# On Edge-Balance Index Sets of $L$-product of Cycles with Stars, Part I

Chao-Chih Chou
General Education Center
St. John's University
Tamsui, Taipei Shien, Taiwan
chao@mail.sju.edu.tw

Meghan Galiardi
Department of Mathematics
Stonehill College
Easton, MA 02357, USA
mgaliardi@students.stonehill.edu

Man Kong
Department of Electrical Engineering and Computer Science
University of Kansas
Laurence, KS 66045, USA
kong@eecs.ku.edu

Sin-Min Lee
Department of Computer Science
San Jose State University
San Jose, CA 95192, USA
lee@cs.sjsu.edu

Daniel Perry
Department of Mathematics
Stonehill College
Easton, MA 02357, USA
dperry1@students.stonehill.edu

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Hsin-Hao Su
Department of Mathematics
Stonehill College
Easton, MA 02357, USA
hsu@stonehill.edu

**Abstract**

Let $G$ be a simple graph with vertex set $V(G)$ and edge set $E(G)$, and let $\mathbb{Z}_2 = \{0,1\}$. Any edge labeling $f$ induces a partial vertex labeling $f^+ : V(G) \to \mathbb{Z}_2$ assigning 0 or 1 to $f^+(v)$, $v$ being an element of $V(G)$, depending on whether there are more 0-edges or 1-edges incident with $v$, and no label is given to $f^+(v)$ otherwise. For each $i \in \mathbb{Z}_2$, let $v_f(i) = |\{v \in V(G) : f^+(v) = i\}|$ and let $e_f(i) = |\{e \in E(G) : f(e) = i\}|$. An edge-labeling $f$ of $G$ is said to be edge friendly if $\{|e_f(0) - e_f(1)| \leq 1$. The edge-balance index set of the graph G is defined as $EBI(G) = \{|v_f(0) - v_f(1)| :$ $f$ is edge-friendly.$\}$. In this paper, we investigate and present results concerning the edge-balance index sets of $L$-product of cycles with stars.

**Keywords and phrases:** vertex labeling, edge labeling, friendly labeling, cordiality, edge-balance index set, L-products, cycles, stars.

**AMS 2000 MSC:** 05C78, 05C25

# 1 Introduction

In [5], Kong and second author considered a new labeling problem of graph theory. Let $G$ be a simple graph with vertex set $V(G)$ and edge set $E(G)$, and let $\mathbb{Z}_2 = \{0,1\}$. An edge labeling $f : E(G) \to \mathbb{Z}_2$ induces a vertex partial labeling $f^+ : V(G) \to \mathbb{Z}_2$ defined by $f^+(v) = 0$ if the edges labeled 0 incident on $v$ is more than the number of edges labeled 1 incident on $v$, and $f^+(v) = 1$ if the edges labeled 1 incident on $v$ is more than the number of edges labeled 0 incident on $v$. $f^+(v)$ is not defined if the number of edges labeled by 0 is equal to the number of edges labeled 1. For $i \in \mathbb{Z}_2$, let $v_f(i) = |\{v \in V(G) : f^+(v) = i\}|$, and let $e_f(i) = |\{e \in E(G) : f(e) = i\}|$.

With these notations, we now introduce the notion of an edge-balanced graph.

**Definition 1.** An edge labeling $f$ of a graph $G$ is said to be ***edge-friendly*** if $|e_f(0) - e_f(1)| \leq 1$. A graph $G$ is said to be an ***edge-balanced*** graph if there is an edge-friendly labeling $f$ of $G$ satisfying $|v_f(0) - v_f(1)| \leq 1$.

Chen, Lee, et al in [1] proved that all connected simple graphs except the star $K_{1,2k+1}$, where $k \geq 0$ are edge-balanced.

**Definition 2.** The ***edge-balance index set*** of the graph $G$, $\mathrm{EBI}(G)$, is defined as $\{|v_f(0) - v_f(1)| : \text{the edge labeling f is edge-friendly.}\}$.

We will use $v(0)$, $v(1)$, $e(0)$, $e(1)$ instead of $v_f(0)$, $v_f(1)$, $e_f(0)$, $e_f(1)$, provided there is no ambiguity.

**Example 1.** $\mathrm{EBI}\,(nK_2)$ is $\{0\}$ if $n$ is even and $\{2\}$ if $n$ is odd.

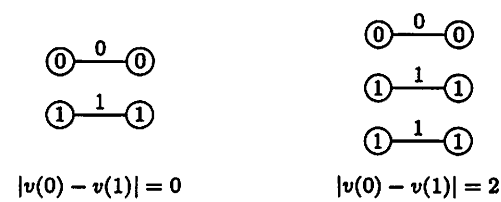



Figure 1: The edge-balance index set of $2K_2$ and $3K_2$

For any $n \geq 1$, we denote the tree with $n+1$ vertices of diameter two by $\mathrm{St}(n)$. The star has a center $c$ and $n$ appended edges from $c$.

