On Edge-Balance Index Sets of Flux Capacitors and L-products of Stars with Cycles

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Abstract

Let G be a simple graph with vertex set $V(G)$ and edge set $E(G)$, and let $\mathbb{Z}_2 = \{0,1\}$. Any edge labeling f induces a partial vertex labeling $f^+ : V(G) \to \mathbb{Z}_2$ assigning 0 or 1 to $f^+(v)$, v being an element of $V(G)$, depending on whether there are more 0-edges or 1-edges incident with v, and no label is given to $f^+(v)$ otherwise. For each $i \in \mathbb{Z}_2$, let $v_f(i) = |\{v \in V(G) : f^+(v) = i\}|$ and let $e_f(i) = |\{e \in E(G) : f(e) = i\}|$. An edge-labeling f of G is said to be edge friendly if $|e_f(0) - e_f(1)| \leq 1$. The edge-balance index set of the graph G is defined as $EBI(G) = \{|v_f(0) - v_f(1)| : f \text{ is edge-friendly.}\}$. In this paper, we investigate and present results concerning the edge-balance index sets of flux capacitors and L-products of stars with cycles.

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1 Introduction

In [5], Kong and Lee considered a new labeling problem of graph theory. Let G be a simple graph with vertex set $V(G)$ and edge set $E(G)$, and let $\mathbb{Z}_2 = \{0,1\}$. An edge labeling $f : E(G) \to \mathbb{Z}_2$ induces a vertex partial labeling $f^+ : V(G) \to \mathbb{Z}_2$ defined by $f^+(v) = 0$ if the number of edges labeled 0 incident on v is more than the number of edges labeled 1 incident on v, and $f^+(v) = 1$ if the number of edges labeled 1 incident on v is more than the number of edges labeled 0 incident on v. Note that $f^+(v)$ is not defined if the number of edges labeled by 0 is equal to the number of edges labeled 1. For $i \in \mathbb{Z}_2$, let $v_f(i) = |\{v \in V(G) : f^+(v) = i\}|$, and let $e_f(i) = |\{e \in E(G) : f(e) = i\}|$.

With these notations, we now introduce the notion of an edge-balanced graph.

Definition 1. An edge labeling f of a graph G is said to be ***edge-friendly*** if $|e_f(0) - e_f(1)| \le 1$. A graph G is said to be an ***edge-balanced*** graph if there is an edge-friendly labeling f of G satisfying $|v_f(0) - v_f(1)| \le 1$.

Chen, Lee, et al in [1] proved that all connected simple graphs, except the star $K_{1,2k+1}$ where $k \ge 0$, are edge-balanced.

Definition 2. The ***edge-balance index set*** of the graph G, EBI(G), is defined as $\{|v_f(0) - v_f(1)| :$ the edge labeling f is edge-friendly.$\}$.

We will use $v(0)$, $v(1)$, $e(0)$, $e(1)$ instead of $v_f(0)$, $v_f(1)$, $e_f(0)$, $e_f(1)$, provided there is no ambiguity.

Example 1. EBI (nK_2) is $\{0\}$ if n is even and $\{2\}$ if n is odd.

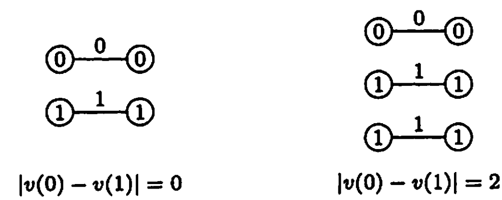

Figure 1: The edge-balance index set of $2K_2$ and $3K_2$

For any $n \ge 1$, we denote the tree with $n+1$ vertices of diameter two by St(n). The star has a center c and n appended edges from c.

Example 2. The edge-balance index set of the star $\mathrm{St}(n)$ is

$$\mathrm{EBI}\,(\mathrm{St}(n)) = \begin{cases} \{0\} & \text{if } n \text{ is even,} \\ \{2\} & \text{if } n \text{ is odd.} \end{cases}$$

Example 3. In [12], Lee, Lo and Tao showed that

$$\mathrm{EBI}\,(P_n) = \begin{cases} \{2\} & \text{if } n \text{ is 2,} \\ \{0\} & \text{if } n \text{ is 3,} \\ \{1,2\} & \text{if } n \text{ is 4.} \\ \{0,1\} & \text{if } n \text{ is odd and greater than 3,} \\ \{0,1,2\} & \text{if } n \text{ is even and greater than 4.} \end{cases}$$

Figure 2 shows the EBI of P_3 and P_4.

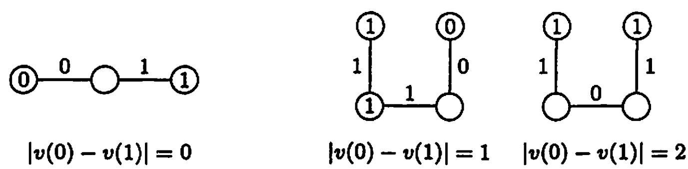

Figure 2: The edge-balance index set of P_3 and P_4

Example 4. After an exhaustive search, we see that the edge-balance index set of a tree with six vertices is $\{0, 1, 2\}$. Figure 3 shows three edge-balance indexes.

