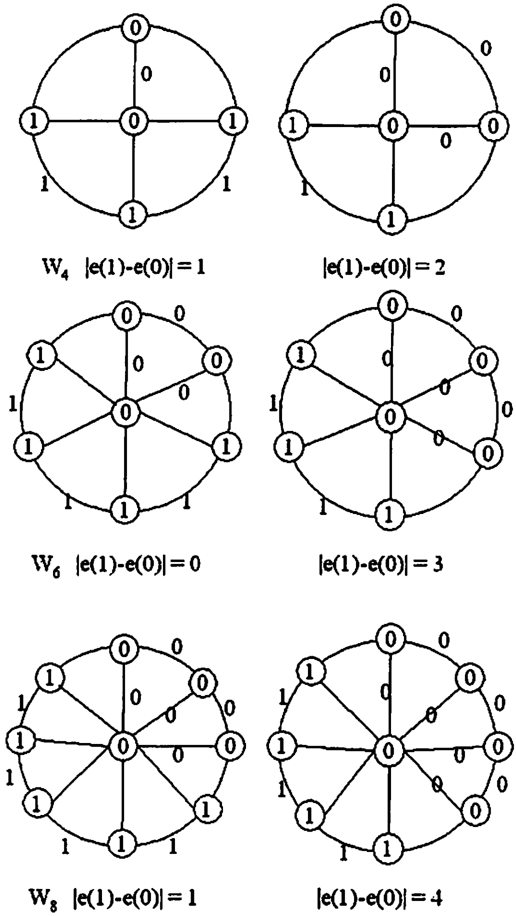


Figure 5.

3. On balance index sets of H(D(m, n)), with m + n even

Without loss of generality, let $m \le n$ in D(m, n).

<u>**Theorem 3.1.**</u> The balance index set of H(D(m, n)) is $\{(n - m)/2,\ ((n + m)/2) - 2\}$ if m + n is even.

Proof. Let m + n = 2k.

There are 2k + 2 vertices. Without loss of generality, assume that the vertex u has label 0.

First label the vertex v by 0. Assume j of the vertices $u_1, \ldots, u_m$ are labeled by 0, and the other (m – j) vertices are labeled by 1. By friendliness, (k – j – 1) of the vertices $v_1, \ldots, v_n$ are labeled by 0, and the other (n – k + j + 1) vertices are labeled by 1. For the double star, e(0) = k and e(1) = 0. For the cycle, e(0) – e(1) = v(0) – v(1) = (k – 1) – (m + n – k + 1) = –2. Thus for the entire graph, e(0) – e(1) = k – 2 = ((m + n)/2) – 2.

Then label the vertex v by 1. Assume j of the vertices $u_1, \ldots, u_m$ are labeled by 0, and the other (m – j) vertices are labeled by 1. By friendliness, (k – j) of the vertices $v_1, \ldots, v_n$ are labeled by 0, and the other (n – k + j) vertices are labeled by 1. For the double star, e(0) = j and e(1) = n – k + j, making e(0) – e(1) = k – n = (m – n)/2. For the cycle, e(0) – e(1) = v(0) – v(1) = k – (m + n – k) = 0. Thus for the entire graph, e(0) – e(1) = (m – n)/2, with absolute value (n – m)/2.□

Corollary 3.2. The balance index set of H(D(1, 2k + 1)) is {k – 1, k} for k ≥ 1.

Example 5. Figure 6 shows BI(H(D(1, 2k + 1))) for k = 1, 2.

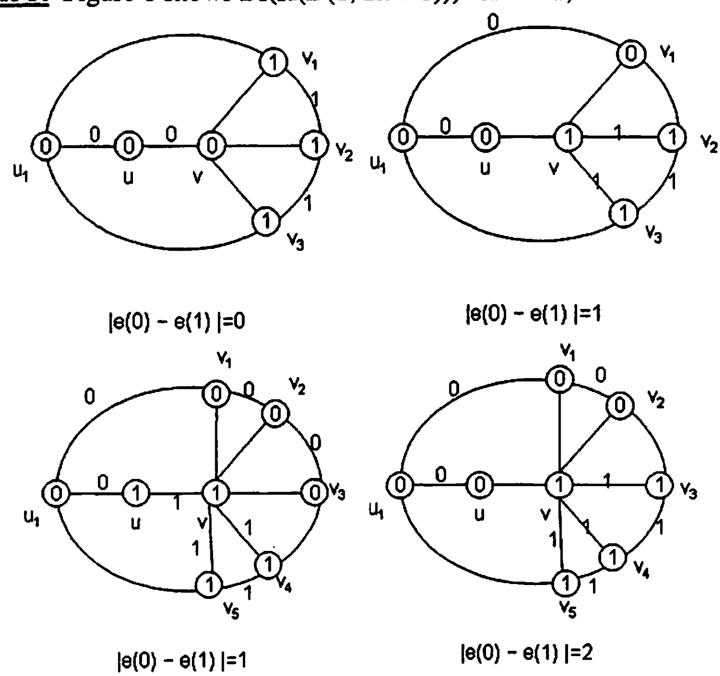


Figure 6.