**Example 2.** The edge-balance index set of the star $\mathrm{St}(n)$ is

$$\mathrm{EBI}\,(\mathrm{St}(n)) = \begin{cases} \{0\} & \text{if } n \text{ is even,} \\ \{2\} & \text{if } n \text{ is odd.} \end{cases}$$

**Example 3.** In [12], Lee, Lo and Tao showed that

$$\mathrm{EBI}\,(P_n) = \begin{cases} \{2\} & \text{if } n \text{ is 2,} \\ \{0\} & \text{if } n \text{ is 3,} \\ \{1,2\} & \text{if } n \text{ is 4,} \\ \{0,1\} & \text{if } n \text{ is odd and greater than 3,} \\ \{0,1,2\} & \text{if } n \text{ is even and greater than 4.} \end{cases}$$

Figure 2 shows the EBI of $P_3$ and $P_4$.

**Example 4.** Figure 3 shows that the edge-balance index set of a tree with six vertices is $\{0,1,2\}$.

The edge-balance index sets can be viewed as the dual of balance index sets. The balance index sets of graphs were considered in [4, 6, 8, 9, 10, 11, 13, 15]. Let $G$ be a simple graph with vertex set $V(G)$ and edge set

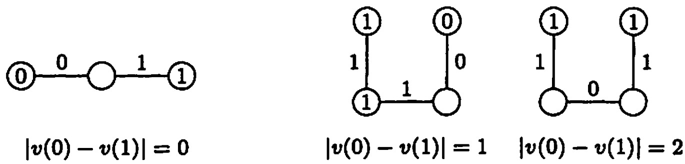



Figure 2: The edge-balance index set of $P_3$ and $P_4$

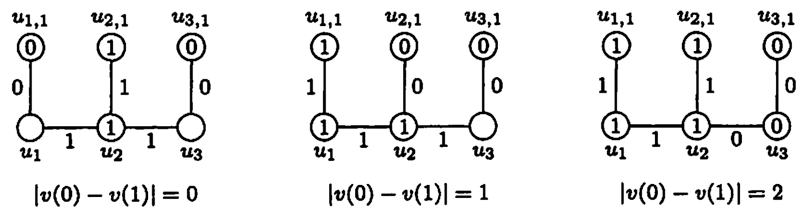



Figure 3: The edge-balance index set of a tree with six vertices

$E(G)$, and let $\mathbb{Z}_2 = \{0, 1\}$. A labeling $f : V(G) \to \mathbb{Z}_2$ induces an edge partial labeling $f^* : E(G) \to A$ defined by $f^*(vw) = f(v)$, if and only if $f(v) = f(w)$ for each edge $vw \in E(G)$. For $i \in \mathbb{Z}_2$, let $v_f(i) = \text{card}\{v \in V(G) : f(v) = i\}$ and $e_{f^*}(i) = \text{card}\{e \in E(G) : f^*(e) = i\}$. A labeling $f$ of a graph $G$ is said to be ***friendly*** if $|v_f(0) - v_f(1)| \leq 1$. If $|e_f(0) - e_f(1)| \leq 1$ then $G$ is said to be ***balanced***. The ***balance index set*** of the graph $G$, $\text{BI}(G)$, is defined as $\{|e_f(0) - e_f(1)| : \text{the vertex labeling } f \text{ is friendly}\}$.

Edge-balance index sets of trees, flower graphs and $(p, p+1)$-graphs were considered in [7, 12, 14].

Let $H$ be a connected graph with a distinguished vertex $s$. Construct a new graph $G \times_L (H, s)$ as follows: take $|V(G)|$ copies of $(H, s)$ and identify each vertex of $G$ with $s$ of a single copy of $H$. We call the resulting graph the ***L-product*** of $G$ and $(H, s)$. In this paper, exact values of the edge-balance index sets of $L$-product of cycles with stars, $C_n \times_L (\text{St}(m), c)$), where $c$ is the center of the star graph and $m$ is odd or 2 are presented.

# 2 On edge-balance index sets of $L$-product of $C_n$ with St(1)

In this section we first consider the edge-balance index sets of the $C_n \times_L (\text{St}(1), c)$.

**Lemma 2.1.** *Let $n = 3k + r$ where $r = 0$, 1, or 2. The highest edge-balance index of $C_n \times_L (St(1), c)$ is $2k$.*

**Proof.** By looking at the structure of $C_n \times_L (\mathrm{St}(1), c)$, we notice that if you label one edge in $C_n$ and the two edges of St(1)s adjacent to the adjacent two vertices 0, then you have four 0-vertices in three 0-edges. This is the most effective way to gain the most 0-vertices with the least 0-edges.

To get the maximal edge-balance index, we label the above package as many as we can. Since each one requires three 0-edges, we can have as many as $k$ packages. Because $C_n$ has $n = 3k + r$ edges, we have enough rooms for $k$ packages. We position each package separately in $C_n$ by inserting a 1-edge in between. These 1-edges do not alter any vertex labeling since they connect two 0-vertices with two 0-edges each. We also label two end edges on $C_n$ 1. They do not alter any already labeled vertices since they connect to two 0-vertices with two 0-edges each. This step occupies $3k$ 0-edges and $k+1$ 1-edges. Also, it labels $2k$ adjacent vertices on $C_n$ 0 and $2k$ edges of St(1)s 0. The number of 0-vertices created so far is $4k$.

The rest $r$ 0-edges can label at most two 0-vertices by placing them in the edges of other St(1)s. So, we totally have $4k + r$ 0-vertices. Note here that these two 0-edges do not affect the labeling of the vertices on $C_n$ they connect to since the other two edges must be labeled 1 by our setting.

