On the Balance Index Sets of Bi-regular and Tri-regular Graphs

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Abstract

The degree set $\mathcal{D}(G)$ of a graph G is the set of degrees of its vertices. It has been shown that when the cardinality of $\mathcal{D}(G)$ is 1 (i.e. G regular) or 2 (i.e. G bi-regular), the balance index set of G has simple structures. In this work we determine the balance index sets of unicyclic graphs and subclasses of $(p, p+1)$ graphs to demonstrate the application of this recent result. In addition, we give an explicit formula for the balance index sets of subclasses of complete tri-bipartite graphs G ($|\mathcal{D}(G)| = 3$). Structural properties regarding the balance index sets of a general graph G and application examples are also presented.

1 Introduction

Throughout this paper, G=($V(G)$,$E(G)$) represents an undirected finite simple graph ([1, 2]). The vertex set of G is $V(G)$ and the edge set of G is $E(G)$. Unless otherwise stated, G has no isolated vertices.

Liu, Tan and the second author ([3]) studied balanced graphs and introduced the concept of *balance indices*. The collection of all balance indices of a graph G is called its *balance index set*. It is a helpful tool in studying the relationship between a vertex labeling of G and its induced (partial) edge labeling. Since its inception, there has been considerable interest in the determination of balance index sets of many classes of graphs. One

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intriguing direction that grows out from these studies is the characterization of the structure of balance index sets. It has been shown ([4]) that the balance index sets of a graph G is completely determined by its degree sequence. One can further deduce that the balance index set of a regular graph is a singleton and the balance index set of a bi-regular graph can be computed by taking the absolute values of the terms of an arithmetic progression. Furthermore, the formula of the arithemetic progression can be determined explicitly.

In general, what can we say about the structure of the balance index set of a general graph G ? If G is tri-regular, can one give an explicit formula for the balance index set of G ? These are the major motivations of this work.

The arrangement of this paper is as follows. Basic notations, definitions and examples regarding balance index sets are presented in Section 1.1. Additional examples and a summary of prior results are presented in Section 1.2 and 1.3 respectively. The determination of the balance index set of the complete tri-partite graph $K_{a,b,c}$ is shown in Section 2. Structural properties are represented in Section 3. Concluding remarks are stated in the final section. Graphical illustrations of the balance index sets of several classes of graphs are included in the Appendix.

1.1 Basic Definitions and Examples

Definition 1 *A vertex labeling of $G = (V(G), E(G))$ is a function f from $V(G)$ to $\{0,1\}$. For each f, it induces a partial function $f^* : E(G) \to \{0,1\}$ as follows:*

$$\forall (u,v) \in E(G), \quad f^*((u,v)) = \begin{cases} 0 & \text{if } f(u) = f(v) = 0, \\ 1 & \text{if } f(u) = f(v) = 1, \\ \text{undefined} & \text{otherwise.} \end{cases}$$

We refer f^* as the induced partial edge labeling of G via f.

Notation 1 *Let f be a vertex labeling of G and f^* be its induced partial edge labeling. We use $v_f(0)$, $v_f(1)$, $e_f(0)$ and $e_f(1)$ to denote the following quantities:*

$$\begin{aligned} v_f(i) &= |\{v \in V(G) : f(v) = i\}| && (i = 0 \text{ or } 1) \\ &= \textit{Number of vertices in } G \textit{ that are labeled } i \textit{ by } f && (i = 0 \text{ or } 1) \\ e_f(i) &= |\{e \in E(G) : f^*(e) = i\}| && (i = 0 \text{ or } 1) \\ &= \textit{Number of edges in } G \textit{ that are labeled } i \textit{ by } f^* && (i = 0 \text{ or } 1) \end{aligned}$$

We will denote the set $\{v : f(v) = 0\}$ *by* $V[f, 0]$ *and* $\{v : f(v) = 1\}$ *by* $V[f, 1]$. *When the labeling* f *is clear under context, we will drop the reference to* f *and simply write* $v(0)$, $v(1)$, $e(0)$, $e(1)$, $V[0]$ *and* $V[1]$ *for these quantities and objects .*

We use the quantities and objects stated in Notation 1 to study the relationships between the vertex labelings and their induced partial edge labelings *quantitatively.*

Definition 2 *A vertex labeling* f *of* G *is said to be friendly when*

$$|v_f(0) - v_f(1)| \leq 1 \quad (\textit{i.e. } G \textit{ is evenly labeled by } 0's \textit{ and } 1's \textit{ via } f).$$

G is said to be balanced if there exists a vertex labeling f *of* G *such that*

$$|v_f(0) - v_f(1)| \leq 1 \quad (\textit{i.e. } f \textit{ is friendly}) \quad \textit{and} \quad |e_f(0) - e_f(1)| \leq 1.$$

Example 1. ([6]) G is a complete graph ($G = K_n$ for some $n \geq 3$): G is balanced iff $n = 3$ or n is even. A more informative way to describe *balancedness* of a graph G (w.r.t. friendly labelings f) is to study the quantity $|e_f(0) - e_f(1)|$. For complete graphs, we have:

$$|e_f(0) - e_f(1)| = \begin{cases} 0 & n \text{ is even}, \\ \frac{n-1}{2} & n \text{ is odd}. \end{cases}$$

This example motivates us to describe the balancedness of a graph G by characterizing all the possible values of $|e_f(0) - e_f(1)|$, where f ranges over all friendly labelings of G.

Definition 3 *The balance index set* $\mathrm{BI}(G)$ *and the signed balanced index set* $\mathrm{SBI}(G)$ *are:*

$$\mathrm{BI}(G) = \{|e_f(0) - e_f(1)| : f \textit{ is a friendly labeling}\} \tag{1}$$

$$\mathrm{SBI}(G) = \{e_f(0) - e_f(1) : f \textit{ is a friendly labeling}\} \tag{2}$$

The balance index sets of many different classes of graphs have been determined. A partial list of these results are shown in Table 1.

