New Bounds on Some Ramsey Numbers*

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Abstract. We derive a new upper bound of 26 for the Ramsey number $R(K_5 - P_3, K_5)$, lowering the previous upper bound of 28. This leaves $25 \leq R(K_5 - P_3, K_5) \leq 26$, improving on one of the three remaining open cases in Hendry's table, which listed Ramsey numbers for pairs of graphs (G, H) with G and H having five vertices.

We also show, with the help of a computer, that $R(B_2, B_6) = 17$ and $R(B_2, B_7) = 18$ by full enumeration of (B_2, B_6)-*good* graphs and (B_2, B_7)-*good* graphs, where B_n is the book graph with n triangular pages.

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1 Introduction

For graphs G and H, a (G,H)-*good* graph is a graph that does not contain G as a subgraph and whose complement does not contain H, and a $(G,H;n)$-*good* graph is a (G,H)-*good* graph on n vertices. The Ramsey number $R(G,H)$ is the smallest integer n such that no $(G,H;n)$-*good* graph exists. We define $\mathcal{R}(G,H)$ as the set of all (G,H)-*good* graphs and $\mathcal{R}(G,H;n)$ as the set of all $(G,H;n)$-*good* graphs. The values and best known bounds for various types of Ramsey numbers are gathered in the dynamic survey *Small Ramsey Numbers* [8].

For two graphs D and F define $D+F$ to be the graph obtained by joining each vertex in D to each vertex in F. If n is a positive integer, we define $B_n = K_2 + \overline{K}_n$ to be the book graph with n pages. We will refer to this K_2 as the 'spine' of a book graph. For the two cases we study, it was known that $17 \leq R(B_2, B_6) \leq 18$ [9] and $R(B_2, B_7) \leq 20$ [2].

In 1989, Hendry [3] compiled a table of Ramsey numbers for connected graphs G and H where both G and H have five vertices. Here, for the number $R(K_5 - P_3, K_5)$ we show that the only possible values are 25 or 26 (note that $K_5 - P_3$ is a K_4 with an additional vertex connected to two of its nodes). The previous upper bound, $R(K_5 - P_3, K_5) \leq 28$, is from Hendry's table and the lower bound is implied by the result $R(K_4, K_5) = 25$ [7]. This latter result is also essential to our improvement of the upper bound to 26. The computations related to the number $R(K_5 - P_3, K_5)$ required only a few hours of a standard desktop computer, while those related to book graphs were more cpu intensive, and were completed in a few days.

2 Enumerations for $R(K_5 - P_3, K_5)$

In order to obtain the new upper bound for $R(K_5 - P_3, K_5)$, it is helpful to enumerate the sets $\mathcal{R}(K_4 - P_3, K_5)$ and $\mathcal{R}(K_5 - P_3, K_4)$. It is known that $R(K_4 - P_3, K_5) = 14$ and $R(K_5 - P_3, K_4) = 18$ [1]. Using straightforward algorithms, the 1092 graphs in $\mathcal{R}(K_4 - P_3, K_5)$ and the 3454499 graphs in $\mathcal{R}(K_5 - P_3, K_4)$ were enumerated. We tested the correctness of these algorithms by exactly reproducing the publicly available sets $\mathcal{R}(K_4, K_4)$ and $\mathcal{R}(K_3, K_5)$ [4].

The program *nauty* [5] was used to eliminate isomorphisms. The data are summarized in Tables I and II.

n	$\vert\mathcal{R}(K_5 - P_3, K_4; n)\vert$	# Edges	Contains K_4	# Edges
1	1	0	0	
2	2	0-1	0	
3	4	0-3	0	
4	10	1-6	1	6
5	26	2-8	2	6-7
6	92	3-12	8	6-12
7	391	5-16	29	7-12
8	2228	7-21	149	8-16
9	15452	9-27	751	10-19
10	107652	12-31	3946	12-24
11	557005	15-36	10649	15-28
12	1455946	18-40	6780	18-32
13	1184231	33-45	0	
14	130816	41-50	0	
15	640	50-55	0	
16	2	60	0	
17	1	68	0	

Table I. Statistics of $\mathcal{R}(K_5 - P_3, K_4)$.

The last two columns offer counts and the corresponding edge ranges of all $(K_5 - P_3, K_4)$-*good* graphs which contain K_4 as a subgraph. In other words, those graphs which are $(K_5 - P_3, K_4)$-*good* but not (K_4, K_4)-*good.*

n	$\vert\mathcal{R}(K_4 - P_3, K_5; n)\vert$	# Edges	Contains K_3	# Edges
1	1	0	0	
2	2	0-1	0	
3	4	0-3	1	3
4	8	0-4	1	3
5	15	1-6	2	3-4
6	36	2-9	4	3-6
7	78	3-12	7	4-7
8	190	4-16	11	5-9
9	308	6-17	18	6-12
10	326	8-20	13	8-13
11	110	10-22	5	10-15
12	13	12-24	1	12
13	1	26	0	

Table II. Statistics of $\mathcal{R}(K_4 - P_3, K_5)$.

