ON RATIONAL FUNCTIONS WITH GOLDEN RATIO AS FIXED POINT

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Abstract

The existence of an equivalence subset of rational functions with Fibonacci numbers as coefficients and the Golden Ratio as fixed point is proven. The proof is based on two theorems establishing basic relationships underlying the Fibonacci Sequence, Pascal's Triangle and the Golden Ratio.

1 Introduction

Echevarria[3, 4] showed that the Golden Ratio induces two alternative mappings of the set of paired Fibonacci numbers into the set of binomial coefficients. More material on this topic can be found in [2,6,8,9]. In the first mapping, the variant of the Fibonacci Sequence without the initial zero was used. In the second mapping, the zero was retained. It is known that Pascal's Triangle and Fibonacci Sequence are related mathematically[1]. It is also known that the Golden Ratio and the Fibonacci Sequence are mathematically related in a number of very interesting ways[7]. Ghyka[5] explains that the ancient Greeks, who discovered the Golden Ratio, were familiar with the Fibonacci Sequence as one ramification of numerical operations on the Golden Ratio. No mention is made, in the article mentioned[3, 4], regarding possible relationships between the two alternative mappings. The ratio of a Fibonacci number to the number that precedes it in the sequence approaches the Golden Ratio at the limit of the sequence[7]. It is also a mathematical fact that any Fibonacci number can be represented by a general formula, known as Binet's Formula that incorporates the Golden Ratio[1]. Letting $F(n)$ represent the n^{th} number of the Fibonacci sequence,

$$F(n) = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$

The following theorems, each establishing an alternative mapping of paired Fibonacci numbers into the binomial coefficients, and each mapping being induced by the Golden Ratio, are due to Echevarria[3, 4].

Theorem 1.1 *Let $F(n)$ be the Fibonacci Sequence defined by $F(0) = 0$, $F(1) = 1$ and $F(n) = F(n-1) + F(n-2)$ $\forall n \geq 2$ and $x = \varphi = \frac{1+\sqrt{5}}{2}$. Then $\xi : \{F(n) \cdot F(n+1)\} \rightarrow n^{th}$ row of Pascal's Triangle such that as $x = \varphi$, $F(n) + F(n+1)x = (x+1)^n$ is one-to-one.*

Theorem 1.2 *Let $F(n)$ be the Fibonacci Sequence defined by $F(0) = 0$, $F(1) = 1$ and $F(n) = F(n-1) + F(n-2)$ $\forall n \geq 2$ and $x = \varphi = \frac{1+\sqrt{5}}{2}$. Then $\xi : \{F(n) \cdot F(n+1)\} \rightarrow n^{th}$ row of Pascal's Triangle such that as $x = \varphi$, $F(n) + F(n+1)x = x(x+1)^n$ is one-to-one and onto.*

2 An Equivalence Subset of Rational Functions with the Golden Ratio as Fixed Point

Consider a subset of rational functions in the variable x of the form $\frac{F(n+1)+F(n+2)x}{F(n)+F(n+1)x}$, where x is any real number. The following theorem shows that, for specific values of $F(n)$, $F(n+1)$, and $F(n+2)$ (in fact, infinite in number), this subset of rational functions forms an infinite equivalence subset, equivalence being defined by a fixed point relationship with the Golden Ratio.

Theorem 2.1 *Let $F(n)$ be the Fibonacci Sequence defined by $F(0) = 0$, $F(1) = 1$ and $F(n) = F(n-1)+F(n-2)$ $\forall n \geq 2$ and $x = \varphi = \frac{1+\sqrt{5}}{2}$. Then $x = \frac{F(n+1)+F(n+2)x}{F(n)+F(n+1)x}, x = \varphi$ is an equivalence subset of rational function with Golden Ratio as fixed point. Also $lim_{n\to\infty}\frac{F(n+1)+F(n+2)x}{F(n)+F(n+1)x} = \varphi$.*

Proof: From Theorems 1 and 2, the two systems of equations given below hold when $x = \varphi$.

$1 + 1x = x + 1$, $2 + 3x = (x+1)^2$, $\cdots$, $F(n) + F(n+1)x = (x+1)^n$

$0 + 1x = x(1)$, $1 + 2x = x(x+1)$, $\cdots$, $F(n+1) + F(n+2)x = x(x+1)^n$

Since $F(n+2) = F(n) + F(n+1)$, dividing the n^{th} equation of the second system by the n^{th} equation of the first system, we obtain:

$$\frac{x(x+1)^n}{(x+1)^n} = \frac{F(n+1)+F(n+2)x}{F(n)+F(n+1)x} \quad , \quad x = \frac{F(n+1)+F(n+2)x}{F(n)+F(n+1)x}$$

