Partition Types

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ABSTRACT. For a graph G having chromatic number k, an equivalence relation is defined on the set X consisting of all proper vertex k-colorings of G. This leads naturally to an equivalence relation on the set $\mathcal{P}$ consisting of all partitions of $V(G)$ into k independent subsets of color classes. The notion of a partition type arises and the algebra of types is investigated.

1. INTRODUCTION AND NOTATION

The graphs considered in this paper are finite, undirected, and simple. For a given graph G, the vertex and edge sets of G are denoted by $V(G)$ and $E(G)$, respectively. For convenience, the vertex set is often written as $V(G) = \{1, 2, \ldots, n\}$. The order of G is the cardinality of $V(G)$ and is denoted by $n = |V(G)|$. It is tacitly assumed that the two usages of n will not create any confusion. Any vertex having degree $n-1$ will be called a terminal vertex. A subset I of $V(G)$ is independent provided that no two distinct vertices in I are adjacent. The maximum cardinality of an independent subset of $V(G)$ is denoted by $\alpha(G)$. For a subset X of $V(G)$ or $E(G)$, the induced subgraph of G by X is denoted $G[X]$. A graph G is vertex k-critical whenever $\chi(G) = k$ and $\chi(G-v) = k-1$ for every vertex $v \in V(G)$, where $\chi(G)$ is the chromatic number of G. A complete subgraph K having order r is called an r-clique and will be denoted by K_r. Lastly, a critical r-clique of G is a subgraph K_r having the property that $\chi(G-K_r) = \chi(G) - r$. A critical r-clique will be denoted by K_r^c.

This paper focuses on partitions admitting critical r-cliques especially when $r \geq 2$. A graph in which every subgraph isomorphic to K_2 is a critical 2-clique deserves special mention. A graph G is said to be vertex double-critical provided that $\chi(G-u-v) = \chi(G) - 2$ for every adjacent pair of vertices u, v. This definition arises out of its relation to the Erdős-Lovász Tihany Conjecture in [2]. One particular case of this conjecture is equivalent to the statement that the only vertex double-critical graph is the complete graph; it is often referred to as the Erdős-Lovász double-critical conjecture. A discussion of this conjecture can be found in [6]. In [9], Stiebitz has shown that K_5 is the only 5-chromatic vertex double-critical graph. Related results for quasi-line graphs are given in [1]. To date, the Erdős-Lovász double-critical conjecture remains open for k-chromatic graphs with $k \geq 6$. In Theorem 6 of [7] and in [5], the edge analogue of the Erdős-Lovász double-critical conjecture is proved. Studying the properties of the pairs of vertices u, v satisfying $\chi(G-u-v) = \chi(G) - 2$, and more generally critical r-cliques, motivates the notion of a partition type.

2. Partition Types

Let G be a graph with $\chi(G) = k$ and fix a set $C = \{c_1, c_2, \ldots, c_k\}$ consisting of k distinct colors. The group of all permutations of C will be denoted by S and the group of all automorphisms of G will be denoted by $\mathcal{G}$. Hence, $S = \mathrm{Perm}(C)$ and $\mathcal{G} = \mathrm{Aut}(G)$. A proper vertex k-coloring of G is a function $f : V(G) \to C$ such that $f(i) \neq f(j)$ whenever $ij \in E(G)$. Let X denote the set of all proper k-colorings of G. Define a relation on X by the following rule: $f_1 \sim f_2$ if and only if there exist $\alpha \in S$ and $\sigma \in \mathcal{G}$ such that $\alpha \circ f_2 = f_1 \circ \sigma$. Whenever there is no possibility of confusion, we shall adopt the usual convention and write $\alpha \circ f$ as αf.

Proposition 1. *$\sim$ is an equivalence relation on X.*

Proof. Denote the identity element of $\mathcal{G}$ by $e_{\mathcal{G}}$ and the identity element of S by e_S. Clearly, $e_S f = f e_{\mathcal{G}}$ for every $f \in X$. Hence, $f \sim f$ for every $f \in X$ and so $\sim$ is reflexive. To see that $\sim$ is symmetric, suppose that $f_1, f_2 \in X$ with $f_1 \sim f_2$. Then there exist $\alpha \in S$ and $\sigma \in \mathcal{G}$ such that $\alpha f_2 = f_1 \sigma$. But then $\alpha^{-1} \in S$ and $\sigma^{-1} \in \mathcal{G}$ with $\alpha^{-1} f_1 = f_2 \sigma^{-1}$. Thus $f_2 \sim f_1$. Finally, suppose that $f_1, f_2, f_3 \in X$ with $f_1 \sim f_2$ and $f_2 \sim f_3$. Then there exist $\alpha, \beta \in S$ and $\sigma, \tau \in \mathcal{G}$ such that $\alpha f_2 = f_1 \sigma$ and $\beta f_3 = f_2 \tau$. As a result,

$$(\alpha\beta) f_3 = \alpha(\beta f_3) = \alpha(f_2 \tau) = (\alpha f_2)\tau = (f_1 \sigma)\tau = f_1(\sigma\tau).$$