Corollary 3.3. The balance index set of H(D(n, n)) is
(1) {0}, if n = 2 and
(2) {0, n – 2} for n ≥ 3.

Example 6. Figure 7 shows BI(H(D(n, n))) for n = 2, 3 and 5.

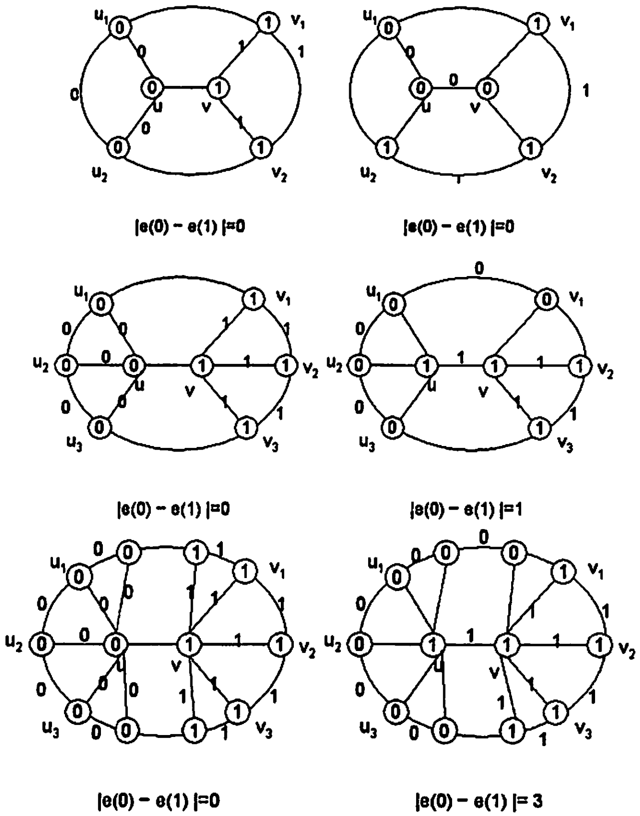


Figure 7.

Corollary 3.4. The balance index set of H(D(2, 2k)) is $\{k-1\}$ for $k \geq 1$.

Example 7. Figure 8 shows $BI(H(D(2, 2k))) = \{2\}$, for $k = 3$.

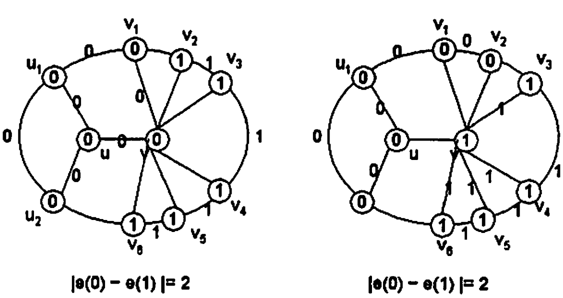
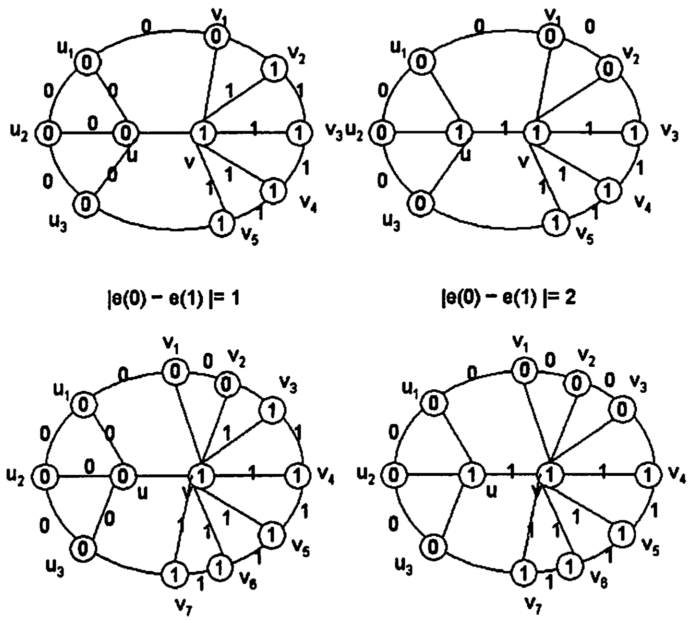


Figure 8.