Here, the last two columns offer counts and the corresponding edge ranges of all $(K_4 - P_3, K_5)$-*good* graphs which contain K_3 as a subgraph. In other words, those graphs which are $(K_4 - P_3, K_5)$-*good* but not (K_3, K_5)-*good.*

3 $R(K_5 - P_3, K_5) \leq 26$

Given a vertex x in a $(K_5 - P_3, K_5)$-*good* graph F, define F_x^+ to be the subgraph induced by the vertices adjacent to x and F_x^- to be the subgraph induced by the vertices non-adjacent to (and not including) x. Clearly, F_x^+ is $(K_4 - P_3, K_5)$-*good* and F_x^- is $(K_5 - P_3, K_4)$-*good*. Because $R(K_4 - P_3, K_5) = 14$ and $R(K_5 - P_3, K_4) = 18$ [1], the degree of a vertex in a $(K_5 - P_3, K_5; 26)$-*good* graph is bounded by 8 and 13, inclusive.

Walker [10] proved a result similar to that in Lemma 1 below for complete graphs. The proof from [10] still holds for our case as follows.

Lemma 1 *If n_i is the number of vertices of degree i in a $(K_5 - P_3, K_5; n)$-*good *graph and $E(G, H, n)$ denotes the maximum number of edges in a $(G, H; n)$-*good *graph then*

$$0 \leq \sum_{i=8}^{13} (2E(K_4 - P_3, K_5, i) + 2E(K_5 - P_3, K_4, n - i - 1) + 3i(n - i - 1) - (n - 1)(n - 2))n_i.$$

Using $n = 26$ in Lemma 1, along with our data from Tables I and II, yields the constraint

$$0 \leq -12n_8 - 7n_9 + 3n_{11} + 3n_{12}, \tag{1}$$

and we know

$$26 = n_8 + n_9 + n_{10} + n_{11} + n_{12} + n_{13}. \tag{2}$$

It is easy to see that there is no nonnegative integer solution with $n_8 \geq 6$.

A similar approach for $n = 27$ yields an inequality similar to (1) with all negative coefficients, proving there is no $(K_5 - P_3, K_5; 27)$-*good* graph. When $n = 25$, we cannot draw any useful conclusions from the resulting inequality.

Lemma 2 *The sum of the degrees of the vertices of any K_4 contained in a $(K_5 - P_3, K_5; 26)$-*good *graph cannot exceed* 34. *Furthermore, any K_4 in a $(K_5 - P_3, K_5; 26)$-*good *graph must have at least two vertices of degree 8.*

Proof: Let F be a $(K_5 - P_3, K_5; 26)$-*good* graph with K_4 as a subgraph. Let $X = \{x_j\}_{j=1}^{4}$ be the vertex set of the K_4. To avoid creating $K_5 - P_3$, the neighborhoods of each vertex x_j, other than the vertices in X, must be disjoint. By counting the vertices adjacent to each vertex x_j that are not in X, we have

$$\sum_{j=1}^{4}(deg(x_j) - 3) + 4 \leq 26.$$

So,

$$\sum_{j=1}^{4} deg(x_j) \leq 34.$$

Because the minimum degree of a vertex is 8, this inequality will hold only if there are at least two vertices in X of degree 8.

□

Lemma 3 *If a* $(K_5 - P_3, K_5; 26)$-good *graph has two* K_4*'s, then they must be disjoint.*

Proof: Let F be a $(K_5 - P_3, K_5; 26)$-*good* graph with two K_4's that share a vertex. Let L denote the vertex set of the two K_4's. Note that if they shared more than one vertex, a $K_5 - P_3$ would be created. By Lemma 2, L must have at least three vertices of degree 8. Observe, from (1) and (2), that there can be no more than 5 vertices of degree 8.

Case 1: Suppose there are exactly three vertices of degree 8 in L. One of these must be the shared vertex. In order to comply with Lemma 2, each K_4 must have two vertices of degree 9, for a total of four vertices of degree 9. However, by (1), it cannot be the case that both $n_8 \geq 3$ and $n_9 \geq 4$.

Case 2: Assume there are exactly four vertices of degree 8 in L. By (1), there can be at most one vertex of degree 9. The remaining vertices must be of degree 10 or greater. But with the assumption that L has exactly four vertices of degree 8, every configuration of the degrees contradicts Lemma 2.

Case 3: Let there be exactly five vertices of degree 8 in L. Then, by (1), there can be at most one vertex of degree less than or equal to 10. This requires L to contain a vertex of degree greater than or equal to 11, which is impossible by Lemma 2.