Consequently, $f_1 \sim f_3$ and it follows that $\sim$ is transitive. Therefore, $\sim$ is an equivalence relation on X. ■

Remark 1. *By considering the group action of the product group $S \times \mathcal{G}$ on X by the rule $(\alpha, \sigma) \cdot f = \alpha f \sigma^{-1}$, the equivalence relation of Proposition 1 is in fact the equivalence relation on X induced by this group action. It is a straightforward to confirm that this is a well-defined group action.*

Proposition 2. *Let $f_1, f_2 \in X$. Then $f_1 \sim f_2$ if and only if $f_1 \sim \beta f_2$ for all $\beta \in S$.*

Proof. Let $f_1, f_2 \in X$. Suppose that $f_1 \sim \beta f_2$ for all $\beta \in S$. Simply choosing $\beta = e_S$ implies that $f_1 \sim f_2$. Conversely, suppose that $f_1 \sim f_2$. Then there exist $\alpha \in S$ and $\sigma \in \mathcal{G}$ such that $\alpha f_2 = f_1 \sigma$. Now observe that for any $\beta \in S$, $(\alpha\beta^{-1})(\beta f_2) = \alpha f_2 = f_1 \sigma$. Therefore, $f_1 \sim \beta f_2$. ■

Let $f \in X$. In order to compute $|\bar{f}|$, the cardinality of the equivalence class of f, let P_f denote the partition of $V(G)$ induced by f. Because f is a k-coloring of G, the partition P_f can be expressed in terms of the color classes determined by f as: $P_f = \left\{P_f^{c_1}, P_f^{c_2}, \ldots, P_f^{c_k}\right\}$, where $P_f^{c_j}$ represents the jth color class for $j = 1, 2, \ldots, k$. Define $\mathcal{P}$ to be the set of all partitions of $V(G)$ induced by the elements of X, i.e., $\mathcal{P} = \{P_f \mid f \in X\}$. Next, define an action of $\mathcal{G}$ on $\mathcal{P}$ by the rule:

$$\sigma \cdot P_f = \sigma P_f = \left\{\sigma\left(P_f^{c_1}\right), \sigma\left(P_f^{c_2}\right), \ldots, \sigma\left(P_f^{c_k}\right)\right\}.$$

Indeed, the defining properties of a group action are easily verified: For all $\sigma, \tau \in \mathcal{G}$ and for all $P_f \in \mathcal{P}$,

$$\begin{array}{ll} \text{I.} & e_{\mathcal{G}} \cdot P_f = P_f \text{ and} \\ \text{II.} & \sigma \cdot (\tau \cdot P_f) = (\sigma\tau) \cdot P_f. \end{array}$$

This group action naturally induces an equivalence relation on the set $\mathcal{P}$ defined by the following rule: $P_{f_1} \sim P_{f_2}$ if and only if there exists $\sigma \in \mathcal{G}$ such that $P_{f_2} = \sigma \cdot P_{f_1}$. Let it be understood that there are now two equivalence relations both of which will be denoted by $\sim$, unambiguous by context. Let the stabilizer of P_f in $\mathcal{G}$ be denoted by $\mathcal{G}_{P_f}$. Then by the orbit-stabilizer theorem,

$$\left|\overline{P_f}\right| = [\mathcal{G} : \mathcal{G}_{P_f}] = \frac{|\mathcal{G}|}{|\mathcal{G}_{P_f}|},$$

i.e., the cardinality of the orbit of P_f equals the index of the stabilizer of P_f in $\mathcal{G}$. Now there is a means by which $|\bar{f}|$ can be computed and is given by the next proposition.

Proposition 3. *Let* $f \in X$. *Then* $|\bar{f}| = |S| \cdot \left|\overline{P_f}\right|$.

Proof. Let $f \in X$. Then for all $\alpha \in S$ and for all $\sigma \in \mathcal{G}$,

$$P_{\alpha f} = \left\{P_f^{\alpha(c_1)}, P_f^{\alpha(c_2)}, \ldots, P_f^{\alpha(c_k)}\right\} = P_f$$

and also

$$\begin{aligned} P_{f\sigma} &= \left\{P_{f\sigma}^{c_1}, P_{f\sigma}^{c_2}, \ldots, P_{f\sigma}^{c_k}\right\} \\ &= \left\{\sigma\left(P_f^{c_1}\right), \sigma\left(P_f^{c_2}\right), \ldots, \sigma\left(P_f^{c_k}\right)\right\} \\ &= \sigma P_f. \end{aligned}$$