Corollary 3.5. The balance index set of H(D(3, 2k + 1)) is {k – 1, k} for k ≥ 1.

Example 8. Figure 9 shows BI(H(D(3, 2k + 1))) for k = 2 and 3.



|e(0) – e(1) |= 1 |e(0) – e(1) |= 2

|e(0) – e(1) |= 2 |e(0) – e(1) |= 3

Figure 9.

4. On balance index sets of H(D(m, n)), with m + n odd

Without loss of generality, let m ≤ n in D(m, n).

Theorem 4.1. The balance index set of H(D(m, n)) is {| (m + n – 7)/2 |, (m + n – 1)/2, | (n – m – 3)/2 |, (n – m + 3)/2} if m + n is odd.

Proof. Let m + n = 2k + 1.

There are 2k + 3 vertices. Without loss of generality, assume that the vertex u is labeled by 0.

First we consider the case that v is labeled by 0. Assume j of the vertices u_1, … , u_m are labeled by 0, and the other (m – j) vertices are labeled by 1. By friendliness, either (k – j – 1) or (k – j) of the vertices v_1, … , v_n are labeled by 0, and the other (n – k + j + 1) or (n – k + j) vertices are labeled by 1 respectively. For the double star, e(0) = k or k + 1 and e(1) = 0, making e(0) – e(1) = k or k + 1 respectively. For the cycle, e(0) – e(1) = v(0) – v(1) = (k – 1) –

$(m + n - k + 1) = -3$, or $k - (m + n - k) = -1$ respectively. Thus for the entire graph, $e(0) - e(1) = (m + n - 7)/2$ or $(m + n - 1)/2$ respectively.

Then we consider the case that v is labeled by 1. Assume j of the vertices u_1, ..., u_m are labeled by 0, and the other $(m - j)$ vertices are labeled by 1. By friendliness, either $(k - j)$ or $(k - j + 1)$ of the vertices $v_1, \ldots, v_n$ are labeled by 0, and the other $(n - k + j)$ or $(n - k + j - 1)$ vertices are labeled by 1 respectively. For the double star, $e(0) = j$ and $e(1) = n - k + j$ or $n - k + j - 1$, making $e(0) - e(1) = k - n$ or $k - n + 1$ respectively. For the cycle, $e(0) - e(1) = v(0) - v(1) = k - (m + n - k) = -1$, or $(k + 1) - (m + n - k - 1) = 1$ respectively. Thus for the entire graph, $e(0) - e(1) = (m - n - 3)/2$ or $(m - n + 3)/2$.□

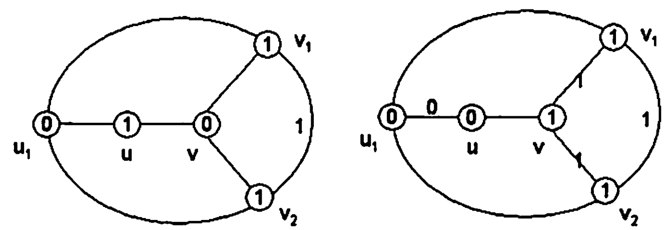
<u>**Corollary 4.2.**</u> The balance index set of H(D(1, 2k)) is

(1) {1, 2} for k = 1,

(2) {0, 1, 2, 3} for k = 2, and

(3) {k – 3, k – 2, k, k + 1} for $k \geq 3$.

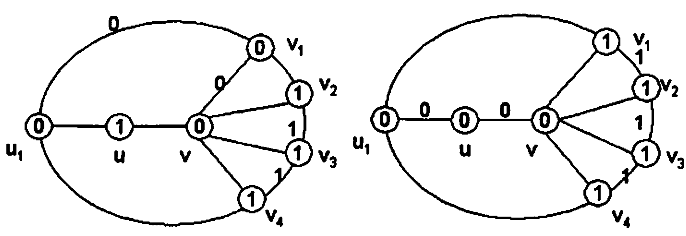
<u>**Example 9.**</u> Figure 10 shows that BI(H(D(1, 2))) = {1, 2}.



|e(0) – e(1) |= 1 |e(0) – e(1) |= 2

Figure 10.