After filling in the rest 1-edges, the rest $2n - (4k + r) = (6k + 2r) - (4k + r) = 2k + r$ vertices are all labeled 1. Thus, the edge-balance index is $(4k + r) - (2k + r) = 2k$. This completes the proof. □

**Theorem 2.2.** *The edge-balance index set of* $C_n \times_L (St(1), c))$ *is* $\{0, 2, \cdots, 2k\}$ *for* $n = 3k$, $3k + 1$ *and* $3k + 2$.

**Proof.** In $C_n \times_L (\mathrm{St}(1), c)$, there are totally $2n$ vertices. Thus, for a friendly edge labeling, $e(0) = n = e(1)$. Also, all vertices on $C_n$ are of order 3 and all vertices on St(1) are of order 1. Because all vertices are of odd order, every vertex must be labeled either 0 or 1. Since the total number of vertices is even, the possible edge-balance indexes are all even.

First, we label all edges on $C_n$ 1 and all edges on St(1) 0. Obviously, its edge-balance index is 0. Name all the vertices on $C_n$ $v_i$ where $1 \leq i \leq n$. Also name all the leaf vertices $u_i$ if it connects to the vertex $v_i$. Note that all $v_i$ are labeled 1 and all $u_i$ are labeled 0.

Now, exchange the 1-label of the edge $v_1v_2$ with the 0-label of the edge $v_nu_n$. This turns the label of the vertex $u_n$ into 1 and the labeled of $v_1$ and $v_2$ into 0. The new edge labeling has edge-balance index 2.

If we continue by exchanging the 1-label of the edge $v_{2t+1}v_{2t+2}$ with the 0-label of the edge $v_{n-t}u_{n-t}$, we obtain a new edge labeling each time to increase the edge-balance index by 2. After repeating $k$ times, we archive the maximal edge-balance index $2k$. Thus, the edge-balance index set of $C_n \times_L (\mathrm{St}(1), c)$ is $\{0, 2, 4, \ldots, 2k\}$. □

**Example 5.** Figure 4 shows that $\mathrm{EBI}(C_n \times_L (\mathrm{St}(1), c)) = \{0, 2\}$ for $n =$

3, 4, 5.

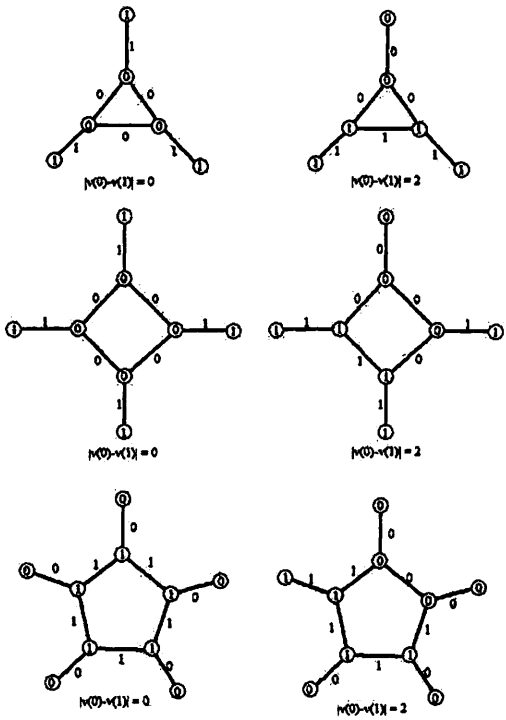
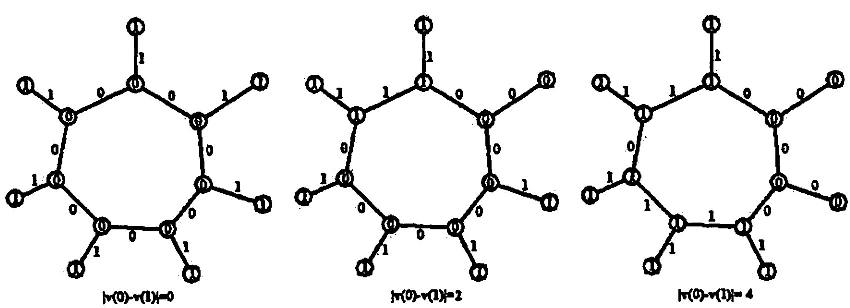