Moreover, for all $f_1, f_2 \in X$, it is asserted that $f_1 \sim f_2$ if and only if $P_{f_1} \sim P_{f_2}$. To prove the assertion, observe that if $f_1 \sim f_2$, then there exist $\alpha \in S$ and $\sigma \in \mathcal{G}$ such that $\alpha f_2 = f_1\sigma$. Thus, it follows that

$$\begin{aligned} f_1 \sim f_2 &\longrightarrow \alpha f_2 = f_1\sigma \\ &\longrightarrow P_{\alpha f_2} = P_{f_1\sigma} \\ &\longrightarrow P_{f_2} = \sigma P_{f_1} \\ &\longrightarrow P_{f_2} = \sigma \cdot P_{f_1} \\ &\longrightarrow P_{f_1} \sim P_{f_2}. \end{aligned}$$

Conversely, if $P_{f_1} \sim P_{f_2}$, then there exists $\sigma \in \mathcal{G}$ such that $P_{f_2} = \sigma \cdot P_{f_1}$. Hence, for some $\alpha \in S$,

$$\begin{aligned} P_{f_1} \sim P_{f_2} &\longrightarrow P_{f_2} = \sigma \cdot P_{f_1} \\ &\longrightarrow P_{f_2} = \sigma P_{f_1} \\ &\longrightarrow P_{f_2} = P_{f_1\sigma} \\ &\longrightarrow \alpha f_2 = f_1\sigma \\ &\longrightarrow f_1 \sim f_2. \end{aligned}$$

From this result together with Proposition 2, Proposition 3 follows. ■

Remark 2. *With regard to the remark following Proposition 1, it should be noted that Proposition 3 also follows from the following computation:*

$$\left|\bar{f}\right| = \frac{|S \times \mathcal{G}|}{\left|(S \times \mathcal{G})_f\right|} = \frac{|S| \cdot |\mathcal{G}|}{\left|(S \times \mathcal{G})_f\right|} = |S| \cdot \frac{|\mathcal{G}|}{\left|\mathcal{G}_{P_f}\right|} = |S| \cdot \left|\overline{P_f}\right|.$$

It is possible to exhibit bijection from the stablizer of f in $S \times \mathcal{G}$ to the stablizer of P_f in $\mathcal{G}$. This establishes the fact that $\left|(S \times \mathcal{G})_f\right| = \left|\mathcal{G}_{P_f}\right|$.

Recall that a graph G is uniquely k-colorable whenever $|\mathcal{P}| = 1$, i.e., every k-coloring of G induces the same partition of $V(G)$. In this case, analysis of the graph under certain conditions is trivial, at least in terms of colorations. For instance, it is not difficult to verify that if G is uniquely k-colorable and vertex k-critical, then G must be isomorphic to a complete graph. In general, a given graph G has a multitude of k-colorings, and consequently a multitude of partitions of $V(G)$ into k independent subsets. The objective is to systematically organize the set $\mathcal{P}$ via the equivalence relation defined above so that what need to be analyzed are the equivalence classes of $\mathcal{P}$ and not the totality of all objects in $\mathcal{P}$.

Definition 1. *A graph G is said to be (k, γ)-type colorable whenever $\chi(G) = k$ and $|\mathcal{P}/\sim| = \gamma$. If k is understood, G is simply called γ-type colorable.*

Example 1. *Any path, cycle, star, or wheel is 1-type colorable. Also, the Petersen graph is 1-type colorable.*

Can general properties of a graph be determined based on its type parameter? For instance, do all 1-type colorable graphs possess a high degree of symmetry? As stated above in [6], the long standing double-critical conjecture of Lovász speculates that a vertex double-critical graph must be isomorphic to a complete graph, and hence uniquely k-colorable. Is it at least possible to prove that a vertex double-critical graph must be 1-type colorable? Or, is it [at least] possible to establish the existence of an edge $e = uv$ such that $\chi(G - u - v) \geq \chi(G) - 1$ whenever G is γ-type colorable with $\gamma \geq 2$? Can the structure of the automorphism group of G be determined for small γ? These are just some of the questions one might consider when considering types as defined in Definition 2 below.

Definition 2. *Let $\overline{P_f} \in \mathcal{P}/\sim$, where $P_f = \left\{P_f^{c_1}, P_f^{c_2}, \ldots, P_f^{c_k}\right\}$. An element $\overline{P_f}$ of $\mathcal{P}/\sim$ is called a type. The type sequence of $\overline{P_f}$ is defined to be the ordered non-decreasing k-tuple of positive integers $T = (t_1, t_2, \ldots, t_k)$, subject to the conditions $t_i = \left|P_f^{c_i}\right|$ for $i = 1, 2, \ldots, n$ and $\sum_{i=1}^{k} t_i = n$. Additionally, the type of an element P of $\mathcal{P}$ is defined as the type of the equivalence class containing P and the type of a k-coloring f of G is defined to be the type of the equivalence class of the image of f under the map $f \longmapsto P_f$.*

Let $T_1, T_2, \ldots, T_\gamma$ be the type sequences for a (k, γ)-type colorable graph G, where $T_i = (t_i^1, t_i^2, \ldots, t_i^k)$ for $i = 1, 2, \ldots, \gamma$. It should be noted that it is possible for distinct types to give rise to the same type sequence, i.e., it is possible for $T_i = T_j$, whenever $\overline{P_{f_i}} \neq \overline{P_{f_j}}$. As a result, it is possible to have two different k-colorings which fail to be equivalent but inducing partitions of $V(G)$ whose corresponding color classes have the same cardinality. Indeed, the analysis of types is non-trivial.