<u>**Example 10.**</u> Figure 11 shows BI(H(D(1, 2k))) for k = 2 and 3.



|e(0) – e(1) |= 0 |e(0) – e(1) |= 1

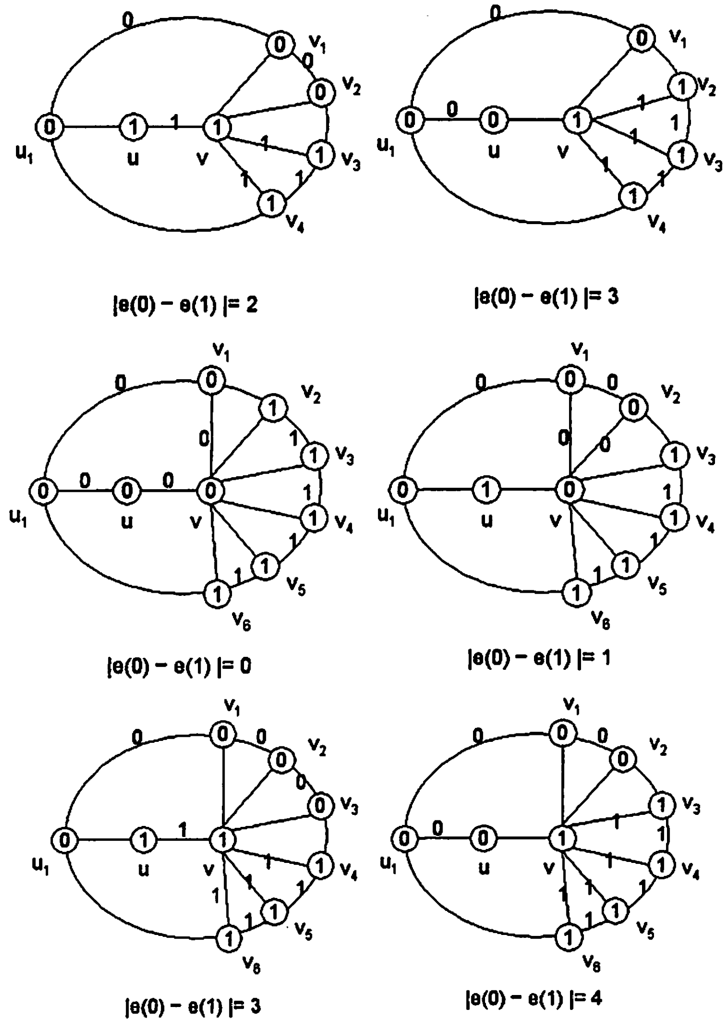


Figure 11.

Corollary 4.3. The balance index set of H(D(2, 2k + 1)) is

(1) $\{1, 2\}$ for $k = 1$, and

(2) $\{k - 2, k + 1\}$ for $k \geq 2$.

Example 11. Figure 12 shows BI(H(D(2, 2k + 1))) for $k = 1$.

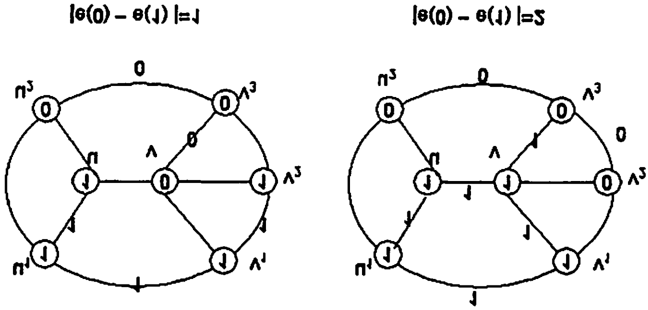


Figure 12.

Example 12. Figure 13 shows that BI(H(D(2, 5))) = {0, 3} and BI(H(D(2, 7))) = {1, 4}.