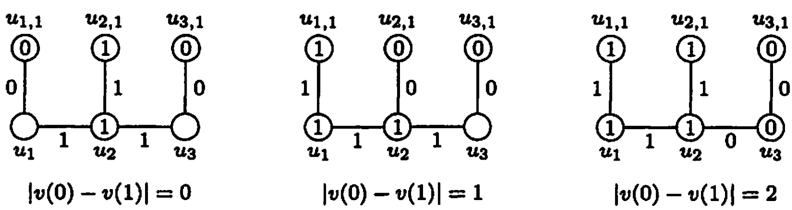

Figure 3: The edge-balance index set of a tree with six vertices

The edge-balance index sets can be viewed as the dual of balance index sets. The balance index sets of graphs were considered in [4, 6, 8, 9, 10, 11, 13, 15]. Let G be a simple graph with vertex set $V(G)$ and edge set $E(G)$, and let $\mathbb{Z}_2 = \{0, 1\}$. A labeling $f : V(G) \to \mathbb{Z}_2$ induces an edge partial labeling $f^* : E(G) \to A$ defined by $f^*(vw) = f(v)$, if and only if $f(v) = f(w)$

for each edge $vw \in E(G)$. For $i \in \mathbb{Z}_2$, let $v_f(i) = |\{v \in V(G) : f(v) = i\}|$ and $e_{f^*}(i) = |\{e \in E(G) : f^*(e) = i\}|$. A labeling f of a graph G is said to be ***friendly*** if $|v_f(0) - v_f(1)| \leq 1$. If $|e_f(0) - e_f(1)| \leq 1$ then G is said to be ***balanced***. The ***balance index set*** of the graph G, $\mathrm{BI}(G)$, is defined as $\{|e_f(0) - e_f(1)| : \text{the vertex labeling } f \text{ is friendly}\}$.

Edge-balance index sets of trees, flower graphs, wheels, fans, and $(p, p+1)$-graphs were considered in [2, 3, 7, 12, 14].

Let H be a connected graph with a distinguished vertex s. Construct a new graph $G \times_L (H, s)$ as follows: take $|V(G)|$ copies of (H, s) and identify each vertex of G with s of a single copy of H. We call the resulting graph the ***L-product*** of G and (H, s).

A ***flux capacitor*** graph is composed of two different types of graphs (a star graph and a cycle). A flux capacitor graph, $\mathrm{FC}(n, m)$, is a star graph $\mathrm{St}(n)$ where on each outer vertex there is a C_m graph.

In this paper, exact values of the edge-balance index sets of flux capacitor graphs, $\mathrm{FC}(n, m)$, and L-product of stars with cycles, $\mathrm{St}(n) \times_L C_m$, are presented.

2 On Edge-Balance Index Set of Flux Capacitor Graphs $\mathrm{FC}(n, m)$

Since a cycle must have at least three edges, we have $m \geq 3$. Since a star graph must have at least one edge, we have $n \geq 1$. In general, the Flux Capacitor graph has $e(\mathrm{FC}(n, m)) = n(m + 1)$ edges.

2.1 The Highest Edge-balance Index of $\mathrm{FC}(n, m)$

To find the edge-balance index set of $\mathrm{FC}(n, m)$, we determine the highest edge-balance index first. The following notations and propositions are borrowed from [2]. You can also find them in [3].

Notation 1. Let C_n be a cycle with a vertex set $\{c_1, c_2, \ldots, c_n\}$. Let f be an edge labeling on C_n (not necessarily edge-friendly). We denote the numbers of edges labeled 0 or 1 by f by $e_C(0)$ or $e_C(1)$, respectively. We also denote the number of vertices labeled 0, 1, or not labeled by f^+ by $v_C(0)$, $v_C(1)$, or $v_C(\times)$, respectively.

If we add an edge to a vertex in a cycle to turn it into degree three, then there are two possible cases:

A If the vertex was already labeled, then the label of the vertex is not changed after adding an edge because at least two edges are labeled by the same number.

B If the vertex was not labeled then the label of the vertex is the same as the label assigned to the new edge.

For later reference, we call these rules as Rule A and Rule B.

Proposition 2.1. *In a cycle C_n with an edge labeling f (not necessarily edge-friendly), assume that $v_C(\times) = 2k > 0$. Then*

$$v_C(1) = e_C(1) - k.$$

Proposition 2.2. *In a cycle C_n with an edge labeling f (not necessarily edge-friendly), assume that $v_C(\times) = 2k > 0$. Then*

$$v_C(0) = n - e_C(1) - k.$$

We note here that when $v_C(\times) = 0$, i.e., either $e_C(0) = n$ or $e_C(1) = n$, the above propositions are still true.

For a finite disjoint union of cycles, we can calculate $v_C(0)$ and $v_C(1)$ for each cycle C and then add all up to get

Theorem 2.3. *In a finite disjoint union of cycles $\cup_i C^i_{n_i}$ (for notational convenience, we still call it C) with an edge labeling f (not necessarily edge-friendly), we have*

$$v_C(0) - v_C(1) = \sum_i n_i - 2e_C(1). \tag{1}$$

This equation suggests that the edge-balance index of an edge friendly labeling of $\mathrm{FC}(n, m)$ is determined by

1. the label of the center s_0,
2. the equation (1), and
3. the labels of the vertices shared by both the star and cycles.