Thus, if a $(K_5 - P_3, K_5; 26)$-*good* graph has two K_4's, then they may not share a vertex. □

Theorem 1 $R(K_5 - P_3, K_5) \leq 26.$

Proof: Let F be a $(K_5 - P_3, K_5; 26)$-*good* graph. There must exist at least one K_4 or else the graph would be $(K_4, K_5; 26)$-*good*, contradicting $R(K_4, K_5) = 25$. Fix a vertex from the K_4. The remaining 25 vertices must also contain at least one K_4. By Lemma 3, these two K_4's must be disjoint. Since the K_4's are disjoint, Lemma 2 implies that there are at least four vertices of degree 8. By (1), there can then be at most one vertex of degree 9. Thus, at least one of the K_4's must contain two vertices of degree 10 or greater, which contradicts Lemma 2. □

Our approach was not effective at further lowering the upper bound, but it is possible that an approach similar to that taken in [6] or [7] could prove successful. We also attempted to construct a $(K_5 - P_3, K_5; 25)$-*good* graph by extending the set of 350904 known $(K_4, K_5; 24)$-*good* graphs. We then tried altering the neighborhoods of specific vertices from graphs in $\mathcal{R}(K_4, K_5; 24)$ to construct new $(K_5 - P_3, K_5; 24)$-*good* graphs. These efforts were not successful, but they were also not exhaustive.

4 Two Ramsey Numbers for Books

Fully enumerating the sets $\mathcal{R}(B_2, B_6)$ and $\mathcal{R}(B_2, B_7)$ gives justification for Theorems 2 and 3 below. Data for $(B_2, B_6; n)$-*good* graphs are presented in Table III. Data for $(B_2, B_7; n)$-*good* graphs are presented in Table IV.

Theorem 2 $R(B_2, B_6) = 17$.

We use a one-vertex extension algorithm similar to that described in [7]. Any new vertex added to a $(B_2, B_6; n)$-*good* graph must be prevented from covering any K_2 contained in a K_3 or any P_3. Additionally, it must hit any $\overline{K}_{1,6}$, and the 'spine' of any $\overline{B}_5$. The algorithm ultimately yields all vertex sets to which the new vertex can connect.

These results were checked using a separate one-vertex extension algorithm which added a vertex to a $(B_2, B_6; n)$-*good* graph and joined it in every possible way. The resulting set of graphs was then filtered to remove all graphs which were not $(B_2, B_6; n+1)$-*good*. The two algorithms produced identical results.

Theorem 3 $R(B_2, B_7) = 18$.

The first one-vertex extension algorithm used for Theorem 2 was modified slightly to generate $\mathcal{R}(B_2, B_7)$. We applied the second extension algorithm to generate graphs on up to 12 vertices and to generate graphs on greater than 16 vertices. Because the number of intermediate graphs is too large and this algorithm is very slow, we were unable to generate those graphs on 13 through 16 vertices due to time and space constraints. The two algorithms yielded identical results for the cases tested.

edges	number of vertices n																
e	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	sum
0	1	1	1	1	1	1	1										7
1		1	1	1	1	1	1										6
2			1	2	2	2	2										9
3			1	3	4	5	5										18
4				2	6	9	10	2									29
5					5	14	20	9									48
6					3	17	37	35	2								94
7						12	50	87	9								158
8						6	55	161	44								266
9						2	45	235	173	2							457
10							22	272	534	16							844
11							6	229	1166	111							1512
12							1	138	1724	640							2503
13								49	1742	2575	1						4367
14								12	1247	6913	24						8196
15								2	611	12057	383						13053
16								1	197	13515	3606						17319
17									41	9821	18290						28152
18									10	4679	51619	16					56324
19									1	1443	84728	161					86333
20									1	309	82705	2530					85545
21										58	48951	24822					73831
22										12	18101	114410					132523
23										3	4412	254684	3				259102
24										2	812	295854	24				296692
25										1	152	190280	615				191048
26											36	71277	10254				81567
27											11	16779	65668				82458
28											4	2991	173717				176712
29											1	561	209420				209982
30											1	158	124637				124796
31												50	38747	15			38812
32												18	6751	431			7200
33												6	863	3314			4183
34												3	216	8561			8780
35												2	103	8655			8760
36												1	58	3845			3904
37													24	835			859
38													9	99			108
39													3	11	6		20
40													2		6		8
41													1		18		19
42													1	1	41		43
43														1	31		32
44														1	11		12
45														1	4		5
46														1			1
47														1			1
48														1		1	2
49														1		1	2
50																1	1
sum	1	2	4	9	22	69	255	1232	7502	52157	313837	974603	631116	25774	117	3	2006703

Table III. Number of $(B_2, B_6; n)$-*good* graphs with e edges.