Graph Classes	Balance Index Set BI(G)
Complete Graphs ([6])	$\forall n \geq 3,\ \mathrm{BI}(K_n) = \begin{cases} \{0\} & n \text{ even} \\ \{\frac{n-1}{2}\} & n \text{ odd} \end{cases}$
Near Cycles : ([5])	$C_n(t)$ $(3 \leq t \leq n-1)$: $C_n(t)$ is the graph which is obtained by appending an edge (c_1, c_t) $(3 \leq t \leq n-1)$ to the cycle $C_n = (V(C_n), E(C_n))$ $\forall 3 \leq t < n,\ \mathrm{BI}(C_n(t)) = \begin{cases} \{0,1\} & n \text{ even} \\ \{0,1,2\} & n \text{ odd} \end{cases}$
Stars: ([6])	$\mathrm{BI}(St(n)) = \begin{cases} \{k\} & n = 2k+1, \\ \{k-1, k\} & n = 2k. \end{cases}$
Regular Graphs: ([4])	$\mathrm{BI}(G) = \begin{cases} \{\frac{m}{2}\} & \lvert V(G)\rvert \text{ is odd,} \\ \{0\} & \text{otherwise.} \end{cases}$

Table 1: Examples of Balance Index Sets

1.2 Additional Examples

In this section, we present two additional examples for illustrations. Interested readers may find graphical illustrations of the balance indices of specific classes of graphs (discussed in this work) in the Appendix.

Example 2. *The unicyclic graphs $C_n \cdot N_a$:* A unicyclic graph is a connected graph in which the number of its edges equals to the number of its vertices. A unicyclic graph that is formed by adding a pendant edges on each vertex of the cycle C_n is denoted as $C_n \cdot N_a$. It is also referred as the corona of cycles with null graphs in the literature.

One may determine the balance index sets for the graphs $C_n \cdot N_a$ directly. The following Lemma demonstrates the determination of $C_n \cdot N_a$ when n is even or a is odd.

Lemma 1.1 *Let $n, a > 0$. n is even or a is odd. Then*

$$\mathrm{BI}(C_n \cdot N_a) = \{|(a+1)(m - \frac{n}{2})| : 0 \leq m \leq n\}.$$

Proof. First note that $e_f(1) - e_f(0) = v_f(1) - v_f(0)$ for any vertex labeling f over a cycle C. This equality is obviously true when all the vertices have the same label. For vertex labeling that uses both 0 and 1 as labels, the labeling partitions C into an equal number of alternate 1-vertex blocks and 0-vertex blocks. In each of these blocks, the number of 1-edges (resp. 0-edges) is one less than the number of 1-vertices (resp. 0-vertices). The edges between blocks are unlabeled. Hence, $e_f(1) - e_f(0) = v_f(1) - v_f(0)$.

To determine BI(G) where $G = C_n \cdot N_a$, since $|V(G)| = n(a+1)$, $v(1) = v(0) = \frac{(a+1)n}{2}$ for any friendly vertex labeling. Next, suppose that on the cycle, there are m vertices labeled 1 and $(n-m)$ vertices labeled 0 $(m = 0, 1, \ldots, n)$. On the cycle alone, $e(1) - e(0) = v(1) - v(0) = 2m - n$. Among the $a(n-m)$ pendant vertices adjacent to the 0-vertices on C, let k of them are labeled 0 (The actual value or range of k is irrelevant, as k will be canceled out before the end of the proof). Then the other $(a(n-m)-k)$ of these pendant vertices are labeled 1. Now consider the other am pendant vertices adjacent to the 1-vertices on the cycle. By vertex-friendliness,

$$\frac{(a+1)n}{2} - m - [a(n-m)-k] = [(a-1)(m - \frac{n}{2}) + k] \qquad (3)$$

of them are labeled 1 and $[m + \frac{(a-1)n}{2} - k]$ of them are labeled 0. Thus for pendant edges, $[(a-1)(m - \frac{n}{2}) + k]$ of them are labeled 1, k of them are labeled 0, and the others are unlabeled. For the entire graph,

$$e(1) - e(0) = 2m - n + [(a-1)(m - \frac{n}{2}) + k] - k = (a+1)(m - \frac{n}{2}). \qquad (4)$$

Let m run from 0 to n, the result follows.

By using similar techniques, the remaining cases can be determined directly. We will revisit this example in the next subsection and show how some recent results can help to determine an explicit formula for BI($C_n \cdot N_a$)

Example 3. *Connected $(p, p+1)$ graphs with minimum degree equals 2:* Their degree sequences are either of the form $(4, 2, \ldots, 2)$ or $(3, 3, 2, \ldots, 2)$. Consider the following subcases:

1. The one-point union of two cycles of length l_1 and l_2, joined at the center c. We denote it $UC(l_1, l_2)$. It has $p = l_1 + l_2 - 1$ vertices.

2. The theta graph $\theta(l_1, l_2, l_3)$, which consists of three paths of length l_1, l_2, l_3 joined at their endpoints u and v. It has $p = l_1 + l_2 + l_3 - 1$ vertices.

3. The dumbbell graph $DB(l_1, l_2, l_3)$. It consists of two cycles of length l_1 and l_2 connected by a path of length l_3 at its endpoints u and v, and has $p = l_1 + l_2 + l_3 - 1$ vertices.

In the coming section, we will use recent results to determine their balance index sets.

1.3 Previous Work

It has been shown in ([4]) that the balance index set of a graph G is completed determined by its degree sequence. We use the following definition and notations to introduce these results.