Figure 4: $\mathrm{EBI}(C_n \times_L (\mathrm{St}(1), c))$ for $n = 3, 4, 5$

**Example 6.** Figure 5 shows that $\mathrm{EBI}(C_n \times_L (\mathrm{St}(1), c)) = \{0, 2, 4\}$ for $n = 6, 7, 8$.



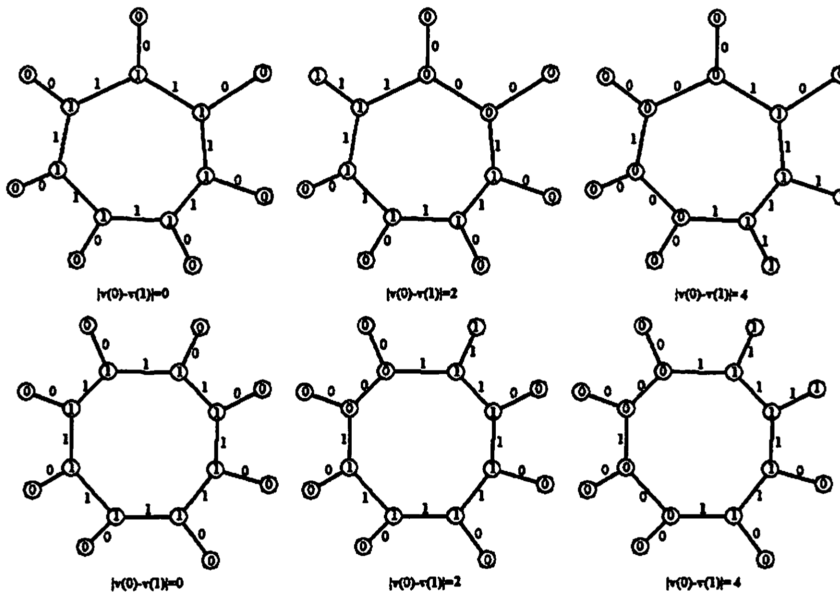



Figure 5: $\mathrm{EBI}(C_n \times_L (\mathrm{St}(1), c))$ for $n = 6, 7, 8$

# 3 On edge-balance index sets of $L$-product of $C_n$ with St(2)

**Lemma 3.1.** *The highest edge-balance index of $C_n \times_L (St(2), c)$ is*

1. $3k$ *if* $n = 4k$;
2. $3k + 2$ *if* $n = 4k + 1$;
3. $3k + 1$ *if* $n = 4k + 2$;
4. $3k + 3$ *if* $n = 4k + 3$.

**Proof.** To find the highest edge-balance index, we try to construct a labeling with the least 1-vertices possible. The strategy is to have as many 1-edges connected to a vertex labeled 0 or unlabeled as possible. When a vertex labeled 1 cannot avoid, label all its edges 1.

By the above strategy, we first label all edges in $C_n$ 1. Then, fill a copy of St(2) by as many 1-edges as possible until we run out of 1-edges. All the rest edges are labeled 0. In this construction, 0-edges are all in St(2)s.

To calculate the edge-balance index of this labeling, we consider the following four cases:

1. When $n = 4k$, the total number of edges is $4k + 2(4k) = 12k = 3n$ with $e(0) = 6k = e(1)$. To fill $C_n$ with 1-edges, we use $n = 4k$ 1-edges. There are $6k - 4k = 2k$ 1-edges left for labeling St(2)s. Thus, there are $k$ copies of St(2) are filled by 1-edges and all other $n - k = 4k - k = 3k$ copies are filled by 0-edges. This tells us that there are $k$ vertices labeled 1 on $C_n$ and $2k$ vertices labeled 1 on St(2)s. Also, because all $6k$ 0-edges are in St(2)s, there are $6k$ vertices labeled 0. Since the $C_n$ is labeled by just 1-edges, the vertex of $C_n$ connected to the St(2) filled by 0-edges is unlabeled. This implies that there are $3k$ unlabeled vertices. So, the edge-balance index is $6k - (k + 2k) = 3k$.

2. When $n = 4k + 1$, the total number of edges is $(4k + 1) + 2(4k + 1) = 12k + 3$. We first construct an edge labeling with $e(0) = 6k + 2$ and $e(1) = 6k + 1$. To fill $C_n$ with 1-edges, we use $n = 4k + 1$ 1-edges. There are $(6k + 1) - (4k + 1) = 2k$ 1-edges left for labeling St(2)s. Thus, there are $k$ copies of St(2) are filled by 1-edges and all other $n - k = (4k + 1) - k = 3k + 1$ copies are filled by 0-edges. This tells us that there are $k$ vertices labeled 1 on $C_n$ and $2k$ vertices labeled 1 on St(2)s. Also, because all $6k + 2$ 0-edges are in St(2)s, there are $6k + 2$ vertices labeled 0. Since the $C_n$ is labeled by just 1-edges, the vertex of $C_n$ connected to the St(2) filled by 0-edges is unlabeled. This implies that there are $3k + 1$ unlabeled vertices. So, the edge-balance index is $(6k + 2) - (k + 2k) = 3k + 2$.

3. When $n = 4k + 2$, the total number of edges is $(4k + 2) + 2(4k + 2) = 12k + 6$ with $e(0) = 6k + 3 = e(1)$. To fill $C_n$ with 1-edges, we use $n = 4k + 2$ 1-edges. There are $(6k + 3) - (4k + 2) = 2k + 1$ 1-edges left for labeling St(2)s. Thus, there are $k$ copies of St(2) are filled by 1-edges and one edge of St(2) is labeled 1. All other St(2) edges are labeled 0 which means $(n - k) - 1 = 3k + 1$ copies of St(2) are filled by only 0-edges. Note that one St(2) are split by one of each. This tells us that there are $k + 1$ vertices labeled 1 on $C_n$ and $2k + 1$ vertices labeled 1 on St(2)s. Also, because all $6k + 3$ 0-edges are in St(2)s, there are $6k + 3$ vertices labeled 0. Since the $C_n$ is labeled by just 1-edges, the vertex of $C_n$ connected to the St(2) filled by 0-edges is unlabeled. This implies that there are $3k + 1$ unlabeled vertices. So, the edge-balance index is $(6k + 3) - (k + 1 + 2k + 1) = 3k + 1$.