3. The Algebra of Types

This section develops a few basic results concerning types. Before proceeding, some additional terminology is required. The automorphism group $\mathcal{G}$ acts on $V(G)$ and consequently partitions $V(G)$ into orbits. The orbit of $v \in V(G)$ is denoted by $\mathcal{G}v$. Also, the stabilizer of $v \in V(G)$ will be denoted by $\mathcal{G}_v$. It follows by the orbit-stabilizer theorem that

$$|\mathcal{G}v| = [\mathcal{G} : \mathcal{G}_v] = \frac{|\mathcal{G}|}{|\mathcal{G}_v|},$$

i.e., the cardinality of the orbit of the vertex v is equal to the index of the stablizer of v in $\mathcal{G}$. Next, for a k-coloring $f \in X$, it is desirable to distinguish the singleton color classes from the remaining color classes in $P_f \in \mathcal{P}$. To this end, the partition P_f is expressed as: $P_f = (\mathcal{A}; \mathcal{B})$. Here, the set $\mathcal{A} = \{\{v_1\}, \{v_2\}, \ldots, \{v_r\}\}$ is the set consisting of all of the singleton color classes of P_f and $\mathcal{B} = \{\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_{k-r}\}$ is the set consisting of the remaining color classes. In terms of the notation defined in the previous section, $P_f^{c_i} = \{v_i\}$ for $i = 1, 2, \ldots, r$ and $P_f^{c_r+j} = \mathcal{B}_j$ for $j = 1, 2, \ldots, k-r$. Moreover, $P_f = \mathcal{A} \cup \mathcal{B}$. For convenience, set $\mathcal{A}^* = \bigcup \mathcal{A}$ and $\mathcal{B}^* = \bigcup \mathcal{B}$.

Theorem 1. *Let G be a graph containing at least one critical vertex. Suppose also that $\overline{P_f} \in \mathcal{P}/\sim$ with $|\overline{P_f}| = 1$ and $\mathcal{A}^*$ is nonempty. Then either $G = K_n$ or there exits at least two distinct orbits of $V(G)$.*

Proof. Consider $P_f = (\mathcal{A}; \mathcal{B})$. By hypothesis, the set $\mathcal{A}^*$ is nonempty. The condition $|\overline{P_f}| = 1$ guarantees that the stabilizer of P_f is the entire group $\mathcal{G}$. Therefore, $\mathcal{G}(\mathcal{A}^*) \subseteq \mathcal{A}^*$ as well as $\mathcal{G}(\mathcal{B}^*) \subseteq \mathcal{B}^*$. Now, if $\mathcal{B}^* = \varnothing$, then clearly $\mathcal{A}^* = V(G)$ so that $G = K_n$. Otherwise, both $\mathcal{A}^*$ and $\mathcal{B}^*$ are nonempty and the facts that $\mathcal{G}(\mathcal{A}^*) \subseteq \mathcal{A}^*$ and $\mathcal{G}(\mathcal{B}^*) \subseteq \mathcal{B}^*$ guarantee the existence of at least two distinct orbits of $V(G)$. ■

There are additional relationships that exist between the orbits of $V(G)$ and the set $\mathcal{A}^*$. They will be investigated below. Of particular interest is the action of the automorphism group $\mathcal{G}$ on the set $\mathcal{P}$ and on the collection of subsets induced by an element of $\mathcal{P}$.

Definition 3. *Let $v \in V(G)$ with $v \in \mathcal{A}^*$. Then v is said to be σ-invariant, provided that $v \in \sigma\mathcal{A}^*$.*

Definition 4. *For $v \in V(G)$ and $P_f = (\mathcal{A}; \mathcal{B}) \in \mathcal{P}$ subject to $v \in \mathcal{A}^*$, define the set*

$$\mathcal{H}(P_f\,;\,v) = \{\sigma \in \mathcal{G} : v \in \sigma\mathcal{A}^*\}$$

and associate with P_f the set $\mathcal{W} = (\mathcal{G}v) \bigcap (\mathcal{A}^*)$.

Observe that for every automorphism $\sigma \in \mathcal{G}$,

$$\sigma(\mathcal{W}) = \sigma\left((\mathcal{G}v) \bigcap (\mathcal{A}^*)\right) = \sigma(\mathcal{G}v) \bigcap \sigma(\mathcal{A}^*) = \mathcal{G}v \bigcap \sigma(\mathcal{A}^*).$$

Thus it is evident that each critical clique, within a particular type, contains the same number of vertices from each orbit that is represented in the corresponding set of singleton color classes.