Since the label of a vertex shared by both the star and a cycle is governed by Rule A and B, to find the highest edge-balance index in the form of $v(0) - v(1)$ of $\mathrm{FC}(n, m)$, we follow the same principle in the proof of 2.4 to maximize $v(0)$ and minimize $v(1)$. The equation (1) tells us that the smaller $e_C(1)$ gives us the larger $v_C(0) - v_C(1)$. This suggests that a smaller $e_C(1)$ is more likely to produce a larger edge-balance index. Thus, we label all the edges of the star 1 to get as few number of 1-edges left for cycles as possible. It uses n 1-edges to produce only one 1-vertex in the center. To further minimize $v(1)$, we try not to produce any 1-vertex in the vertex incident to the star. By Rule A and B, we have to label 0 to the two edges of a cycle incident to the vertex of the star. Since all edges in the star labeled 1, by Rule A, it will not affect the labeling of its adjacent vertex. By Theorem 2.3, $v_C(0) - v_C(1) = nm - 2e_C(1)$, where $e_C(1)$ is the number of 1-edges in cycles. With the center labeled 1, we can conclude that the edge-balance index of an edge friendly labeling of $\mathrm{FC}(n, m)$ is

$$v(0) - v(1) = nm - 2e_C(1) - 1, \tag{2}$$

where $e_C(1)$ is the number of 1-edges in cycles.

Generally speaking, this method works when the highest edge-balanced index in the form of $v(0) - v(1)$ is greater than 0 because we assume that $e(0) \geq e(1)$ which usually generates more 0-vertices then 1-vertices and we also avoid to create 1-vertex unless it is not avoidable. But, since we start with labeling the center 1, in $\mathrm{FC}(1, 3)$ case, it leads to more 1-vertices than 0-vertices due to only four edges to use.

2.2 On Edge-Balance Index Set of $\mathbf{FC}(n, 3)$

Theorem 2.4. *The edge-balance index set of $FC(n, 3)$ is*

$$EBI(FC(n,3)) = \begin{cases} \{0, 1, \ldots, n-1\} & \text{if } n \geq 2, \\ \{0, 1\} & \text{if } n = 1. \end{cases}$$

Proof. In particular, $e(\mathrm{FC}(n, 3)) = 4n$. Since it is even, $e(0) = e(1) = 2n$. As we discussed in the end of section 2.1, to get the highest edge-balance index, we label all edges of $\mathrm{St}(n)$ 1. Then, there are n 1-edges left for cycles, i.e., $e_C(1) = n$. By Equation (2), the highest edge-balance index is

$$v(0) - v(1) = 3n - 2e_C(1) - 1 = n - 1.$$

Knowing this, we can outline a procedure to find the maximum difference in vertices for any graph $\mathrm{FC}(n, 3)$ where $v(0) \geq v(1)$. First, we label all edges in $\mathrm{St}(n)$ by n 1-edges. This causes the center s_0 labeled 1 and leaves us with n 1-edges label. The remaining n 1-edges can be placed in the outer edges of each C_3 so that they do not produce any additional vertices labeled 1. We then label all n degree 3 vertices in the C_3's by 0. This will use all $2n$ 0-edges to produce exactly n 0-vertices. The remaining vertices will stay unlabeled because they will each incident to one 1-edge and one 0-edge. This labeling gives us one 1-vertex and n 0-vertices, implying that the edge-balance index is $n-1$. This is the highest edge-balance index in the form of $v(0)-v(1)$.

If any 1-edge was switched with a 0-edge, no more 0-vertices could be produced without creating at least one 1-vertex. A 0-edge can easily be switched with any 1-edge causing the edge-balance index to decrease in a number of different ways. However, we will look at the method for causing it to decrease by 1. If on one cycle the 1-edge is switched with either 0-edges, another vertex labeled 0 is added and the existing 0-vertex on that cycle is changed to a 1-vertex. Thus $v(0)$ stays the same and $v(1)$ is increased by 1. Assuming $v(0) > v(1)$ this will cause the edge-balance index to decrease by 1. The edge-balance index of this labeling is now $v(0)-v(1) = n-2$.

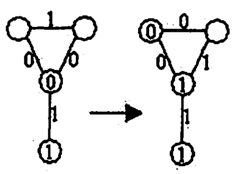

Figure 4: Transformation of a vertex of a star: increases $v(1)$ by 1, decreases the edge-balance index by 1

Using this process multiple times results in the full spectrum of EBI(FC$(n,3)$). We can go through each C_3 and decrease the difference in vertices by 1 with each C_3 changed. Since there are n cycles on the graph, we can decrease the edge-balance index by one n times, and end up with edge-balance indexes all the way to 0. So EBI(FC$(n,3)$) $= \{0,1,2,\ldots,n-1\}$.

When $n=1$, this switching creates an edge-balance index -1 in the form of $v(0)-v(1)$. Therefore, EBI(FC$(n,3)$) $= \{0,1\}$. Figure 4 also demonstrates two edge-balanced indexes of FC$(n,3)$. □

Example 5. Figure 5 demonstrates edge friendly labelings for EBI(FC$(3,3)$) $= \{0,1,2\}$.