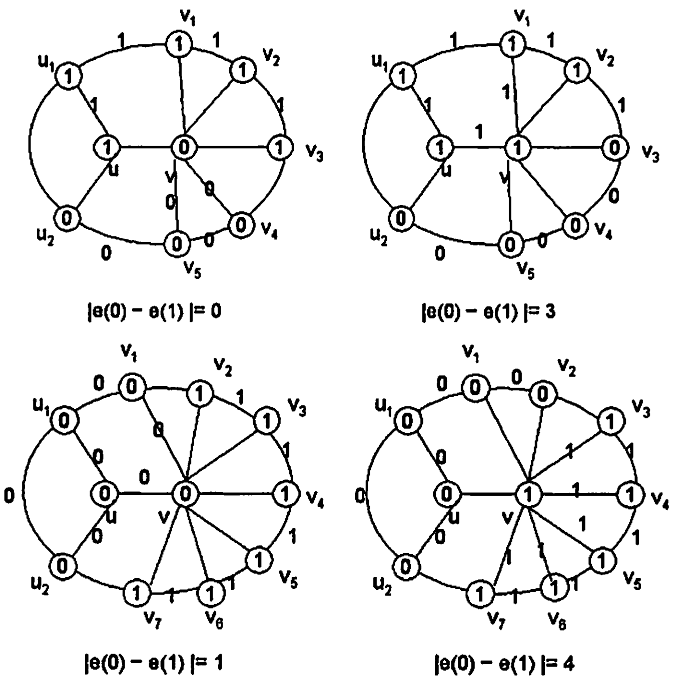


Figure 13.

Corollary 4.4. The balance index set of H(D(3, 2k)) is

(1) {0, 1, 2, 3} for k = 2, and

(2) {k − 3, k − 2, k, k + 1} for k ≥ 3.

Example 13. Figure 14 shows BI(H(D(3, 2k))) for k = 2.

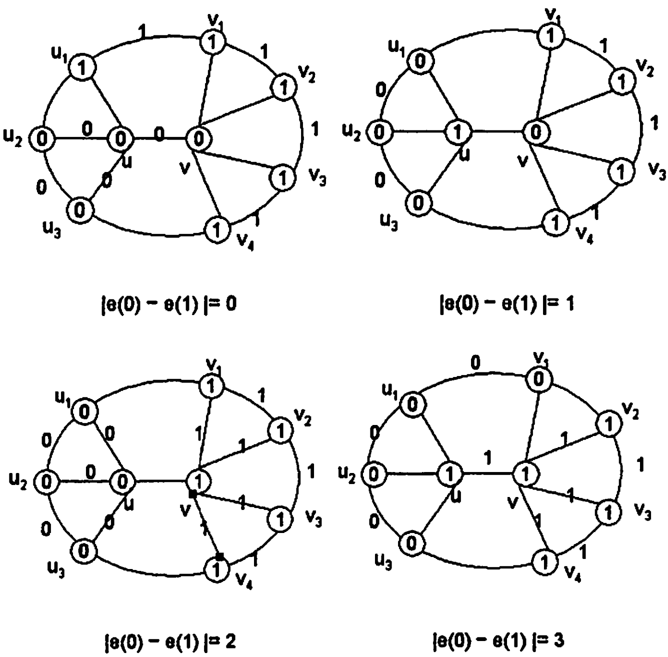
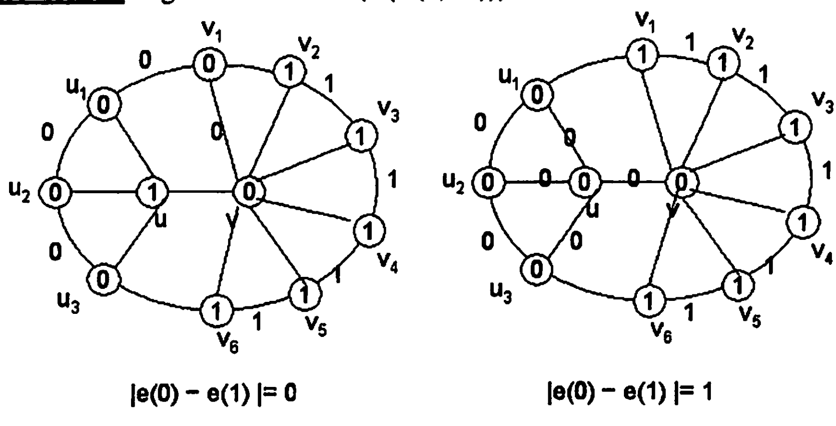
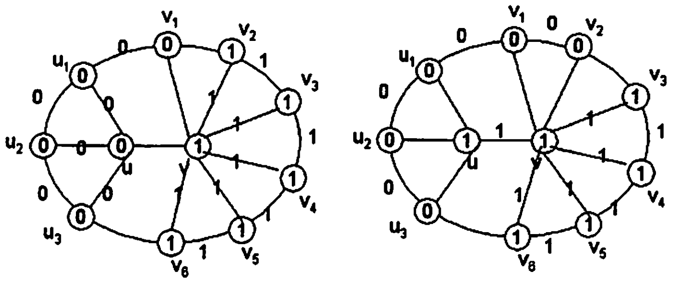


Figure 14.