4. When $n = 4k + 3$, the total number of edges is $(4k + 3) + 2(4k + 3) = 12k + 9$. We first construct an edge labeling with $e(0) = 6k + 5$ and $e(1) = 6k + 4$. To fill $C_n$ with 1-edges, we use $n = 4k + 3$ 1-edges. There are $(6k + 4) - (4k + 3) = 2k + 1$ 1-edges left for labeling St(2)s. Thus, there are $k$ copies of St(2) are filled by 1-edges and one edge of St(2) is labeled 1. All other St(2) edges are labeled 0 which means

$(n-k)-1 = 3k+2$ copies of St(2) are filled by only 0-edges. Note that one St(2) are split by one of each. This tells us that there are $k+1$ vertices labeled 1 on $C_n$ and $2k+1$ vertices labeled 1 on St(2)s. Also, because all $6k+5$ 0-edges are in St(2)s, there are $6k+5$ vertices labeled 0. Since the $C_n$ is labeled by just 1-edges, the vertex of $C_n$ connected to the St(2) filled by 0-edges is unlabeled. This implies that there are $3k+2$ unlabeled vertices. So, the edge-balance index is $(6k+5)-(k+1+2k+1) = 3k+3$.

Since, in this construction, all 0-edges are on St(2)s, any exchange with 1-edges on $C_n$ will increase the number of vertices labeled 1 by 1. Thus, this labeling yields the maximal edge-balance index. □

**Lemma 3.2.** *The edge-balance index set of $C_n \times_L (St(2), c)$ contains*

1. $\{1, 2, \cdots, 3k\}$ *if* $n = 4k$;
2. $\{2, 3, \cdots, 3k+2\}$ *if* $n = 4k+1$;
3. $\{1, 2, \cdots, 3k+1\}$ *if* $n = 4k+2$;
4. $\{2, 3, \cdots, 3k+3\}$ *if* $n = 4k+3$.

**Proof.** Let us start with the edge labeling constructed at Lemma 3.1. The number of unlabeled vertices is listed as follow:

1. $3k$ if $n = 4k$;
2. $3k+2$ if $n = 4k+1$;
3. $3k+1$ if $n = 4k+2$;
4. $3k+3$ if $n = 4k+3$.

Note that these unlabeled vertices are connected to two 0-edges and two 1-edges. Without loss of generality, we can assume that they are lining up sequentially.

First, we work on the unlabeled vertex adjacent to a 1-vertex on $C_n$. By switching a 0-edge on St(2) leaf and a 1-edge adjacent to another unlabeled vertex, we replace a 0-vertex on a leaf by a 1-vertex and change the adjacent unlabeled vertex into a vertex labeled 0. This move reduces the edge-balance index by 1.

Next, we move on to the adjacent vertex. (Now, it is labeled 0.) Repeat the same exchange method as above, we can reduce the edge-balance by 1 again.

This exchange method works until you have turned all the adjacent unlabeled vertices into 0-vertices. Thus, we can reduce the edge-balance index by one the number of unlabeled vertices minus one times. This completes the proof. □

**Theorem 3.3.** *The edge-balance index set of $C_n \times_L (St(2), c))$ is*

1. $\{1, 2, \cdots, 3k\}$ *if* $n = 4k$;
2. $\{2, 3, \cdots, 3k+2\}$ *if* $n = 4k+1$;
3. $\{1, 2, \cdots, 3k+1\}$ *if* $n = 4k+2$;
4. $\{2, 3, \cdots, 3k+3\}$ *if* $n = 4k+3$.

**Proof.** When $n$ is even, we can label all pairs of edges in each copy of St(2) by 0 and 1. Then label $C_n$ by the sequence $0, 1, 0, 1, \cdots$. It is easy to see that the edge-balance index is 0. With the results of Lemma 3.2, we complete the proofs of (1) and (3).

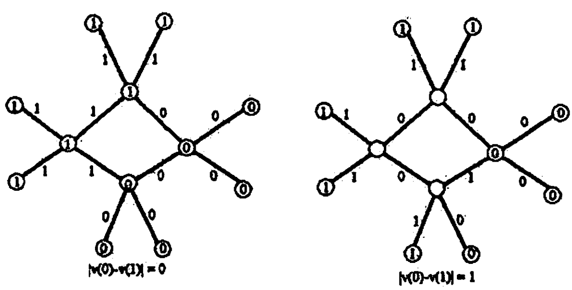
When $n$ is odd, we construct two edge labeling with edge-balance index 0 and 1. Since $n$ is odd, the total number of edges is $n + 2n = 3n$ is also odd. Moreover, if we assume that $n = 2t + 1$, we may have $e(0) = 3t + 1$ and $e(1) = 3t + 2$.

First, we label every pair of edges on St(2) by 0 and 1 and edges on $C_n$ by the sequence $1, 0, 1, 0, \cdots, 1$. It is easy to see that all vertices on $C_n$ are unlabeled except one is labeled 1. The leaves of St(2) provides the equal number of 0-vertices and 1-vertices. Thus, the edge-balance index is $-1$ in the sense of $e(0) - e(1)$. By taking absolute value, we obtain an edge labeling with the edge-balance index 1.

By switching a 1-edge leaf of the unlabeled vertex on $C_n$ with the 0-edge adjacent to the same unlabeled vertex, we replace the 1-vertex on the leaf by a 0-vertex. At the same time, we turn the adjacent unlabeled vertex into a vertex labeled 1. Everything else remains the same. Thus, the edge-balance increases by 1. This creates an edge labeling with the edge-balance index 0.