Definition 4 *The degree set $\mathcal{D}(G)$ of a graph G is the set:*

$$\mathcal{D}(G) = \{d : (\exists v \in V(G))[deg(v) = d]\}.$$

Notation 2 $\mathcal{G}_i = \{G : |\mathcal{D}(G)| = i\}$ $(i \geq 1)$. *Using this notation, $\mathcal{G}_1$ is the class of regular graphs and $\mathcal{G}_2$ is the class of bi-regular graphs.*

Note that for any graph G with 2 or more vertices, $|\mathcal{D}(G)| < |V(G)|$ (since G contains two vertices with the same degree). The main result from ([4]) are summarized in the following Theorem:

Theorem 1.2 *([4]) Let f be a vertex labeling of G. Then*

$$2(e_f(0) - e_f(1)) = \sum_{v \in V[f,0]} \deg(v) - \sum_{v \in V[f,1]} \deg(v). \tag{5}$$

In addition, we have:

1. $G \in \mathcal{G}_1$ *and* $\mathcal{D}(G) = \{m\} \Rightarrow \mathrm{BI}(G) = \begin{cases} \{\frac{m}{2}\} & |V(G)| \text{ is odd,} \\ \{0\} & |V(G)| \text{ is even.} \end{cases}$

2. $G \in \mathcal{G}_2$ *and* $\mathcal{D}(G) = \{a, b\}$. *If*

$$n_a = |\{v \in G : \ deg(v) = a\}| \geq n_b = |\{v \in G : \ deg(v) = b\}|,$$

then $\mathrm{BI}(G) = \{|s_k| : 0 \leq k \leq n_b\}$ *where*

$$s_k = k(b-a) + a\lceil\frac{|V(G)|}{2}\rceil - |E(G)|, \quad 0 \leq k \leq n_b. \tag{6}$$

Although Theorem 1.2 states that $\mathrm{BI}(G) = \{|s_k| : 0 \leq k \leq n_b\}$, It only implies that $\{s_k : 0 \leq k \leq n_b\} \subseteq \mathrm{SBI}(G)$, and in general, the two sets are not equal.

Theorem 1.2 enables us to obtain explicit formulas for sub-classes of bi-regular graphs. We use the examples stated in Section 1.2 as illustrations. The following lemma shows how the balance index sets of the above subclasses of $(p, p+1)$ graphs can be determined directly.

Lemma 1.3 *Let G be a connected $(p, p+1)$ graph with minimum degree equals 2. If $G = UC(l_1, l_2)$, then*

$$\mathrm{BI}(G) = \mathrm{BI}(UC(l_1,l_2)) = \begin{cases} \{1\} & p \text{ is even and} \\ \{0,2\} & p \text{ is odd.} \end{cases} \tag{7}$$

If G is $\theta(l_1, l_2, l_3)$ or $DB(l_1, l_2, l_3)$, then

$$\mathrm{BI}(G) = \begin{cases} \{0,1\} & p \text{ is even and} \\ \{0,1,2\} & p \text{ is odd.} \end{cases} \tag{8}$$

Proof. The graph $UC(l_1, l_2)$ is bi-regular with one vertex, say c, of degree 4. All the other vertices of G are of degree 2. For any friendly labeling of $UC(l_1, l_2)$, $\{v_f(0), v_f(1)\} = \{\lfloor\frac{p}{2}\rfloor, \lceil\frac{p}{2}\rceil\}$. Without loss of generality, assume $v_f(0) = \lfloor\frac{p}{2}\rfloor$. By applying Equation (5), the balance indices for the two cases ($f(c) = 0, 1$) can be computed. We summarize the computation via the following table:

case	$f(c)$	balance index of $UC(l_1, l_2)$
1	0	$\frac{1}{2}\{4 + 2(\lfloor\frac{p}{2}\rfloor - 1) - 2(\lceil\frac{p}{2}\rceil)\} = 1 + (\lfloor\frac{p}{2}\rfloor - \lceil\frac{p}{2}\rceil) = \begin{cases} 1 & p \text{ is even} \\ 0 & p \text{ is odd.} \end{cases}$
2	1	$\lvert\frac{1}{2}\{2(\lfloor\frac{p}{2}\rfloor) - 2(\lceil\frac{p}{2}\rceil - 1) - 4\}\rvert = 1 - (\lfloor\frac{p}{2}\rfloor - \lceil\frac{p}{2}\rceil) = \begin{cases} 1 & p \text{ is even} \\ 2 & p \text{ is odd} \end{cases}$

This proves (7). For (8), the underlying graph G is bi-regular but there are two degree 3 vertices (denoted by c_1 and c_2). The remaining vertices of G are of degree 2. For any friendly labeling of G, $\{v_f(0), v_f(1)\} = \{\lfloor\frac{p}{2}\rfloor, \lceil\frac{p}{2}\rceil\}$ and again, we assume $v_f(0) = \lfloor\frac{p}{2}\rfloor$. Then (8) follows from the following case analysis:

case	$f(c_1)$	$f(c_2)$	The balance index of G
1	0	0	$\lvert\frac{1}{2}\{6 + 2(\lfloor\frac{p}{2}\rfloor - 2) - 2(\lceil\frac{p}{2}\rceil)\}\rvert = \lvert 1 + (\lfloor\frac{p}{2}\rfloor - \lceil\frac{p}{2}\rceil)\rvert = \begin{cases} 1 & p \text{ is even} \\ 0 & p \text{ is odd.} \end{cases}$
2	0	1	$\lvert\frac{1}{2}\{2(\lfloor\frac{p}{2}\rfloor - 1) - 2(\lceil\frac{p}{2}\rceil - 1)\}\rvert = \lvert(\lfloor\frac{p}{2}\rfloor - \lceil\frac{p}{2}\rceil)\rvert = \begin{cases} 0 & p \text{ is even} \\ 1 & p \text{ is odd} \end{cases}$
3	1	1	$\lvert\frac{1}{2}\{2(\lfloor\frac{p}{2}\rfloor) - 2(\lceil\frac{p}{2}\rceil - 2) - 6\}\rvert = \lvert 1 - (\lfloor\frac{p}{2}\rfloor - \lceil\frac{p}{2}\rceil)\rvert = \begin{cases} 1 & p \text{ is even} \\ 2 & p \text{ is odd} \end{cases}$

Next, we will revisit the example of the unicyclic graphs $C_n \cdot N_a$. We demonstrate how Theorem 1.2 helps to determine an explicit formula for the balance index set of unicyclic graphs.