Given a graph G, it would be straightforward, albeit tedious, to compute $\mathcal{H}(P_f\,;\,v)$ for a particular vertex and partition. One need only have at their disposal the automorphism group of G. However, Definition 4 does not provide a useful method for determining $|\mathcal{H}(P_f\,;\,v)|$, a quantity that is of interest. The following proposition circumvents this shortcoming.

Proposition 4. *Let $v \in V(G)$. Then $\mathcal{H}(P_f\,;\,v) = \bigcup_{w \in \mathcal{W}} (\mathcal{G}_v)\tau_w$, where $\tau_w \in \mathcal{G}$ is any automorphism such that $\tau_w(w) = v$ and $w \in \mathcal{W}$. Hence, $|\mathcal{H}(P_f\,;\,v)| = |\mathcal{W}| \cdot |\mathcal{G}_v|$.*

Proof. Let $v \in V(G)$ and consider an arbitrary $\mu \in \mathcal{H}(P_f\,;\,v)$. By the definition of $\mathcal{H}(P_f\,;\,v)$, it follows that $v \in \mu\mathcal{A}^*$. Hence, there exists a vertex $w_0 \in \mathcal{A}^*$ with $v = \mu(w_0)$. As a result, $w_0 = \mu^{-1}(v) \in \mathcal{G}v$ and so $w_0 \in \mathcal{W}$. Now let $\tau_{w_0} \in \mathcal{G}$ be any automorphism such that $\tau_{w_0}(w_0) = v$, where $w_0 \in \mathcal{W}$, and define $\eta = \mu\tau_{w_0}^{-1}$. Observe that $\eta \in \mathcal{G}_v$ since

$$\eta(v) = \left(\mu\tau_{w_0}^{-1}\right)(v) = \mu\left(\tau_{w_0}^{-1}(v)\right) = \mu(w_0) = v.$$

Moreover, $\mu = \eta\tau_{w_0} \in (\mathcal{G}_v)\tau_{w_0} \subseteq \bigcup_{w \in \mathcal{W}} (\mathcal{G}_v)\tau_w$. Therefore,

$$\mathcal{H}(P_f\,;\,v) \subseteq \bigcup_{w \in \mathcal{W}} (\mathcal{G}_v)\tau.$$

For the reverse inclusion, suppose that $\mu \in \bigcup_{w \in \mathcal{W}} (\mathcal{G}_v)\tau_w$. Then there exists a $w_0 \in W$ and automorphisms $\sigma \in \mathcal{G}_v$ and $\tau_{w_0} \in \mathcal{G}$ with $\tau_{w_0}(w_0) = v$ such that $\mu = \sigma\tau_{w_0}$. The fact that $v \in \mu\mathcal{A}^*$ follows from the computation

$$\mu(w_0) = (\sigma\tau_{w_0})(w_0) = \sigma(\tau_{w_0}(w_0)) = \sigma(v) = v.$$

Thus, $\mu \in \mathcal{H}(P_f\,;\,v)$. Hence, $\bigcup_{w \in \mathcal{W}} (\mathcal{G}_v)\tau \subseteq \mathcal{H}(P_f\,;\,v)$. To complete the proof of the proposition, let w_1 and w_2 be distinct elements of $\mathcal{W}$. Observe that for any $\tau_{w_1}, \tau_{w_2} \in \mathcal{G}$ with $\tau_{w_1}(w_1) = \tau_{w_2}(w_2) = v$, it must be that $(\mathcal{G}_v)\tau_{w_1} \bigcap (\mathcal{G}_v)\tau_{w_2} = \varnothing$. If to the contrary there was an automorphism $\beta \in (\mathcal{G}_v)\tau_{w_1} \bigcap (\mathcal{G}_v)\tau_{w_2}$, then both w_1 and w_2 would map to v under β, which is clearly impossible. Consequently, $|\mathcal{H}(P_f\,;\,v)| = |\mathcal{W}| \cdot |\mathcal{G}_v|$. ■

It is also of interest to consider elements of $\mathcal{B}^*$ that remain unmoved by automorphisms of G. The analogous definitions and proposition follow.

Definition 5. *Let $z \in V(G)$ with $v \in \mathcal{B}^*$. Then z is said to be ρ-invariant, provided that $z \in \rho\mathcal{B}^*$.*

Definition 6. *For $z \in V(G)$ and $P_g \in \mathcal{P}$, define the set*

$$\mathcal{K}(P_g\ ;\ z) = \{\rho \in \mathcal{G} : z \in \rho\mathcal{B}^*\}$$

and set $\mathcal{Y} = (\mathcal{G}z) \bigcap (\mathcal{B}^*)$.