There remains to show that $x = \varphi$. Dividing the $(n+1)^{th}$ of the first set of equalities by the n^{th} of the second set of equalities in this proof yields:

$$\frac{(x+1)^{n+1}}{x(x+1)^n} = \frac{F(n+1)+F(n+2)x}{F(n)+F(n+1)x} = \frac{x+1}{x}$$

However,

$$x = \frac{F(n+1)+F(n+2)x}{F(n)+F(n+1)x} = \frac{x+1}{x}$$

We arrive at $x^2 - x - 1 = 0 \to x = \frac{1\pm\sqrt{5}}{2}$. In particular $x = \frac{1+\sqrt{5}}{2} = \varphi$. This concludes the proof of the first part. To prove the other remark, as $n \to \infty$, using the Binet's Formula for $F(n)$:

$F(n) = \frac{x^n - y^n}{x-y}$ where $x = \frac{1+\sqrt{5}}{2}$, $y = \frac{1-\sqrt{5}}{2} \to x+y = 1$ and $x-y = \sqrt{5}$.

$$\lim_{n\to\infty} \frac{F(n+1)}{F(n)} = \lim_{n\to\infty} \frac{x^{n+1} - y^{n+1}}{x^n - y^n} = \lim_{n\to\infty} \frac{x^{n+1}}{x^n} = x.$$

Since $|y| < 1$, as $n \to \infty$ then $y^n \to 0$.

Hence,

$$\lim_{n\to\infty} \frac{F(n+1)+F(n+2)x}{F(n)+F(n+1)x} = \lim_{n\to\infty} \frac{F(n+1)+F(n+2)x}{F(n)+F(n+1)x} \cdot \frac{F(n+1)}{F(n+1)}$$

$$\to \lim_{n\to\infty} \frac{F(n+1)+F(n+2)x}{F(n)+F(n+1)x} \cdot \frac{\frac{1}{F(n+1)}}{\frac{1}{F(n+1)}} = \lim_{n\to\infty} \frac{1 + \frac{F(n+2)x}{F(n+1)}}{\frac{F(n)}{F(n+1)} + x}$$

$$= \frac{1 + x \cdot x}{\frac{1}{x} + x} = \frac{1+x^2}{\frac{1+x^2}{x}} = 1 + x^2 \cdot \frac{x}{1+x^2} = x = \frac{1+\sqrt{5}}{2} = \varphi.$$

This concludes the proof.

Corollary 2.2 *The smallest positive Fibonacci triples satisfying* $\varphi = \frac{F(n+1)+F(n+2)\varphi}{F(n)+F(n+1)\varphi}$ *is (1,1,2).*

Proof: This is immediate from Theorem 2.1. By inspection, the consecutive Fibonacci triples forming the equivalence subset of rational functions are: (1,1,2), (2,3,5), (5,8,13), ... , $(F(n), F(n+1), F(n+2))$ which yields $F(n) = 1, F(n+1) = 1, F(n+2) = 2$ as the smallest positive Fibonacci triples.

3 ALGEBRAIC PROPERTIES AND STRUCTURES

It is interesting to note that the equivalence subset of rational functions is related to the Golden Ratio in more ways than can be determined by the use of algebraic methods. At least two such properties, namely, the absence of extreme values and convergence to the Golden Ratio, can be identified. We show that this equivalence subset of rational functions possesses an algebraic structure, namely, the formation of a commutative semigroup of rational functions under the operation of composition of functions.

3.1 Extreme Values

To verify if the equivalence subset of rational functions has extreme values, one sets:

$$\frac{d}{dx}\left(\frac{F(n+1)+F(n+2)x}{F(n)+F(n+1)x}\right)=0$$

$$\frac{F(n)+F(n+1)x(F(n+2))-(F(n+1)+F(n+2)x(F(n+1))}{(F(n)+F(n+1)x)^2}=0$$

$$\frac{F(n)\cdot F(n+2)-(F(n+1))^2}{(F(n)+F(n+1)x)^2}=0$$

If $F(n)\cdot F(n+2)-(F(n+1))^2\neq 0$, then the equation will not hold for finite values of n. Recall that it is always true that $F(n)\cdot F(n+2)-(F(n+1))^2=\pm 1$. Therefore, the equivalence subset of rational functions has no extreme values for finite values of $F(n)$, $F(n+1)$, and $F(n+2)$.

3.2 Convergence of Real-Valued Rational Functions

Since each rational function of the equivalence class is uniquely determined by the values of $F(n)$, $F(n+1)$, and $F(n+2)$ corresponding to it, one can construct a sequence of rational functions in terms of increasing values for $F(n)$, $F(n+1)$, and $F(n+2)$. The sequence of rational functions converges to the Golden Ratio as the values of the Fibonacci numbers in the function tend to infinity.