Lemma 1.4 *Let $n, a > 0$. Then* $\mathrm{BI}(C_n \cdot N_a) = \{|s_m| : 0 \le m \le n\}$ *where*

$$s_m = m(a+1) - \lfloor \frac{n(a+1)}{2} \rfloor. \qquad 0 \le m \le n.$$

Proof. Let $G = C_n \cdot N_a$. G is bi-regular and $|V(G)| = |E(G)| = n(a+1)$. In addition, G has na vertices of degree 1 and n vertices of degree $a+2$. Now,

- $\mathcal{D}(G) = \{1, a+2\}$.
- $n_1 = |\{v \in G : \deg(v) = 1\}| = na$.
- $n_{a+2} = |\{v \in G : \deg(v) = a+2\}| = n \le n_1$.

By Theorem 1.2, $\mathrm{BI}(C_n \cdot N_a) = \{|s_m| : 0 \le m \le n\}$ where

$$\begin{aligned} s_m &= m(a+1) + \lceil \frac{|V(G)|}{2} \rceil - |E(G)|, && 0 \le m \le n \\ &= m(a+1) - \lfloor \frac{n(a+1)}{2} \rfloor && 0 \le m \le n \\ &\quad \text{since } |E(G)| = |V(G)| = \lceil \frac{|V(G)|}{2} \rceil + \lfloor \frac{|V(G)|}{2} \rfloor. \end{aligned}$$

Observe that when n is even or a is odd, $|V(G)| = n(a+1)$ is even and

$$s_k = m(a+1) - \frac{n(a+1)}{2} = (a+1)[m - \frac{n}{2}] \quad \text{where } m = 0, \ldots, n.$$

This coincides with the result stated in Lemma 1.1.

2 Tri-regular Graphs

Theorem 1.2 provides an explicit formula for $\mathrm{BI}(G)$ when $G \in \mathcal{G}_2$. It is natural to see if analogous results hold when we move "one step" forward. That is, when the graph G is tri-regular $G \in \mathcal{G}_3$ (See Notation 2).

2.1 Complete Tri-partite Graphs

We begin with complete tri-partite graphs $K_{a,b,c}$ $(a, b, c \geq 1)$. They are members of $\mathcal{G}_3$ where the vertex set $V(K_{a,b,c})$ can be partitioned into three disjoint subsets V_x, V_y and V_z :

$$V_x = \{x_1, \ldots, x_a\}, \quad V_y = \{y_1, \ldots, y_b\} \quad \text{and} \quad V_z = \{z_1, \ldots, z_c\}. \qquad (9)$$

An edge e is in $E(K_{a,b,c})$ if and only if the end vertices of e are *not* from the same subset V_x, V_y or V_z of the partition. Note that $K_{a,b,c} \in \mathcal{G}_3$. In particular,

$$\deg(x_i) = b + c, \ \deg(y_j) = a + c \ \text{ and } \deg(z_k) = a + b, \qquad (10)$$

where $1 \leq i \leq a, 1 \leq j \leq b, 1 \leq k \leq c$. When $a + b \leq c$ (i.e. the number of vertices in V_z dominates), we can obtain an explicit formula for $\mathrm{BI}(K_{a,b,c})$.

Theorem 2.1 *If $a + b \leq c$, then*

$$\mathrm{BI}(K_{a,b,c}) = \begin{cases} \{|\frac{1}{2}\{(2s-a)(b+c) + (2t-b)(a+c) - 2(a+b)(s+t) \\ \quad +(a+b)^2\}| : 0 \leq s \leq a, 0 \leq t \leq b\} \\ \hfill \textit{when } a+b+c \textit{ is even.} \\ \{|\frac{1}{2}\{(2s-a)(b+c) + (2t-b)(a+c) - 2(a+b)(s+t) \\ \quad +(a+b)(a+b-1)\}| : 0 \leq s \leq a, 0 \leq t \leq b\} \\ \hfill \textit{when } a+b+c \textit{ is odd.} \end{cases}$$

Proof. First note that $|V(K_{a,b,c})| = a+b+c$. For any friendly labeling f, the cardinality of the set $V[f,0]$ or the set $V[f,1]$ is either $\lceil \frac{a+b+c}{2} \rceil$ or $\lfloor \frac{a+b+c}{2} \rfloor$. Without loss of generality, we assume that $v_f(0) = |V[f,0]| = \lfloor \frac{a+b+c}{2} \rfloor$ and $v_f(1) = |V[f,1]| = \lceil \frac{a+b+c}{2} \rceil$. Let f labels s (resp. t) vertices in V_x (resp. V_y) by 0. Then:

$$\text{Number of vertices in } V_z \text{ labeled 0 by } f = \lfloor \frac{a+b+c}{2} \rfloor - (s+t) \qquad (11)$$

$$\text{Number of vertices in } V_z \text{ labeled 1 by } f = \lceil \frac{a+b+c}{2} \rceil - (a+b) + (s+t) \qquad (12)$$

By Theorem 1.2, the quantity $e_f(0) - e_f(1) = \frac{1}{2}(Q_1 - Q_2)$, where Q_1 equals

$$(b+c)s + (a+c)t + (a+b)(\lfloor \frac{a+b+c}{2} \rfloor - (s+t)) \qquad (13)$$

and Q_2 equals

$$(b+c)(a-s)+(a+c)(b-t)+(a+b)[\lceil\frac{a+b+c}{2}\rceil-(a+b)+(s+t)] \quad (14)$$