Proposition 5. *Let $z \in V(G)$. Then $\mathcal{K}(P_g\ ;\ z) = \bigcup_{y \in \mathcal{Y}} (\mathcal{G}_z)\eta$, where $\eta \in \mathcal{G}$ is any automorphism such that $\eta(y) = z$ and $y \in \mathcal{Y}$. Hence, $|\mathcal{K}(P_g\ ;\ z)| = |\mathcal{Y}| \cdot |\mathcal{G}_z|$.*

Proof. Let $z \in V(G)$ and consider an arbitrary $\rho \in \mathcal{K}(P_g\ ;\ z)$. By the definition of $\mathcal{K}(P_g\ ;\ z)$, it follows that $z \in \rho\mathcal{B}^*$. Hence, there exists a vertex $y_0 \in \mathcal{B}^*$ with $z = \rho(y_0)$. As a result, $y_0 = \rho^{-1}(z) \in \mathcal{G}z$ and so $y_0 \in \mathcal{Y}$. Now let $\eta_{y_0} \in \mathcal{G}$ be any automorphism such that $\eta_{y_0}(y_0) = z$, where $y_0 \in \mathcal{Y}$, and define $\lambda = \rho\eta_{y_0}^{-1}$. Observe that $\lambda \in \mathcal{G}_z$ since

$$\lambda(z) = \left(\rho\eta^{-1}\right)(z) = \rho\left(\eta^{-1}(z)\right) = \rho(y) = z.$$

Moreover, $\rho = \lambda\eta \in (\mathcal{G}_z)\eta \subseteq \bigcup_{y \in \mathcal{Y}} (\mathcal{G}_z)\eta$. So, $\mathcal{K}(P_g\ ;\ z) \subseteq \bigcup_{y \in \mathcal{Y}} (\mathcal{G}_z)\eta$. For the reverse inclusion, we suppose that $\rho \in \bigcup_{y \in \mathcal{Y}} (\mathcal{G}_z)\eta_y$. Then there exists a $y_0 \in \mathcal{Y}$ and automorphisms $\phi \in \mathcal{G}_z$ and $\eta_{y_0} \in \mathcal{G}$ with $\eta_{y_0}(y_0) = z$ such that $\rho = \phi\eta_{y_0}$. The fact that $z \in \rho\mathcal{B}^*$ follows from the computation

$$\rho(y_0) = \left(\beta\eta_{y_0}\right)(y_0) = \beta\left(\eta_{y_0}(y_0)\right) = \beta(z) = z.$$

Thus, $\rho \in \mathcal{K}(P_g\ ;\ z)$. Hence, $\bigcup_{y \in \mathcal{Y}} (\mathcal{G}_z)\eta \subseteq \mathcal{K}(P_g\ ;\ z)$. To complete the proof of the proposition, let y_1 and y_2 be distinct elements of $\mathcal{Y}$. Observe that for any $\eta_{y_1}, \eta_{y_2} \in \mathcal{G}$ with $\eta_{y_1}(y_1) = \eta_{y_2}(y_2) = z$, it must be that $(\mathcal{G}_z)\eta_{y_1} \bigcap (\mathcal{G}_z)\eta_{y_2} = \varnothing$. If to the contrary there existed an automorphism $\gamma \in (\mathcal{G}_z)\eta_{y_1} \bigcap (\mathcal{G}_z)\eta_{y_2}$, then both y_1 and y_2 would map to z under γ, which is clearly impossible. Consequently, $|\mathcal{K}(P_g\ ;\ z)| = |\mathcal{Y}| \cdot |\mathcal{G}_z|$. ■

If reasonable additional conditions are imposed on the set of partitions $\mathcal{P}$, then some other numerical results are obtainable. It will be shown below in Proposition 9 that certain additional conditions imposed on $\mathcal{P}$ seem appropriate for investigation.

Proposition 6. *Let $v \in V(G)$ and let $P_{f_1}, P_{f_2} \in \overline{P_f}$ with $\mathcal{W}_1 \cap \mathcal{W}_2 = \{v\}$. Then $\mathcal{H}(P_{f_1}\ ;\ v) \cap \mathcal{H}(P_{f_2}\ ;\ v) = \mathcal{G}_v$.*

Proof. Clearly $\mathcal{G}_v \subseteq \mathcal{H}(P_{f_1}\ ;\ v) \cap \mathcal{H}(P_{f_2}\ ;\ v)$. For the reverse inclusion, suppose that $\sigma \in \mathcal{H}(P_{f_1}\ ;\ v) \cap \mathcal{H}(P_{f_2}\ ;\ v)$. Then there exist $w_1 \in \mathcal{W}_1$ and $w_2 \in \mathcal{W}_2$ such that $v = \sigma(w_1) = \sigma(w_2)$. Since σ is injective and $\mathcal{W}_1 \cap \mathcal{W}_2 = \{v\}$, it follows $w_1 = w_2 = v$. Therefore, $\sigma \in \mathcal{G}_v$. ■

Corollary 1. *Let $v \in V(G)$. Also, let $P_{f_1}, P_{f_2}, \ldots, P_{f_m} \in \overline{P_f}$ satisfy $\mathcal{W}_i \cap \mathcal{W}_j = \{v\}$ and $|\mathcal{W}_i| = |\mathcal{W}_j|$ for $i \neq j$. Then for the set Z defined by*

$$Z = \bigcup_{i=1}^{m} \mathcal{H}(P_{f_i}\,;\,v),$$

it follows that $|Z| = |\mathcal{G}_v| \cdot (m\lambda - m + 1)$, where $\lambda = |\mathcal{W}_i|$ for each i. Moreover, $|\mathcal{G}v| \geq m\lambda - m + 1$.