This full enumeration of $\mathcal{R}(B_2, B_6)$ shows that $R(B_2, B_6) = 17$, with three critical graphs on 16 vertices.

edges e	8	9	10	11	12	13	14	15	16	17	sum
1	1										7
2	2										11
3	5										23
4	11	1									39
5	23	4	1								67
6	52	22	1								132
7	99	82	5								248
8	167	233	22								483
9	237	523	107								914
10	272	972	457	3							1726
11	229	1484	1683	22							3424
12	138	1846	4886	203	1						7075
13	49	1765	10373	1550	1						13738
14	12	1249	16149	8569	4						25983
15	2	611	18741	33427	36						52817
16	1	197	16340	90836	543						107917
17		41	10479	172098	7749						190367
18		10	4765	227707	66967	3					299452
19		1	1450	211682	335550	16					548699
20		1	310	139383	1030461	202					1170357
21			58	64793	2023072	3598					2091521
22			12	21006	2601178	61936					2684132
23			3	4801	2224981	635231					2865016
24			2	863	1286044	3374689					4661598
25			1	158	510455	9717128	4				10227746
26				38	141805	16254780	278				16396901
27				11	28687	16660092	15645				16704435
28				4	4850	10849383	374556				11228793
29				1	884	4622454	3438516				8061855
30				1	219	1334479	14158181	2			15492882
31					63	277504	29275327	4			29552898
32					23	47385	33114201	28			33161637
33					7	8345	21890140	828			21899320
34					3	1849	8910524	28309			8940685
35					2	517	2355110	407886			2763515
36					1	174	443185	2204437			2647797
37						58	73919	5053747			5127724
38						21	15168	5600518			5615707
39						7	4148	3309850			3314005
40						4	1338	1115058	21		1116421
41						1	499	223498	186		224184
42						1	206	27665	1594		29466
43							81	2535	8037		10653
44							30	516	19995		20541
45							12	383	24083		24478
46							6	299	13259		13564
47							3	204	3735		3942
48							2	118	681	1	802
49							1	70	85		156
50								30	10	1	41
51								12		8	20
52								5		20	25
53								3		22	25
54								2		9	11
55								1		3	4
56								1	1	1	3
57									1		1
58									1		1
59									1		1
60									1		1
61									1		1
62									1		1
63									1		1
64									1		1
sum	1301	9042	85845	977156	10263586	63849857	114071080	17976009	71695	65	207305998

Table IV. Number of $(B_2, B_7; n)$-*good* graphs with e edges. The data for $n \le 7$ is identical to that of Table III, so they are not included.

This full enumeration of $\mathcal{R}(B_2, B_7)$ shows that $R(B_2, B_7) = 18$, with 65 critical graphs on 17 vertices.

One of the three $(B_2, B_6; 16)$-*good* graphs is presented in Figure 1 below. This graph is isomorphic to one previously found by Rousseau [9]. For the remaining two, one can be obtained by adding either of the edges AC or BD; the other by adding both AC and BD.

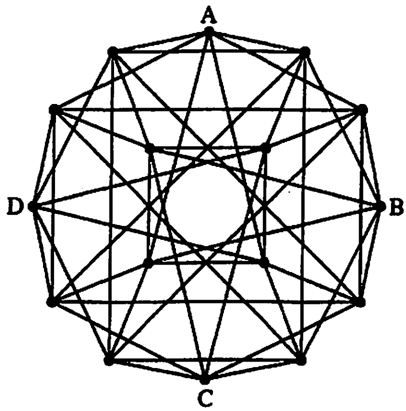

Figure 1. One of three $(B_2, B_6; 16)$-*good* graphs.

Figure 2 shows one $(B_2, B_7; 17)$-*good* graph. To maintain the symmetries present in Figure 2 and to avoid creating ambiguities, the 17th vertex, X, is not shown. The four vertices adjacent to X are indicated as such. Note that there is no vertex in the center of the graph.

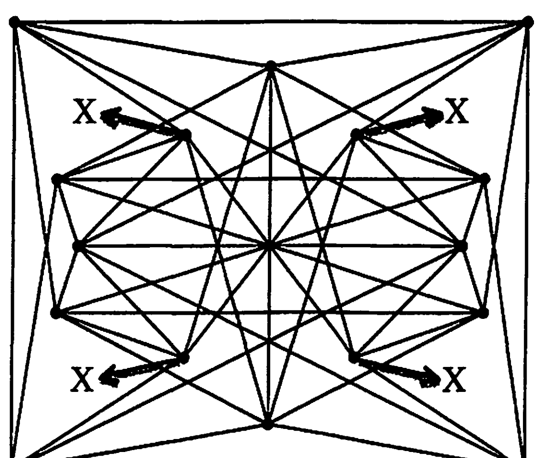

Figure 2. A $(B_2, B_7; 17)$-*good* graph.

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