By (13) and (14) and observe that

$$\lfloor\frac{a+b+c}{2}\rfloor-\lceil\frac{a+b+c}{2}\rceil+(a+b)=\begin{cases} a+b & a+b+c \text{ is even,} \\ a+b-1 & a+b+c \text{ is odd.}\end{cases}$$

$$e_f(0)-e_f(1)=\begin{cases} \frac{1}{2}\{(2s-a)(b+c)+(2t-b)(a+c)-2(a+b)(s+t) \\ \quad +(a+b)^2\} \quad \text{when } a+b+c \text{ is even.} \\ \frac{1}{2}\{(2s-a)(b+c)+(2t-b)(a+c)-2(a+b)(s+t) \\ \quad +(a+b)(a+b-1)\} \quad \text{when } a+b+c \text{ is odd.}\end{cases}$$

Finally, since $a+b \leq c$, the parameter s (resp. t) can range over 0 to a (resp. 0 to b). This completes the proof.

Example 4. Note that Theorem 2.1 can determine the $K_{a,b,c}$ even when $a+b \not\leq c$ (In such cases, we need to enumerate the parameters s, t carefully). We have:

1. $\mathrm{BI}(K_{2,3,4}) = \{0,1,2,3,4,5,6\}$.
2. $\mathrm{BI}(K_{3,4,4}) = \{2,3,4,5\}$.
3. $\mathrm{BI}(K_{2,4,6}) = \{0,2,4,6,8\}$. (See Figure 4)

3 Structural Properties of SBI(G)

We begin our discussions on the structure of SBI(G), the signed balance index set for G.

3.1 Basic Lemmas

In this section, we list some basic properties of SBI(G) as lemmas. These lemmas provide necessary conditions for checking if a number s belongs to SBI(G).

Lemma 3.1 *For any graphs G and for any integer s,*

$$s \in \mathrm{SBI}(G) \Rightarrow -s \in \mathrm{SBI}(G). \quad (15)$$

Proof. Given a graph $G = (V(G), E(G))$ and a friendly labeling f of G, the function $f' : V(G) \to \{0,1\}$ defined by $f'(v) = 1 - f(v)$ is also a friendly labeling and $e_f(0) = e_{f'}(1)$ and $e_f(1) = e_{f'}(0)$. Hence

$$\begin{aligned} s \in \mathrm{SBI}(G) &\Rightarrow (\exists f)[f \text{ is friendly and } s = e_f(0) - e_f(1)] \\ &\Rightarrow (\exists f')[f' \text{ is friendly and } -s = e_{f'}(0) - e_{f'}(1)] \\ &\Rightarrow -s \in \mathrm{SBI}(G). \end{aligned}$$

Lemma 3.2 *Let* $G = (V(G), E(G))$, $|V(G)| = n$. *We write* $V(G) = \{v_1, \ldots, v_n\}$ *where* $\deg(v_1) \geq \deg(v_2) \geq \cdots \geq \deg(v_n)$, *then the maximum value of the set* $\mathrm{SBI}(G)$ *is*

$$\mathrm{SBI}_{\max}(G) = \frac{1}{2}\left(\sum_{i=1}^{r} \deg(v_i) - \sum_{i=r+1}^{n} \deg(v_i)\right) \quad (r = \lceil \frac{|V(G)|}{2} \rceil), \tag{16}$$

and the minimum value of the set $\mathrm{SBI}(G)$ *is* $\mathrm{SBI}_{\min}(G) = -\mathrm{SBI}_{\max}(G)$.

Proof. The equality $\mathrm{SBI}_{\min}(G) = -\mathrm{SBI}_{\max}(G)$ follows immediately from Lemma 3.1. Let h be the vertex labeling defined by

$$h(v_i) = \begin{cases} 0 & 1 \leq i \leq r \\ 1 & r+1 \leq i \leq n \end{cases} \qquad (r = \lceil \frac{|V(G)|}{2} \rceil).$$

The labeling h is friendly because (again, $n = |V(G)|$ and $r = \lceil \frac{|V(G)|}{2} \rceil$)

$$v_h(0) - v_h(1) = 2r - n = 2\lceil \frac{|V(G)|}{2} \rceil - |V(G)| = \begin{cases} 0 & |V(G)| \text{ is even,} \\ 1 & |V(G)| \text{ is odd.} \end{cases}$$

Now, for any friendly labeling f, as the given degree sequence $\{\deg(v_i)\}$ is monotonic non-increasing, we get

$$e_f(0) \leq \sum_{i=1}^{r} \deg(v_i) = e_h(0) \quad \text{and} \quad e_f(1) \geq \sum_{i=r+1}^{n} \deg(v_i) = e_h(1),$$

and (16) follows immediately.

To simplify our notation, we will omit reference to the underlying graph G and denote the quantities $\mathrm{SBI}_{\max}(G)$ by $S_{\max}$ and $\mathrm{SBI}_{\min}(G)$ by $S_{\min}$.