Proof. By Proposition 4, $|\mathcal{H}(P_{f_i}\,;\,v)| = |\mathcal{W}_i| \cdot |\mathcal{G}_v|$ and the condition $\mathcal{W}_i \cap \mathcal{W}_j = \{v\}$ implies $\mathcal{H}(P_{f_i}\,;\,v) \cap \mathcal{H}(P_{f_j}\,;\,v) = \mathcal{G}_v$. Consequently,

$$\begin{aligned}
|Z| &= \left|\bigcup_{i=1}^{m} \mathcal{H}(P_{f_i}\,;\,v)\right| \\
&= \sum_{i=1}^{m} |\mathcal{H}(P_{f_i}\,;\,v)| - (m-1)\,|\mathcal{G}_v| \\
&= \sum_{i=1}^{m} |\mathcal{W}_i| \cdot |\mathcal{G}_v| - (m-1)\,|\mathcal{G}_v| \\
&= m\lambda\,|\mathcal{G}_v| - (m-1)\,|\mathcal{G}_v| \\
&= |\mathcal{G}_v| \cdot (m\lambda - m + 1).
\end{aligned}$$

The inequality $|\mathcal{G}v| \geq m\lambda - m + 1$ follows immediately from the facts that $Z \subseteq \mathcal{G}$ and $|\mathcal{G}| = |\mathcal{G}_v| \cdot |\mathcal{G}v|$. ■

With regard to the conclusion of Corollary 1, consider the case that $m\lambda - m \geq \deg(v)$. Should this inequality hold, then $|\mathcal{G}v| \geq \deg_G(v) + 1$. What can be said in case of equality?

Proposition 7. *If $\mathcal{H}(P_f\,;\,v) = \mathcal{G}$ and G is 1-type colorable, then v is a terminal vertex.*

Proof. Assume that G is 1-type colorable and $\mathcal{H}(P_f\,;\,v) = \mathcal{G}$. The fact that $\mathcal{H}(P_f\,;\,v) = \mathcal{G}$ implies that $\{v\}$ is a singleton color class in some, and, in fact, every partition of the lone type of G. Because v is a critical vertex, $\chi(G - v) = \chi(G) - 1$. Now, if there was a vertex $w \in V(G)$ that is not adjacent to v, then it would be possible to have $\{v, w\}$ being a color class for some k-coloring of G. To see this, simply color $G - v$ using colors $c_1, c_2, \ldots, c_{k-1}$ and then color v using color c_k to obtain a k-coloring of G. As v is the only vertex colored with c_k, it is clear that vertex w could be recolored with color c_k so that $\{v, w\}$ determines a color class, a clear contradiction of the hypotheses. Hence, $\deg(v) = n - 1$. ■

Corollary 2. *If $\mathcal{H}(P_f\,;\,v) = \mathcal{G}$, then either v is a terminal vertex or G contains at least two orbits.*

Proof. Set $P_f = (\mathcal{A}; \mathcal{B})$. The hypothesis $\mathcal{H}(P_f\,;\,v) = \mathcal{G}$ implies that $\mathcal{G}\mathcal{A}^* \subseteq N[v]$. ■

Proposition 8. *If G is a vertex k-critical graph and 1-type colorable, then $(\mathcal{G}v) \bigcap (\mathcal{A}^*)$ is nonempty for every pair (v, P_f). Hence, the number of orbits in $V(G)$ is a lower bound for the critical clique number $\omega_c(G)$.*

Proof. Assume that G is a vertex k-critical graph and 1-type colorable. Select an arbitrary pair (v, P_f), where $v \in V(G)$ and $P_f = (\mathcal{A}; \mathcal{B})$. If $v \in \mathcal{A}^*$, then there is nothing to show. Else, there exists a partition $P_g = P_g(v)$ such that $\mathcal{H}(P_g; v)$ is nonempty. Now, because G is 1-type colorable, there exists an automorphism $\sigma \in \mathcal{G}$ such that $P_f = \sigma \cdot P_g$. Consequently, $\sigma(v) \in \mathcal{A}^*$. ■

4. On Vertex Double-Critical Graphs

Stiebitz in [9] has shown that K_5 is the only 5-chromatic vertex double-critical graph. It would be desirable to prove this result by using partition types. But presently, it is not known whether or not it is possible to give an alternate proof of the result by Stiebitz. The next proposition is a small start. In [4], the maximum order of a critical clique of G is called the critical clique number and is denoted by $\omega_c(G)$. Moreover, it is an elementary fact and consequence of Proposition 10 below that $\omega_c(G) \leq \omega(G)$ with equality holding if and only if G is complete.