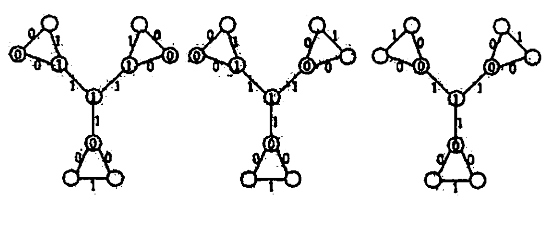

Figure 5: EBI(FC$(3,3)$)

2.3 On Edge-Balance Index Set of FC$(n,4)$

Theorem 2.5. *The edge-balance index set of $FC(n,4)$ is*

$$EBI(FC(n,4)) = \begin{cases} \{0,1,\ldots,n-1\} & \text{if } n \text{ is even,} \\ \{0,1,\ldots,n\} & \text{if } n \text{ is odd.} \end{cases}$$

Proof. A similar argument in the proof of theorem 2.4 is used here depending of the parity of n.

Case 1. EBI(FC$(n,4)$) $= \{0,1,\ldots,n-1\}$ when n is even.

Since n is even, we can assume $n = 2k$, where $k \in \mathbb{N}$. The number of edges in FC$(n,4)$ is $e(\text{FC}(n,4)) = 5n = 10k$. Since it is even, for an edge labeling to be friendly, we have $e(0) = e(1) = 5k$.

We first label the graph by using $2k$ 1-edges to label all the edges of St$(2k)$ resulting in a label of 1 for the center s_0. Then $4k$ 0-edges are placed on each of the $2k$ degree 3 vertices resulting in $2k$ 0-vertices. This leaves $3k$ 1-edges and k 0-edges left to be placed. By Equation (2), the highest edge-balance index is $(2k)4 - 2(3k) - 1 = 2k - 1 = n - 1$. Therefore, EBI(FC$(n,4)$) $\subseteq \{0,1,\ldots,n-1\}$.

To finish creating an edge friendly labeling to achieve the highest edge-balance index, we split the remaining $2k$ C_4's into two equal groups; $C_4(+)$ and $C_4(-)$. Since this split is for labeling purposes only and all C_4's are so far all labeled the same, it is inconsequential which cycles are placed in which group as long as there are k cycles in each group.

In every $C_4(+)$, the two remaining unlabeled edges are labeled by a 0-edge and a 1-edge, creating one 0-vertex. This uses k 0-edges and k 1-edges, leaving no more 0-edges and $2k$ 1-edges. In every $C_4(-)$, the two remaining unlabeled edges are labeled by 1-edges, creating one 1-vertex. This uses the remaining $2k$ 1-edges and creates k more 1-vertices. This edge-friendly labeling results $v(0) - v(1) = 3k - (k+1) = 2k - 1 = n - 1$, the highest edge-balance index.

For each $C_4(+)$, an 1-edge can be switched with a 0-edge adjacent to the vertex of the star. This adds another 0-vertex to the cycle, but also changes the existing degree 3 0-vertex on the cycle to a 1-vertex. Since $v(0)$ stays the same and $v(1)$ is increased by 1, by switching these edges, the edge-balance index is decreased by 1.

For each $C_4(-)$, any one of the 1-edges can be switched with a 0-edge. This changes an 1-vertex into an unlabeled vertex and a 0-vertex into an 1-vertex. Since $v(0)$ is decreased by 1 and $v(1)$ stays the same, the edge-balance index can again be decreased by 1.

Since there are $2k = n$ C_4's, we have enough number of $C_4(+)$ or $C_4(-)$ to create an edge-balance index from $n-2$ to 0 by switching edges of C_4's. Thus, EBI(FC$(n,4)$) $= \{0,1,\ldots,n-1\}$.

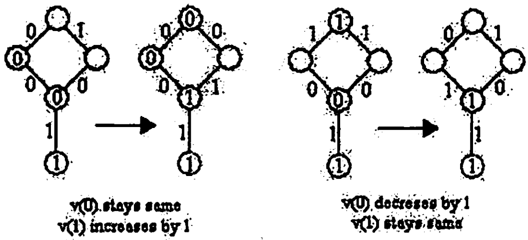

Figure 6: Two types of transformation of FC$(n, 4)$

<u>**Case 2.**</u> EBI(FC$(n, 4)$) $= \{0, 1, \ldots, n\}$ when n is odd.

Since n is odd, we can assume that $n = 2k+1$, where $k \in \mathbb{N}$. The number of edges in FC$(n, 4)$ is $e(\text{FC}(n, 4)) = 5n = 5(2k+1) = 10k+5 = 2(5k+2)+1$. Since it is odd, for an edge labeling to be friendly, we have $e(0) = 5k + 3$ and $e(1) = 5k + 2$.

By the same way of labeling as Case 1, we can create an edge friendly labeling such that the edge-balance index is $v(0)-v(1) = (3k+2)-(k+1) = 2k+1 = n$ with $k+1$ cycles of type $C_4(+)$ and k cycles of type $C_4(-)$. By Equation (2), we have

$$(2k + 1)4 - 2((5k + 2) - (2k + 1)) - 1 = 2k + 1 = n.$$

This confirms the creation of the highest edge-balance index of FC$(n, 4)$.