Lemma 3.3 *Let G be a graph such that $\mathcal{D}(G) = \{d_1, \ldots, d_n\}$. Let $\alpha = gcd\ (d_1, \ldots, d_n)$. Then*

$$\mathrm{SBI}(G) \subseteq \{S_{\min},\ S_{\min} + \alpha,\ S_{\min} + 2\alpha,\ \ldots\ , S_{\max}\}. \tag{17}$$

Proof. Let h be a labeling such that $e_h(0) - e_h(1) = S_{\min}$. Now, for any labeling f,

$$\begin{aligned} e_h(0) - e_h(1) &= \sum_{v \in V[h,0]} \deg(v) - \sum_{v \in V[h,1]} \deg(v) \\ e_f(0) - e_f(1) &= \sum_{v \in V[f,0]} \deg(v) - \sum_{v \in V[f,1]} \deg(v) \\ e_f(0) - e_f(1) - S_{\min} &= \sum_{v \in V[f,0]} \deg(v) - \sum_{v \in V[f,1]} \deg(v) \\ &\quad - \sum_{v \in V[h,0]} \deg(v) + \sum_{v \in V[h,1]} \deg(v). \end{aligned}$$

Since $\alpha \mid \deg(v_i)$ for all $i = 1, \ldots, n$, we have $\alpha \mid [e_f(0) - e_f(1) - S_{\min}]$. This implies that $e_f(0) - e_f(1) = S_{\min} + k\alpha \leq S_{\min} + m\alpha = S_{\max}$. Hence $k \leq m$. Thus

$$\mathrm{SBI}(G) \subseteq \{S_{\min},\ S_{\min} + \alpha,\ S_{\min} + 2\alpha,\ \ldots\ , S_{\max}\}.$$

Lemma 3.4 *Let $G \in \mathcal{G}_3$, $\mathcal{D}(G) = \{d_1, d_2, d_3\}$ and $d_1 > d_2 > d_3$.*

$$s \in \mathrm{SBI}(G) \Rightarrow\ (\exists s' \in \mathrm{SBI}(G))\,[|s - s'| \leq \max\{d_1 - d_2, d_2 - d_3\}]\,. \tag{18}$$

Proof. Let $s \in \mathrm{SBI}(G)$. Let h_s be the vertex labeling that *witnesses* $s \in \mathrm{SBI}(G)$. That is,

$$s = \frac{1}{2}\left(\sum_{v \in V[h_s,0])} \deg(v) - \sum_{v \in V[h_s,1]} \deg(v)\right).$$

Now, $\mathcal{D}(G) = \{d_1, d_2, d_3\}$. Hence, at least two vertices v_0, v_1 such that $v_0 \in V[h_s, 0]$, $v_1 \in V[h_s, 1]$ and $\{\deg(v_0), \deg(v_1)\} = \{d_1, d_2\}$ or $\{d_2, d_3\}$. It is because, in our case, it is always possible to select two vertices v_0 and v_1 from $V[h_s, 0]$ and $V[h_s, 1]$ respectively such that $\deg(v_0) \neq \deg(v_1)$. If it turns out that $\{\deg(v_0), \deg(v_1)\} = \{d_1, d_3\}$, we can find a vertex v with $deg(v) = d_2$. We then replace the vertex v_i by v, where v_i ($i = 0$ or 1) is the one that comes from the same partition as v. Now, let $s' \in \mathrm{SBI}(G)$ that is witnessed by the vertex labeling $h_{s'}$:

$$h_{s'}(v) = \begin{cases} h_s(v) & v \neq v_0, v_1. \\ 1 - h_s(v) & v = v_0 \text{ or } v_1. \end{cases}$$

Then,

$$\begin{aligned} |s - s'| &\leq |\deg(v_0) - \deg(v_1)| \\ &\leq \begin{cases} d_1 - d_2 & \text{if } \{\deg(v_0), \deg(v_1)\} = \{d_1, d_2\} \\ d_2 - d_3 & \text{if } \{\deg(v_0), \deg(v_1)\} = \{d_2, d_3\} \end{cases} \end{aligned}$$

This completes our proof.

3.2 Examples

In this section, we give examples to demonstrate the potential use of the basic lemmas.

Example 5. Let $G \in \mathcal{G}_3$, $\mathcal{D}(G) = \{d_1, d_2, d_3\}$ and $d_1 > d_2 > d_3$. If $\alpha = \gcd(d_1, d_2, d_3) > 1$, then by Lemma 3.3, $1 \notin \mathrm{BI}(G)$. It is interesting to note that, when $\alpha = \gcd(d_1, d_2, d_3) = 1$, 1 may not be a balance index of the graph G. Consider the tri-regular graph G which is formed by the disjoint union of K_8, K_6 and K_4. $\mathcal{D}(G) = \{7, 5, 3\}$ and $\gcd(7, 5, 3) = 1$. However, $1 \notin \mathrm{BI}(G)$.

Proof. Assume the contrary and let f be a friendly vertex labeling such that

$$\frac{1}{2}\left(\sum_{v \in V[f,0]} \deg(v) - \sum_{v \in V[f,1]} \deg(v)\right) = \pm 1 \tag{19}$$

Let n_1, n_2 and n_3 be the number of vertices in $V[f, 0]$ that is of degree 7,5 and 3 respectively. Note that $n_1 + n_2 + n_3 = 9$ ($|V(G)| = 18$), G has 8 vertices of degree 7, 6 vertices of degree 5 and 4 vertices of degree 3. By (19), we have

$$\begin{aligned} 7(2n_1 - 8) + 5(2n_2 - 6) + 3(2n_3 - 4) &= \pm 2 \\ 7n_1 + 5n_2 + 3n_3 &= 49 \pm 1 \\ 4n_1 + 2n_2 + 3(n_1 + n_2 + n_3) &= 49 \pm 1 \\ 4n_1 + 2n_2 &= 22 \pm 1 \end{aligned}$$

Because the LHS of the last expression is even but the RHS must be odd. This leads to a contradiction. Hence, $1 \notin \text{BI}(G)$.

Example 6. (Second largest signed balanced indices) Let $G \in \mathcal{G}_3$, $\mathcal{D}(G) = \{d_1, d_2, d_3\}$ and $d_1 > d_2 > d_3$. Let $\alpha = \gcd(d_1, d_2, d_3)$. Then by Lemma 3.3 and Lemma 3.4, the second largest signed balance indices of G, denoted by S_{sec}, satisfies the following inequalities:

$$S_{\max} - \max\{d_1 - d_2, d_2 - d_3\} \le S_{sec} \le S_{\max} - \alpha \tag{20}$$

4 Concluding Remarks

Equation (5) indicates that the balance index set of a graph G is completely characterized by the degree sequence of G. It also provides a uniform way to compute the balance indices of G. When G is bi-regular, the computation of balance indices is captured by a simple formula. In this work, we consider tri-regular graphs and demonstrate that for complete tripartite graphs, such approach is still possible. We also derive necessary conditions which may help to simplify the computation of balance indices.