Example 14. Figure 15 shows BI(H(D(3, 2k))) for k = 3.





|e(0) − e(1) |= 3 |e(0) − e(1) |= 4

Figure 15.

5. Unsolved problem

In this section, we address the following problem: For what m and n, where $m \le n$, will the numbers in BI(H(D(m, n)) form an arithmetic progression? The following result follows directly from Theorem 3.1.

Theorem 5.1. The numbers in the balance index set of H(D(m, n)) form an arithmetic progression if m + n is even.

However, the situation shows more variation if m + n is odd.

Theorem 5.2. If m + n is odd and $m \le n$, the numbers in the balance index set of H(D(m, n)) form an arithmetic progression

(1) for m = 1: if and only if n = 2, 4,
(2) for m = 2: for all odd n,
(3) for m = 3: if and only if n = 4,
(4) for m = 4: for no odd $n \ge 5$,
(5) for m = 5: if and only if $n \ne 6$,
(6) for m = 6: for no odd $n \ge 7$.
(7) for m = 7: for no even $n \ge 8$,
(8) for m = 8: for all odd $n \ge 11$ but not for n = 9,
(9) for m = 9: for no even $n \ge 10$,
(10) for m = 10: for no odd $n \ge 11$.

Proof. Use Theorem 4.1.

Theorem 5.3. If m + n is odd, $m \ge 9$, and $n \ge m + 1$, the numbers in the balance index set of H(D(m, n)) do not form an arithmetic progression.

Proof. For n = m + 1, BI(H(D(m, n))) = {m − 3, m, 1, 2}. Since $m \ge 9$, the result is obvious. For $n \ge m + 3$, BI(H(D(m, n))) = {(m + n − 7)/2, (m + n − 1)/2, (n − m − 3)/2, (n − m + 3)/2}. An ascending order, the balance indices are (n −

m – 3)/2, (n – m + 3)/2, (m + n – 7)/2, (m + n – 1)/2, with differences 3, m – 5, and 3. Again, since $m \geq 9$, the result follows.

Finally, we propose the following unsolved problem for future study.
Problem. Characterize the trees T such that the numbers in BI(H(T)) form an arithmetic progression.

References

[1] C-C. Chou and S-M. Lee, On the balance index sets of the amalgamation of complete graphs and stars, manuscript.

[2]R.Y. Kim, S-M. Lee and H.K. Ng, On balancedness of some families of graphs, *Journal of Combinatorial Mathematics and Combinatorial Computing*. 66 (2008) 3-16.

[3] H. Kwong and S-M. Lee, On balance index sets of chain sum and amalgamation of generalized theta graphs, *Congressus Numerantium* 187 (2007), 21-32.

[4]H. Kwong, S.M. Lee,S-P. Lo, H-H. Su, and Y-C. Wang On Balance Index Sets of L-Products with Cycles and Complete Graphs,*Journal of Combinatorial Mathematics and Combinatorial Computing* 70 (2009), 85-96.

[5] H. Kwong, S-M. Lee, and D.G. Sarvate, On balance index sets of one-point unions of graphs, *Journal of Combinatorial Mathematics and Combinatorial Computing*. 66 (2008) 113-127..

[6] A.N-T. Lee, S-M. Lee and H.K. Ng, On balance index sets of graphs, *Journal of Combinatorial Mathematics and Combinatorial Computing*. 66 (2008) 135-150.

[7] S.M. Lee, A. Liu and S.K. Tan, On balanced graphs, *Congressus Numerantium* 87 (1992), 59-64.

[8] S.M. Lee, Ho Kuen Ng and Siu-Ming Tong On the balance index set of the chain-sum graphs of cycles, *Utilitas Mathematica*. 77(2008) 113-123.

[9] S.M.Lee, Hsin-Hao Su andYing-Chin Wang,On balance index sets of disjoint union graphs, *Congresses Numerantium* 199 (2009), 97 – 120

[10] S.M. Lee, Y.C. Wang and Y.H. Wen, On the balance index sets of the (p; p+1)-graphs, *J. Combin. Math. Combin. Comput.*, 62 (2007),193-216.