With these two edge labelings, we complete the proofs of (2) and (4). □

**Example 7.** Figure 6 shows that $\mathrm{EBI}(C_4 \times_L (\mathrm{St}(2), c)) = \{0, 1, 2, 3\}$.




**Example 8.** Figure 7 shows that $\mathrm{EBI}(C_5 \times_L (\mathrm{St}(2), c)) = \{0, 1, 2, 3, 4, 5\}$

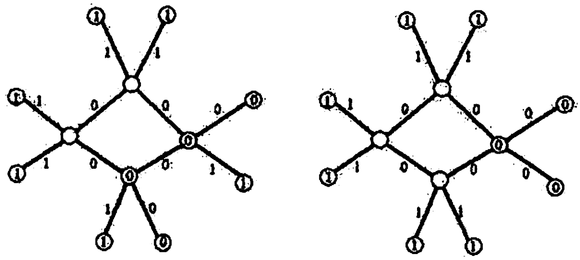


Figure 6: $\mathrm{EBI}(C_4 \times_L (\mathrm{St}(2), c))$

# 4 On edge-balance index sets of $L$-product of $C_n$ with $\mathrm{St}(m)$ where $m$ is odd and greater or equal to three

**Lemma 4.1.** *Consider $C_n \times_L (St(m), c)$ where $m$ is odd and greater or equal to three. Let $n = mk + r$ where $0 \le r < m$ and $r = 2t$ if $r$ is even or $r = 2t + 1$ if $r$ is odd. We also define $\frac{n(m+1)}{2} = mk_0 + r_0$ and $\frac{n(m-1)}{2} = mk_1 + r_1$ where $0 \le r_0, r_1 < m$. Let $T = r_0 r_1 - 2$. The highest edge-balance index is*

1. $n + k$ *if* $r = 0$;
2. $n + k$ *if* $T < 0$ *and* $r$ *is even;*
3. $n + k - 1$ *if* $T < 0$ *and* $r$ *is odd;*
4. $n + k + 2$ *if* $T > 0$ *and* $r$ *is even;*
5. $n + k + 1$ *if* $T > 0$ *and* $r$ *is odd.*

**Proof.** By the same strategy as Lemma 3.1, to find the maximal edge-balance index, we first label all edges in $C_n$ 1. Then, fill a copy of $\mathrm{St}(m)$ by as many 1-edges as possible until we run out of 1-edges. All the rest edges are labeled 0. In this construction, 0-edges are all in $\mathrm{St}(m)$s.

In $C_n \times_L (\mathrm{St}(m), c)$ where $m$ is odd and greater or equal to three, there are totally $n(m+1)$ edges. Since $m$ is odd, $n(m+1)$ is even. Thus, for a friendly labeling, we should have $e(0) = \frac{n(m+1)}{2} = e(1)$. Since we use $n$ 1-edges in $C_n$, there are $\frac{n(m+1)}{2} - n = \frac{n(m-1)}{2}$ 1-edges left for $\mathrm{St}(m)$s. It produces $\frac{n(m-1)}{2}$ vertices on $\mathrm{St}(m)$s. Also, all 0-edges are on $\mathrm{St}(m)$s. Therefore, there are $\frac{n(m+1)}{2}$ vertices on $\mathrm{St}(m)$s.

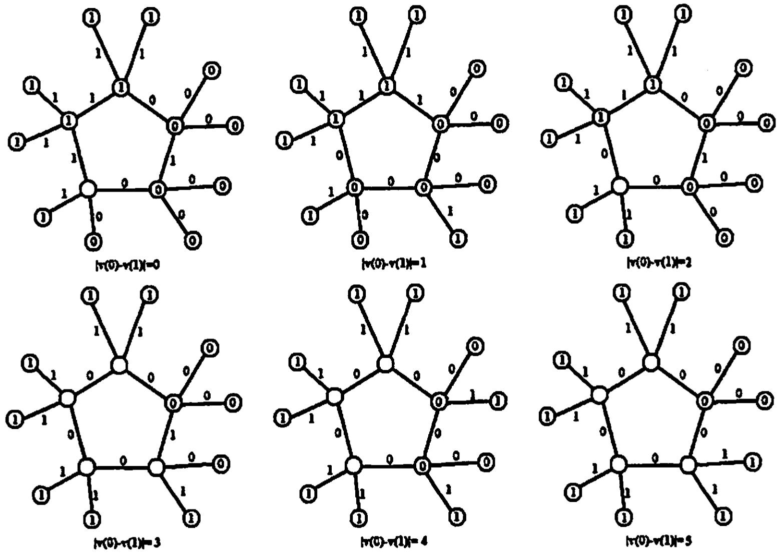


Figure 7: EBI($C_5 \times_L$ (St(2), $c$))

To calculate the edge-balance index of this labeling, we consider the following three cases:

1. When $r = 0$, we have $\frac{mk(m-1)}{2}$ 1-edges on St($m$)s. Since $m$ is odd, $\frac{mk(m-1)}{2}$ is a multiple of $m$. Thus, we have $\frac{k(m-1)}{2} = \frac{n-k}{2}$ vertices labeled 1 on $C_n$. Since the $C_n$ is labeled by just 1-edges and $m \geq 3$, the vertex of $C_n$ connected to the St($m$) filled by 0-edges is labeled 0. This implies that there are $n - \frac{n-k}{2} = \frac{n+k}{2}$ 0-vertices on $C_n$. So, the edge-balance index is

$$\left[\frac{n(m+1)}{2} + \frac{n+k}{2}\right] - \left[\frac{n(m-1)}{2} + \frac{n-k}{2}\right] = n + k.$$