In general, it is of interest of interest to study how the complexity of computing balance indices changes with repsect to the parameter $|\mathcal{D}(G)|$.

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5 Appendix

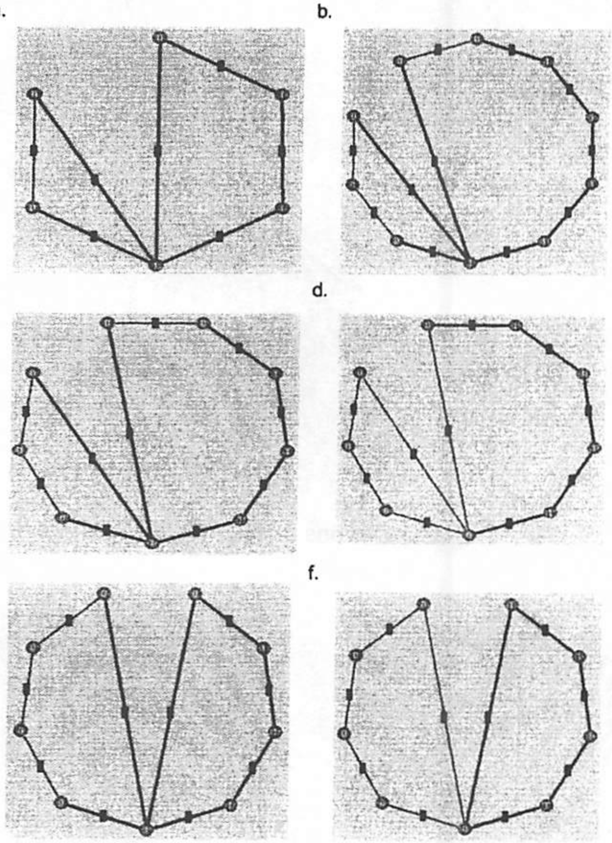

Figure 1: $BI(UC(3,4) = BI(UC(4,7) = \{1\}$ (a,b); $BI(UC(4,6)) = BI(UC(5,5)) = \{0,2\}$ (c,d,e,f).