Since there are $2k + 1 = n$ cycles, we have enough number of $C_4(+)$ or $C_4(-)$ to create an edge-balance index from $n - 1$ to 0 by switching edges of C_4's as Case 1. Thus, EBI(FC$(n, 4)$) $= \{0, 1, \ldots, n\}$. □

Example 6. Figure 7 demonstrates edge friendly labelings for EBI(FC$(2, 4)$) $= \{0, 1\}$.

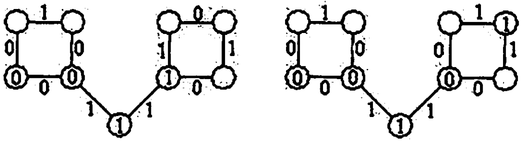

Figure 7: EBI(FC$(2, 4)$) $= \{0, 1\}$

Example 7. Figure 8 demonstrates edge friendly labelings for EBI(FC$(3, 4)$) $= \{0, 1, 2, 3\}$.

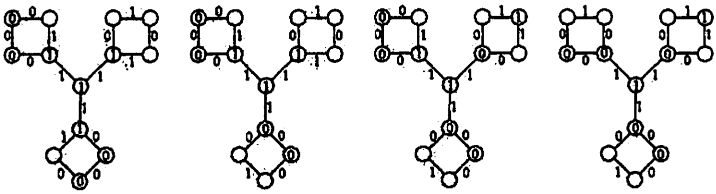

Figure 8: EBI(FC(3, 4)) = {0, 1, 2, 3}

2.4 On Edge-Balance Index Set of FC(n, m)

Theorem 2.6. *For $m \geq 5$, the edge-balance index set of FC(n, m) is*

$$EBI(FC(n,m)) = \begin{cases} \{0,1,\ldots,n-1\} & \text{if } m \text{ is odd,} \\ \{0,1,\ldots,n-1\} & \text{if } n \text{ is even and } m \text{ is even,} \\ \{0,1,\ldots,n\} & \text{if } n \text{ is odd and } m \text{ is even.} \end{cases}$$

Proof of Case 1. EBI(FC(n, m)) = $\{0, 1, \ldots, n-1\}$ when $m \geq 5$ is odd

Since m is odd, we assume that $m = 2k + 1$, where $k \in \mathbb{N}$. The number of edges is $n(m + 1) = n(2k + 1 + 1) = n(2k + 2) = 2n(k + 1)$. Since it is even, we have $e(0) = e(1) = n(k + 1)$.

We start with labeling all the edges of the St(n) 1. This uses n 1-edges and leaves $n(k + 1) - n$ 1-edges for cycles. By Equation (2), we have

$$n(2k+1) - 2(n(k+1) - n) - 1 = n - 1.$$

Each cycle then gets labeled $0, 1, 0, 1, \ldots$ starting at an edge adjacent to the vertex of the star and ending with the other edges adjacent to the vertex. Since m is odd, each cycle will have two 0-edges adjacent to the vertex of the star. Thus, all n vertices incident to the star are labeled 0 and all other vertices of cycles are not labeled. Therefore, this edge-friendly labeling creates the highest edge-balance index $n - 1$.

Since there are n cycles on the graph, we have enough cycles to create an edge-balance index from $n-1$ to 0 by switching edges of cycles as the proof of the Theorem 2.4. Thus, the edge-balance index set is $\{0, 1, \ldots, n - 1\}$. □

Example 8. Figure 9 demonstrates edge friendly labelings for EBI(FC(4, 7)) = $\{0, 1, 2, 3\}$.

Proof of Case 2. EBI(FC(n, m)) = $\{0, 1, \ldots, n-1\}$ when $m \geq 5$ and $n \geq 2$ are both even.

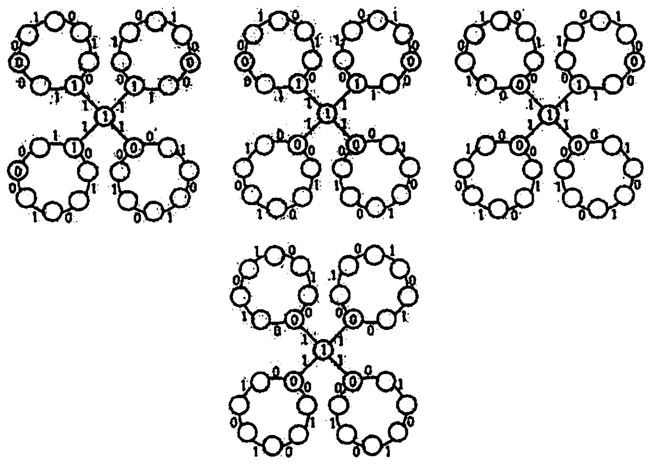

Figure 9: $\text{EBI}(\text{FC}(4,7)) = \{0, 1, 2, 3\}$

Since m is even, assume that $m = 2k$ where $k \in \mathbb{N}$. Since n is even, assume that $n = 2j$ where $j \in \mathbb{N}$. The total number of edge in the graph is $e(\text{FC}(n, m)) = n(m+1) = 2j(2k+1)$. Since it is even, we have $e(0) = e(1) = j(2k+1)$.