2. When $T < 0$, we have $r_0 < r_1 + 2$. This implies that the vertex with mixed 0- and 1-edges has more 1-edges than 0-edges so it is labeled 1. If $r$ is even, then

$$\begin{aligned} \frac{(mk+r)(m-1)}{2} &= m\frac{k(m-1}{2} + tm - t \\ &= m\left(\frac{k(m-1)}{2} + (t-1)\right) + (m-t). \end{aligned}$$

Thus, we have $\frac{k(m-1)}{2} + (t-1) = \frac{n-k-2}{2}$ vertices labeled 1 on $C_n$ with all 1-edges only. Since the $C_n$ is labeled by just 1-edges and $m \geq 3$, the vertex of $C_n$ connected to the $\mathrm{St}(m)$ filled by 0-edges is labeled 0. This implies that there are $n - \frac{n-k-2}{2} - 1 = \frac{n+k}{2}$ 0-vertices on $C_n$. So, the edge-balance index is

$$\left(\frac{n(m+1)}{2} + \frac{n+k}{2}\right) - \left(\frac{n(m-1)}{2} + \frac{n-k-2}{2} + 1\right) = n+k.$$

If $r$ is odd, then

$$\begin{aligned}\frac{(mk+r)(m-1)}{2} &= m\frac{k(m-1)}{2} + \frac{(2t+1)(m-1)}{2}\\ &= m\left(\frac{k(m-1)}{2} + t\right) + \frac{m-1}{2} - t.\end{aligned}$$

Note that since $r < m$, we have $\frac{m-1}{2} > t$. Thus, we have $\frac{k(m-1)}{2} + t = \frac{n-k-1}{2}$ vertices labeled 1 on $C_n$ with all 1-edges only. Since the $C_n$ is labeled by just 1-edges and $m \geq 3$, the vertex of $C_n$ connected to the $\mathrm{St}(m)$ filled by 0-edges is labeled 0. This implies that there are $n - \frac{n-k-1}{2} - 1 = \frac{n+k-1}{2}$ 0-vertices on $C_n$ with all 0-edges on $\mathrm{St}(m)$ only. So, the edge-balance index is

$$\left(\frac{n(m+1)}{2} + \frac{n+k-1}{2}\right) - \left(\frac{n(m-1)}{2} + \frac{n-k-1}{2} + 1\right) = n+k-1.$$

3. When $T > 0$, we have $r_0 > r_1 + 2$. This implies that the vertex with mixed 0- and 1-edges has more 0-edges than 1-edges so it is labeled 0. If $r$ is even, then

$$\begin{aligned}\frac{(mk+r)(m-1)}{2} &= m\frac{k(m-1)}{2} + tm - t\\ &= m\left(\frac{k(m-1)}{2} + (t-1)\right) + (m-t).\end{aligned}$$

Thus, we have $\frac{k(m-1)}{2} + (t-1) = \frac{n-k-2}{2}$ vertices labeled 1 on $C_n$ with all 1-edges only. Since the $C_n$ is labeled by just 1-edges and $m \geq 3$, the vertex of $C_n$ connected to the $\mathrm{St}(m)$ filled by 0-edges is labeled 0. This implies that there are $n - \frac{n-k-2}{2} - 1 = \frac{n+k}{2}$ 0-vertices on $C_n$. So, the edge-balance index is

$$\left(\frac{n(m+1)}{2} + \frac{n+k}{2} + 1\right) - \left(\frac{n(m-1)}{2} + \frac{n-k-2}{2}\right) = n+k+2.$$

If $r$ is odd, then

$$\frac{(mk+r)(m-1)}{2} = m\frac{k(m-1)}{2} + \frac{(2t+1)(m-1)}{2}$$

$$= m\left(\frac{k(m-1)}{2}+t\right)+\frac{m-1}{2}-t.$$

Note that since $r < m$, we have $\frac{m-1}{2} > t$. Thus, we have $\frac{k(m-1)}{2}+t = \frac{n-k-1}{2}$ vertices labeled 1 on $C_n$ with all 1-edges only. Since the $C_n$ is labeled by just 1-edges and $m \geq 3$, the vertex of $C_n$ connected to the St($m$) filled by 0-edges is labeled 0. This implies that there are $n - \frac{n-k-1}{2} - 1 = \frac{n+k-1}{2}$ 0-vertices on $C_n$ with all 0-edges on St($m$) only. So, the edge-balance index is

$$\left(\frac{n(m+1)}{2}+\frac{n+k-1}{2}+1\right)-\left(\frac{n(m-1)}{2}+\frac{n-k-1}{2}\right)=n+k+1.$$

Since, in this construction, all 0-edges are on St($m$)s, any exchange with 1-edges on $C_n$ will increase the number of vertices labeled 1 by 1. Thus, this labeling yields the maximal edge-balance index. □

**Theorem 4.2.** *Consider $C_n \times_L (St(m), c)$ where $m$ is odd and greater or equal to three. Let $n = mk + r$ where $0 \leq r < m$ and $r = 2t$ if $r$ is even or $r = 2t+1$ if $r$ is odd. We also define $\frac{n(m+1)}{2} = mk_0 + r_0$ and $\frac{n(m-1)}{2} = mk_1 + r_1$ where $0 \leq r_0, r_1 < m$. Let $T = r_0 - r_1 - 2$. The edge-balance index set of $C_n \times_L (St(m), c)$ is*