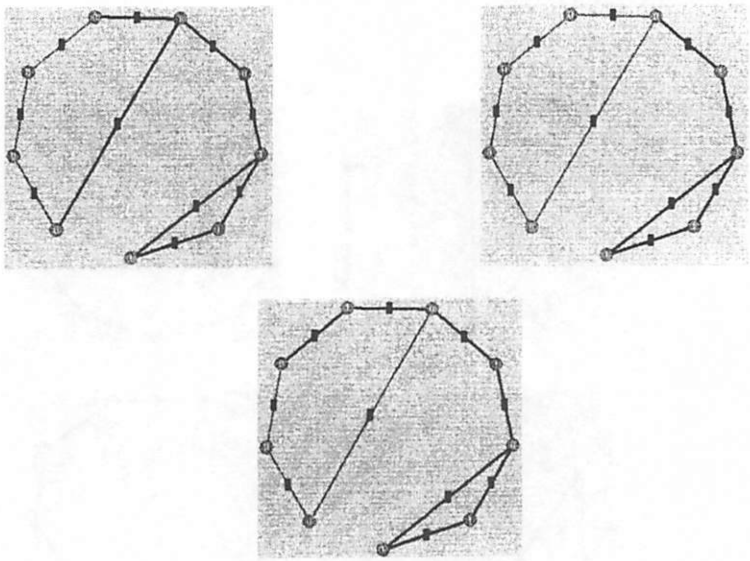

Figure 2: Illustrations for $BI(DB(3,5,2)) = \{0,1,2\}$

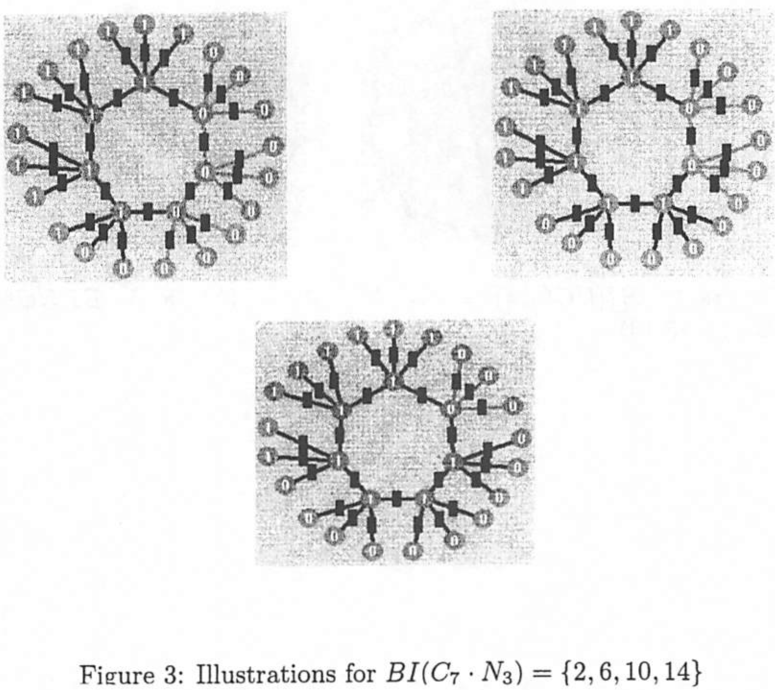

Figure 3: Illustrations for $BI(C_7 \cdot N_3) = \{2,6,10,14\}$

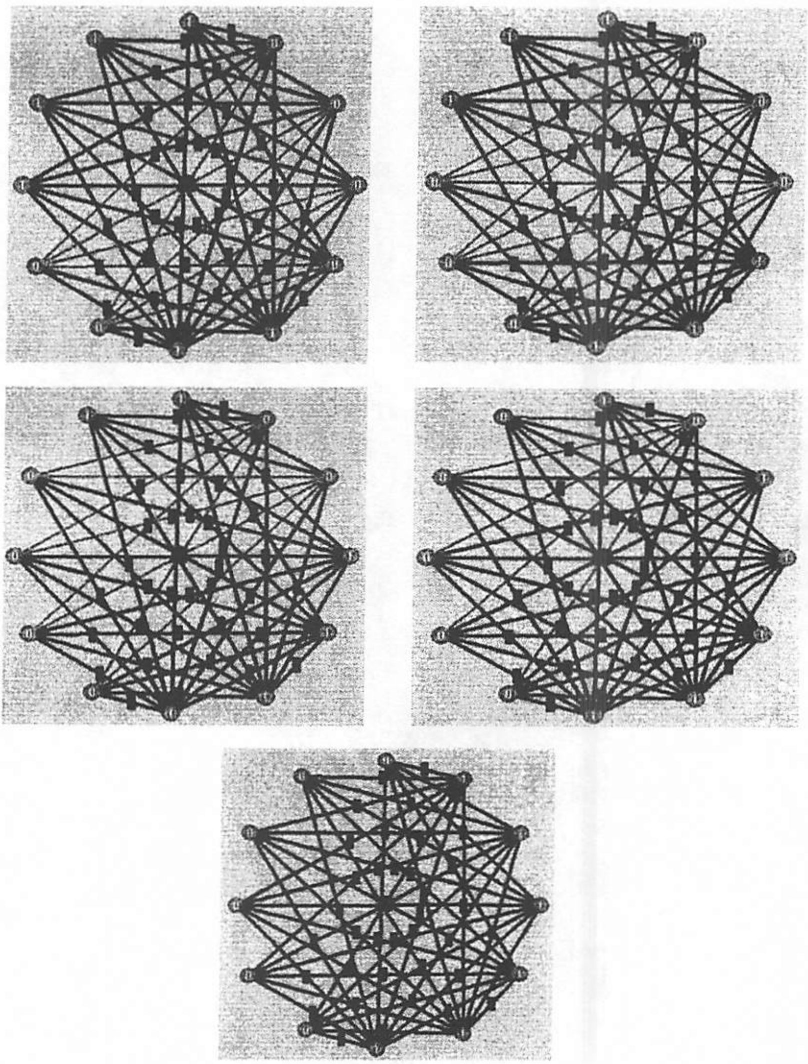

Figure 4: Illustrations for $BI(K_{2,4,6}) = \{0, 2, 4, 6, 8\}$

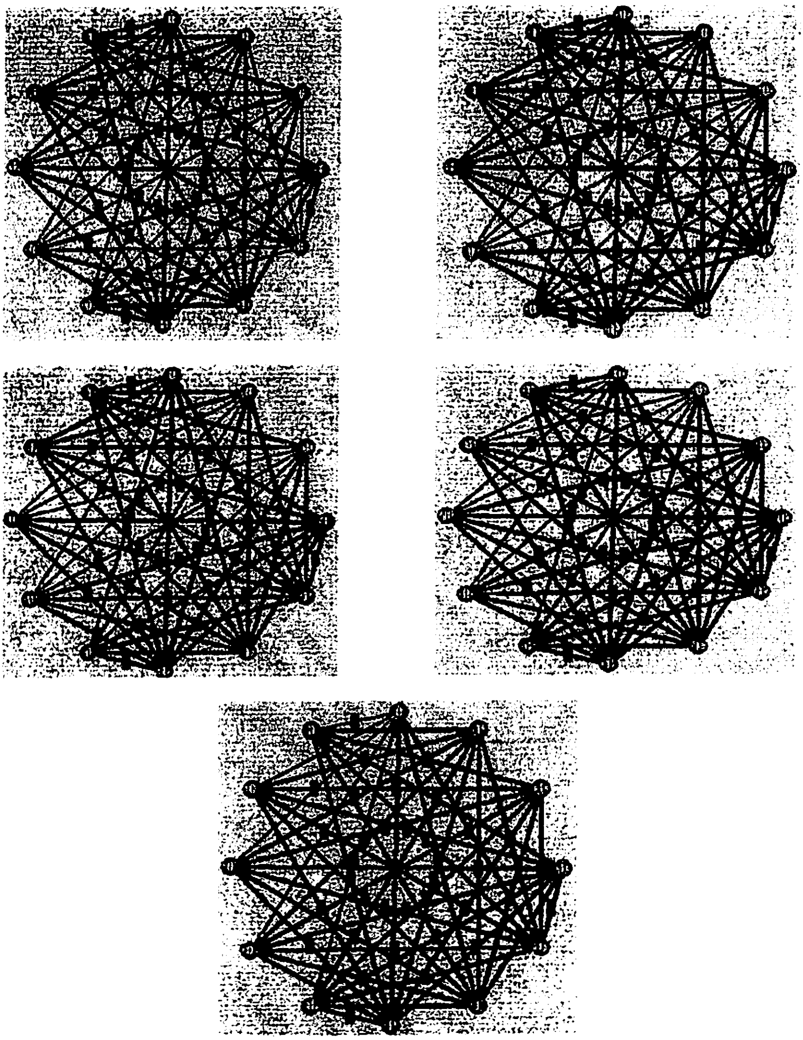

Figure 5: Illustrations for $BI(K_{3,4,5}) = \{0, 1, 2, 3, 4\}$