A similar labeling of Theorem 2.5 is used here. We start with labeling all the edges of the $\text{St}(n)$ 1. This uses n 1-edges and leaves $j(2k+1) - (2j)$ 1-edges for cycles. By Equation (2), we have

$$(2j)(2k) - 2(j(2k+1) - (2j)) - 1 = n - 1.$$

Therefore, the edge-balance index set is a subset of $\{0, 1, \ldots, n-1\}$.

To label the rest of the graph to achieve the highest edge-balance index, the C_m's are split into two types; $C_m(+)$ and $C_m(-)$. There are j C_m's in each group and, at this moment, it doesn't matter which group they are put into as the groups are just two different ways of labeling. In each of the j $C_m(+)$, $k-1$ 0-edges are strung together starting with one of the edges adjacent to a degree 2 vertex adjacent to an edge already labeled 0 and continuing until all $k-1$ 0-edges are used. This will result in $j(k-1)$ 0-vertices using $j(k-1)$ 0-edges with $j(k-2)$ 0-edges remaining. The remaining $k-1$ edges in each $C_m(+)$ will be labeled by 1-edges resulting in $j(k-2)$ 1-vertices. This leaves jk 1-edges remaining. In the j $C_m(+)$'s, $v(0) - v(1) = j(k-1) - j(k-2) = j$.

In each of the j $C_m(-)$, $k-2$ 0-edges are strung together starting with one of the edges adjacent to a degree 2 vertex adjacent to an edge already labeled 0 and continuing until all $k-2$ 0-edges are used. This will result in $j(k-2)$ 0-vertices using the remaining $j(k-2)$ 0-edges. The remaining k edges in each $C_m(-)$ are labeled by 1-edges resulting in $j(k-1)$ 1-vertices. This process uses the remaining jk 1-edges and completes our friendly labeling. In the j $C_m(-)$'s, $v(0)-v(1)=j(k-2)-j(k-1)=-j$.

All the vertices in $C_m(+)$ and $C_m(-)$ cancel out, leaving only the vertices in $\mathrm{St}(n)$. Since only the center s_0 was labeled 1 while every other vertex is labeled 0, we have created the highest edge-balance index $n-1$.

Since there are n cycles on the graph, we have enough cycles to create an edge-balance index from $n-1$ to 0 by switching edges of cycles as the proof of the Theorem 2.5. Thus, the edge-balance index set is $\{0,1,\ldots,n-1\}$. □

Example 9. Figure 10 demonstrates edge friendly labelings for $\mathrm{EBI}(\mathrm{FC}(4,8)) = \{0,1,2,3\}$.

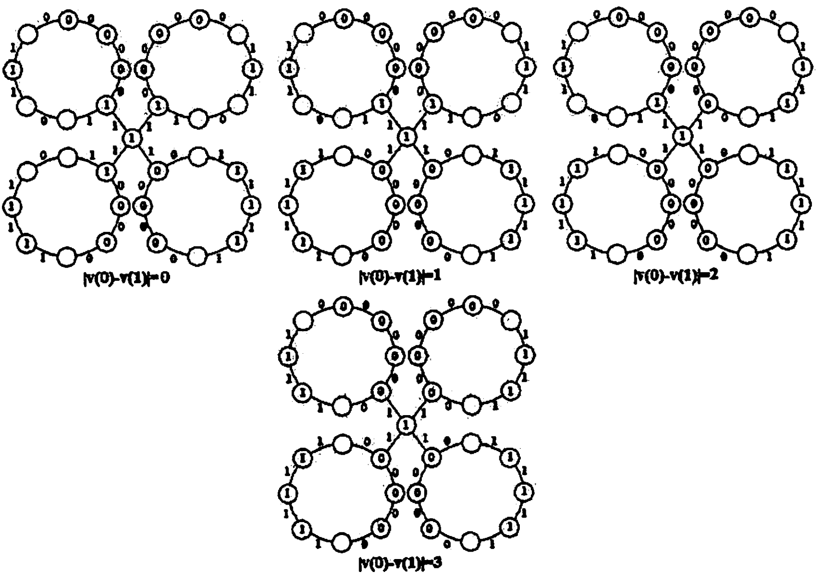

Figure 10: $\mathrm{EBI}(\mathrm{FC}(4,8))=\{0,1,2,3\}$

<u>**Proof of Case 3.**</u> $\mathrm{EBI}(\mathrm{FC}(n,m))=\{0,1,\ldots,n\}$ when $m\geq 5$ is even and $n\geq 1$ is odd.

Since m is even, assume that $m=2k$ where $k\in\mathbb{N}$. Since n is odd, assume that $n=2j+1$ where $j\in\mathbb{N}$. The total number of edges in the graph is $e(\mathrm{FC}(n,m))=n(m+1)=(2j+1)(2k+1)=2(2jk+k+j)+1$. Since it is odd, we have $e(0)=2jk+k+j+1$ and $e(1)=2jk+k+j$.