1. $\{0, 2, 4, 6, \cdots, n+k\}$ *if* $r = 0$;
2. $\{0, 2, 4, 6, \cdots, n+k\}$ *if* $T < 0$ *and* $r$ *is even;*
3. $\{0, 2, 4, 6, \cdots, n+k-1\}$ *if* $T < 0$ *and* $r$ *is odd;*
4. $\{0, 2, 4, 6, \cdots, n+k+2\}$ *if* $T > 0$ *and* $r$ *is even;*
5. $\{0, 2, 4, 6, \cdots, n+k+1\}$ *if* $T > 0$ *and* $r$ *is odd.*

**Proof.** In $C_n \times_L$ (St($m$), $c$), there are totally $n + nm$ vertices. If $m$ is odd, then it has even number of vertices. Also, all vertices on $C_n$ are of order $m+2$ and all vertices on St($m$) are of order 1. As $m$ is odd, all vertices are of odd order. This means every vertex must be labeled either 0 or 1. Since the total number of vertices is even, the possible edge-balance indexes are all even.

Let us start with the edge labeling constructed at Lemma 4.1. Without loss of generality, we can assume that all 0-vertices on $C_n$ are lining up sequentially.

First, we work on the vertex mixed with 0-edges and 1-edges on $C_n$. Note that if $r = 0$, then we skip this step. By switching a 0-edge on St($m$) leaf and a 1-edge adjacent to another 0-vertex, we replace a 0-vertex on leaf by a 1-vertex. Everything else remains the same. This move reduces the edge-balance index by 2.

Second, we work on the 0-vertex adjacent to the previous vertex. Note that if $r = 0$, then we start working on the 0-vertex adjacent to a 1-vertex on $C_n$. By switching a 0-edge on a St($m$) leaf and a 1-edge adjacent to another 0-vertex, we replace a 0-vertex on a leaf by a 1-vertex. Everything else remains the same. This move reduces the edge-balance index by 2.

Next, we move on to the adjacent vertex. Repeat the same exchange method as above, we can reduce the edge-balance by 2 again.

This exchange method works until you run out of all 0-vertices on $C_n$. Thus, we can reduce the edge-balance index by two the number of 0-vertices times. By the proof of Lemma 4.1, the number of 0-vertices on $C_n$ is listed as follow:

1. $\frac{n+k}{2}$ if $r = 0$;
2. $\frac{n+k}{2}$ if $T < 0$ and $r$ is even;
3. $\frac{n+k-1}{2}$ if $T < 0$ and $r$ is odd;
4. $\frac{n+k}{2} + 1$ if $T > 0$ and $r$ is even;
5. $\frac{n+k-1}{2} + 1$ if $T > 0$ and $r$ is odd.

Since, as numerical values, they are all half of the highest edge-balance index, we have enough 0-vertices on $C_n$ to reduce the edge-balance index by two each time all the way down to 0.

This completes the proof. □

**Example 9.** For $C_8 \times_L (\mathrm{St}(3), c)$, we have $n = 8$ and $m = 3$. Thus, $k = 2$, $r = 2$ and $t = 1$. Also, we have $r_0 = 1$ since $\frac{n(m+1)}{2} = 16 = 5 \times 3 + 1$ and $r_1 = 2$ since $\frac{n(m-1)}{2} = 8 = 2 \times 3 + 2$. Therefore, $T = 1 - 2 - 2 = -3 < 0$. By Theorem 4.2(2), EBI $= \{0, 2, 4, \cdots, 8 + 2\}$. Figure 8 shows that $\mathrm{EBI}(C_8 \times_L (St(3), c)) = \{0, 2, 4, 6, 8, 10\}$.

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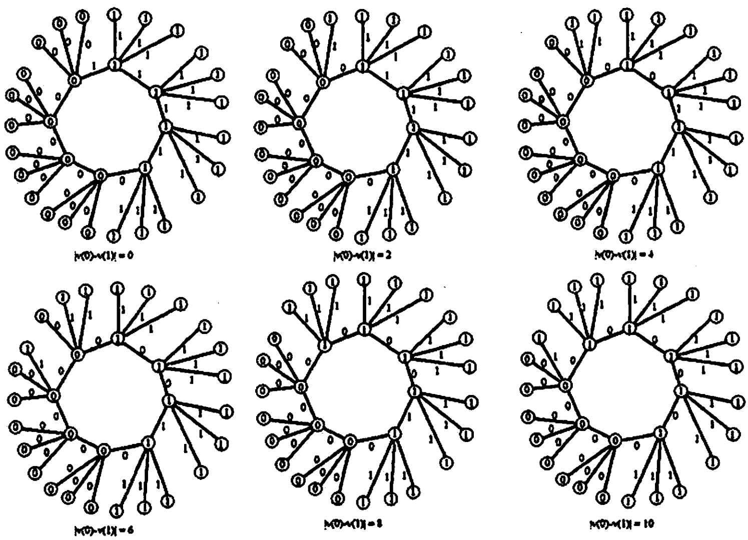

Figure 8: EBI($C_8 \times_L (St(3), c)$)


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