3.3 A Commutative Semigroup of Rational Functions

We show that the equivalence subset of rational functions forms a commutative semigroup under the operation of composition of functions. In addition, the infinite number of elements is generated by a single element under the same operation, converging to a limit. The general form of the rational function is $\frac{F(n+1)+F(n+2)x}{F(n)+F(n+1)x}$ where $F(n)$, $F(n+1)$, and $F(n+2)$ are consecutive Fibonacci numbers and x represents any real number other than $-\frac{F(n)}{F(n+1)}$. It will suffice to show that this subset of rational functions is closed, associative and commutative under composition of functions, and that it is contained a cyclic subgroup that converges to a limit.

3.4 Closure, Associativity and Commutativity

Associativity follows immediately from associativity of composition of real valued functions. We show closure and commutativity simultaneously. Given functions $\frac{F(n+1)+F(n+2)x}{F(n)+F(n+1)x}$ and $\frac{F(m+1)+F(m+2)x}{F(m)+F(m+1)x}$, composition following the order of appearance of the functions gives:

$$\frac{F(n+1)+F(n+2)\cdot\left(\frac{F(m+1)+F(m+2)x}{F(m)+F(m+1)x}\right)}{F(n)+F(n+1)\cdot\left(\frac{F(m+1)+F(m+2)x}{F(m)+F(m+1)x}\right)}$$

$$=\frac{F(n+1)\cdot(F(m)+F(m+1)x)+F(n+2)\cdot(F(m+1)+F(m+2)x)}{F(n)\cdot(F(m)+F(m+1)x)+F(n+1)\cdot(F(m+1)+F(m+2)x)}$$

$$=\frac{F(n+1)^{*}+F(n+2)^{*}x}{F(n)^{*}+F(n+1)^{*}x}$$

where $F(n)^{*}=F(n)\cdot F(m)+F(n+1)\cdot F(m+1)$; $F(n+1)^{*}=F(n)\cdot F(m+1)+F(n+1)\cdot F(m)+F(n+1)\cdot F(m+1)=F(n+1)\cdot F(m)+F(n+2)\cdot F(m+1)$; $F(n+2)^{*}=F(n)^{*}+F(n+1)^{*}$. The fact that $F(n)^{*}$ and $F(n+1)^{*}$ are consecutive Fibonacci numbers follows from the number theoretic result that $F(n+m)=F(n)F(m+1)+F(m)F(n-1)$. Reversing the order of composition gives:

$$\frac{F(m+1)+F(m+2)\cdot\left(\frac{F(n+1)+F(n+2)x}{F(n)+F(n+1)x}\right)}{F(m)+F(m+1)\cdot\left(\frac{F(n+1)+F(n+2)x}{F(n)+F(n+1)x}\right)}$$

$$=\frac{F(m+1)\cdot(F(n)+F(n+1)x)+F(m+2)\cdot(F(n+1)+F(n+2)x)}{F(m)\cdot(F(n)+F(n+1)x)+F(m+1)\cdot(F(n+1)+F(n+2)x)}$$

$$=\frac{F(n+1)^{*}+F(n+2)^{*}x}{F(n)^{*}+F(n+1)^{*}x}$$

The equivalence subset of rational functions is, therefore, closed, associative and commutative under the composition of functions, which makes it a commutative semigroup under the operation.

3.5 Absence of an Identity Element

Letting $g(x) = \frac{F(n+1)+F(n+2)x}{F(n)+F(n+1)x}$, an identity element under composition of functions should yield $fg(x) = gf(x) = g(x)$. However, it is known that this fixed point relationship does not generally hold, but only when

$$\frac{F(n+1)+F(n+2)x}{F(n)+F(n+1)x} = \varphi.$$

3.6 The Generator

The element $\frac{(1+x)}{x}$ where $F(n)$, $F(n+1)$, and $F(n+2)$ are the first three Fibonacci numbers 0, 1 and 1, generates all the others under composition of functions. The second element (corresponding to the second, third and fourth Fibonacci numbers) is $\frac{(1+2x)}{(1+x)}$ which is $\frac{1+\frac{(1+x)}{x}}{\frac{(1+x)}{x}}$. Given any arbitrary $\frac{(F(n+1)+F(n+2)x)}{F(n)+F(n+1)x}$, replacing x with $\frac{(1+x)}{x}$ yields

$$\frac{(F(n+2)+(F(n+1)+F(n+2))x)}{F(n+1)+F(n+2)x}$$

which is the element immediately following the arbitrary element. Thus, $\frac{(1+x)}{x}$ generates all the rational functions in the semigroup under composition of functions.

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