A similar labeling of Theorem 2.5 is used here. We start with labeling all the edges of the $\mathrm{St}(n)$ with 1. This uses n 1-edges and leaves $(2jk+k+j)-(2j+1)$ 1-edges for cycles. By Equation (2), we have

$$(2j+1)(2k)-2((2jk+k+j)-(2j+1))-1=n.$$

Therefore, the edge-balance index set is a subset of $\{0,1,\ldots,n\}$.

For $n=1$, the total number of edges is $e(\mathrm{FC}(1,m))=1(m+1)=2k+1$. Since it is odd, we have $e(0)=k+1$ and $e(1)=k$. Since there is only one edge in $\mathrm{St}(1)$, we have $e_C(1)=k-1$. By Equation (2), we have the highest edge-balance index is $(1)(2k)-2(k-1)-1=1$. Therefore, the edge-balance index set is a subset of $\{0,1\}$.

By a similar labeling, with n cycles on the graph, we have enough cycles to create an edge-balance index from $n-1$ to 0 by switching edges of cycles as the proof of the Theorem 2.5. Thus, the edge-balance index set is $\{0,1,\ldots,n\}$. □

Example 10. Figure 11 demonstrates edge friendly labelings for $\mathrm{EBI}(\mathrm{FC}(3,8))=\{0,1,2,3\}$.

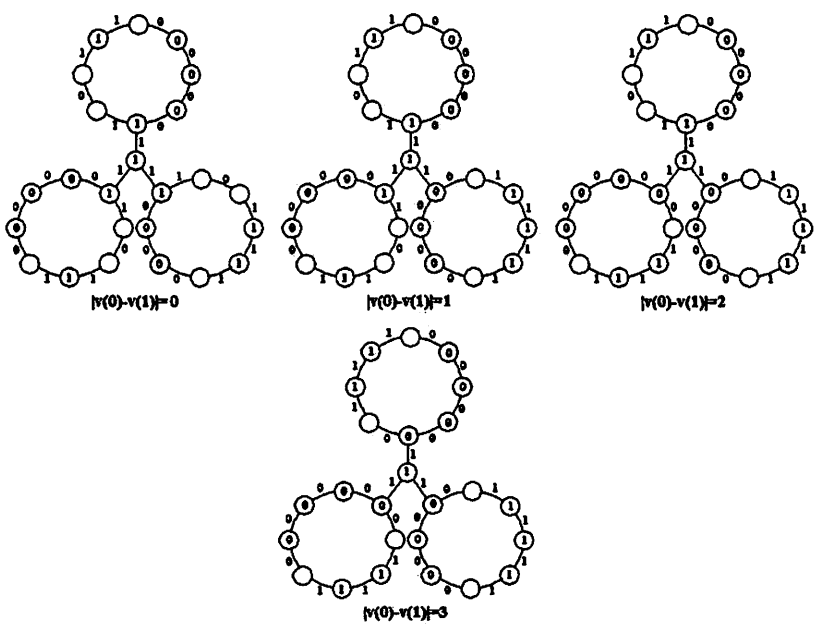

Figure 11: $\mathrm{EBI}(\mathrm{FC}(3,8))=\{0,1,2,3\}$

3 On Edge-Balance Index Set of L-product of Stars By Cycles

The $\mathrm{St}(n) \times_L C_m$ can be constructed from $\mathrm{FC}(n+1, m)$ by contracting any edge incident to the center s_0 of the star. In $\mathrm{FC}(n+1, m)$, let us call the center s_0, the vertex merged into the center v and the contracting edge e.

By the same principle from the precious section, to determine the edge-balance index sets of $\mathrm{St}(n) \times_L C_m$, we would like to find the highest edge-balance index first. The Equation (1) is still true here and a similar argument from the derivation of the Equation (2) works too. The only difference is, since the center s_0 is also a vertex of a cycle, it is included in Equation (1). Thus, the edge-balance index of an edge friendly labeling of $\mathrm{St}(n) \times_L C_m$ is

$$v(0) - v(1) = (n+1)m - 2e_C(1), \tag{3}$$

where $e_C(1)$ is the number of 1-edges in cycles.

Theorem 3.1. *The edge-balance index set of $St(n) \times_L C_m$ is*

$$EBI(St(n) \times_L C_m) = \begin{cases} \{0, 1, \ldots, n+1\} & \textit{if } n \textit{ is odd or } m \textit{ is odd;} \\ \{0, 1, \ldots, n\} & \textit{if both } n \textit{ and } m \textit{ are even.} \end{cases}$$

Proof. The number of edges of $\mathrm{St}(n) \times_L C_m$ is $(n+1)m + n$. Since we fill all edges of the star 1, $e_C(1)$ is the number of 1-edges minus n. With these information, the highest edge-balance index of an edge friendly labeling can be determined by the Equation (3). Therefore, depending on the parity of n and m, for all integers $s, t \geq 1$, we have the following table:

n	m	$e_C(1)$	Highest edge-balance index
$2t+1$	$2s+1$	$2st+2s$	$2t+2=n+1$
$2t$	$2s+1$	$2st+s$	$2t+1=n+1$
$2t+1$	$2s$	$2st+2s-t-1$	$2t+2=n+1$
$2t$	$2s$	$2st+s-t$	$2t=n$
1	m	$m-1$	2

Figure 12: The highest edge-balance index of $\mathrm{St}(n) \times_L C_m$

Therefore, we can conclude that the edge-balance index set of $\mathrm{St}(n) \times_L C_m$ is a subset of

$$\mathrm{EBI}(\mathrm{St}(n) \times_L C_m) \subseteq \begin{cases} \{0, 1, \ldots, n+1\} & \text{if } n \text{ is odd and } m \text{ is odd,} \\ \{0, 1, \ldots, n+1\} & \text{if } n \text{ is even and } m \text{ is odd,} \\ \{0, 1, \ldots, n+1\} & \text{if } n \text{ is odd and } m \text{ is even,} \\ \{0, 1, \ldots, n\} & \text{if } n \text{ is even and } m \text{ is even.} \end{cases}$$

We note here that, in order to get the highest edge-balance index, the above discussion requires that all edges adjacent to the center of the star in $\mathrm{St}(n) \times_L C_m$ labeled 1. We call this special edge friendly labeling h.

Because of the contraction, an edge friendly labeling of $\mathrm{FC}(n+1, m)$ can be used to construct an edge friendly labeling of $\mathrm{St}(n) \times_L C_m$. From Theorem 2.6, we know the edge-balance index set of $\mathrm{FC}(n+1, m)$ is

$$\mathrm{EBI}(\mathrm{FC}(n+1,m)) = \begin{cases} \{0,1,\ldots,n\} & \text{if } m \text{ is odd,} \\ \{0,1,\ldots,n+1\} & \text{if } n \text{ is even and } m \text{ is even,} \\ \{0,1,\ldots,n\} & \text{if } n \text{ is odd and } m \text{ is even.} \end{cases}$$

We note here that only when both n and m are even, the number of edges of $\mathrm{St}(n) \times_L C_m$ is even. Otherwise, $\mathrm{St}(n) \times_L C_m$ has odd number of edges.

When the number of edges of $\mathrm{St}(n) \times_L C_m$ is odd, the number of edges of $\mathrm{FC}(n+1, m)$ is even. Therefore, for an edge friendly labeling of $\mathrm{FC}(n+1, m)$, after contracting, it becomes an edge friendly labeling of $\mathrm{St}(n) \times_L C_m$. Let f be the edge friendly lebeling of $\mathrm{FC}(n+1, m)$ with the highest edge-balance index n we constructed in the proof of Theorem 2.6.

When $n > 2$, the contraction eliminates the vertex merged into the center, which is labeled 0. Thus, it creates an edge friendly labeling of $\mathrm{St}(n) \times_L C_m$ with the edge-balance index $n-1$. Since there are still n cycles not being merged into the center, by the switching we used in the proof of Theorem 2.6, we can create edge-balance indexes from $n-2$ all the way to 0. Also, since h has the edge-balance index $n+1$, the same switching idea creates edge-balance indexes n. Therefore, we can conclude that

$$\mathrm{EBI}(\mathrm{St}(n) \times_L C_m) = \{0, 1, \ldots, n+1\} \text{ if } n \text{ is odd or } m \text{ is odd.}$$

When $n = 2$ and m is odd, the contraction eliminates the vertex merged into the center, which is labeled 0. But, at the same time, the center becomes unlabeled, Thus, it creates an edge friendly labeling of $\mathrm{St}(n) \times_L C_m$ with the edge-balance index 2. Similarly, by switching edges on two cycles, we create edge-balance indexes 1 and 0. With h provides the highest edge-balance index 3, we get

$$\mathrm{EBI}(\mathrm{St}(2) \times_L C_m) = \{0, 1, 2, 3\},$$

where m is odd.

When $n = 1$, h provides the highest edge-balance index 2. With two cycles, by switching, we get

$$\mathrm{EBI}(\mathrm{St}(1) \times_L C_m) = \{0, 1, 2\}.$$

If both n and m are even, the number of edges of $\mathrm{FC}(n+1, m)$ is odd. Since we always assume that $v(0) > v(1)$, we need to remove a 0-edge to create an edge friendly labeling of $\mathrm{St}(n) \times_L C_m$. Similarly, let f be the edge friendly labeling of $\mathrm{FC}(n+1, m)$ with the highest edge-balance index $n+1$ we constructed in the proof of Theorem 2.6. By switching a 1-edge adjacent to the center with a 0-edge which shared a vertex v of the previous 1-edge, we get another edge friendly labeling, g, of $\mathrm{FC}(n+1, m)$ with the edge-balance index n. Then, the contraction eliminates v, which is labeled 0 to create an edge friendly labeling of $\mathrm{St}(n) \times_L C_m$ with the edge-balance index $n-1$. Again, since there are still n cycles not being merged into the center, by the switching we used in the proof of Theorem 2.6, we can create edge-balance indexes from $n-2$ all the way to 0. Again, h provides the highest edge-balance index n. Thus,

$$\mathrm{EBI}(\mathrm{St}(n) \times_L C_m) = \{0, 1, \ldots, n\} \text{ if both } n \text{ and } m \text{ are even.}$$

This completes the proof. □

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