Proposition 9. *If G is 5-chromatic, vertex double-critical, and 1-type colorable, then $\mathcal{G}$ acts transitively on $V(G)$.*

Proof. Assume that G is 5-chromatic and vertex double-critical. By a more general result in [4], any k-chromatic vertex double-critical graph which contains K_{k-1} as a subgraph is complete. Hence, if $\omega_c(G) = 5$, 4, or 3, then $\omega(G) \geq 4$ and there would be nothing to show. Therefore, it can be assumed that $\omega_c(G) = 2$. Now suppose to the contrary that $\mathcal{G}$ does not act transitively on $V(G)$ so that there are at least two orbits of $V(G)$. Observe that if there are three or more orbits, then by Proposition 8 it would follow that $\omega_c(G) \geq 3$. Hence, it can further be assumed that there are only two orbits of $V(G)$. In this case, notice that these two orbits cannot both be independent subsets of $V(G)$. Else, G would be 2-chromatic contradicting the fact that G is 5-chromatic. Thus, there are adjacent vertices u and v contained in the same orbit. Since G is vertex double-critical, there exists a partition $P_f = (\mathcal{A}; \mathcal{B})$ such that $\{u, v\} \subseteq \mathcal{A}^*$. Again by Proposition 8, it would follow that $\omega_c(G) \geq 3$ as all orbits must have a representative in $\mathcal{A}^*$. However, this contradicts the additional assumption that $\omega_c(G) = 2$. Consequently, $\mathcal{G}$ acts transitively on $V(G)$. ■

Definition 7. *For a graph G, let*

$$C = \{e \in E(G) \mid e = uv \text{ and } \chi(G - u - v) = \chi(G) - 2\}$$

and define the core of G, denoted by Core(G), *to be the induced subgraph* $G[C]$.

It is immediate from the definition above that the Erdős-Lovász double-critical conjecture is equivalent to the statement: Core$(G) = G$ if and only if $G = K_n$. By determining properties of Core(G), it is believed that the

Erdős-Lovász double-critical conjecture can be proved in the affirmative. By a careful analysis of Core(G) in conjunction with the equivalence relation defined above, would it be possible to construct an edge not in C under the assumption that G is not complete? Helpful with this analysis is the following theorem from [4] which generalizes the fact that if v is a critical vertex of a k-chromatic graph G, then v must be adjacent to at least one vertex from each color class in any $(k-1)$-coloring of $G - v$.

Proposition 10. *Let K_r^c be a maximal critical r-clique of G and suppose that $G \neq K_n$. Then*

$$C_i \cap \left[\bigcap_{v \in K_r^c} N(v) \right] \neq \emptyset$$

for each color class C_i with $|C_i| \geq 2$.

Proof. Set $V(K_r^c) = \{v_1, v_2, \ldots, v_r\}$ and suppose, without loss of generality, that vertex v_i is colored with color $c_{\chi(G)-r+i}$ for $i = 1, 2, \ldots, r$. Now consider $P = \mathcal{P}(V(K_r^c))$, the power set of $V(K_r^c)$. For each $X \in P$, define the set

$$C_i(X) = \{w \in C_i \ : \ N(w) \cap V(K_r^c) = X\}.$$

Then $\{C_i(X)\}_{X \in P}$ forms a partition of C_i. To see this, select any vertex $w \in C_i$ and set $X_w = N(w) \cap V(K_r^c)$. Clearly, $X_w \in P$ and $w \in C_i(X_w)$ so that

$$C_i = \bigcup_{X \in P} C_i(X).$$

Moreover, suppose $z \in C_i(X_1) \cap C_i(X_2)$. Then $N(z) \cap V(K_r^c) = X_1$and $N(z) \cap V(K_r^c) = X_2$ implying that $X_1 = X_2$. Thus, $C_i(X_1) \cap C_i(X_2) = \emptyset$ for $X_1 \neq X_2$. Hence, $\{C_i(X)\}_{X \in P}$ forms a partition of C_i. Also note that for each $X \in P$, $X \neq V(K_r^c)$, there exists a vertex $v_{i_0} = v_{i_0}(X) \in V(K_r^c)$ such that $v_{i_0} \notin X$. Now suppose to the contrary that $C_i(V(K_r^c)) = \emptyset$. If this is the case, then for each $X \in P$, $X \neq V(K_r^c)$, the set $C_i(X)$ can be recolored using color $c_{\chi(G)-r+i_0}$. This would yield a proper coloring of G using fewer than $\chi(G)$ colors which is impossible. Therefore, it must be that $C_i(V(K_r^c)) \neq \emptyset$, i.e., there exists a vertex $w \in C_i$ that is adjacent to every vertex in $V(K_r^c)$. ■

5. Concluding Remarks

In conclusion, it is noted that Seinsche in [8] has shown that $\chi(G) = \omega(G)$ whenever G contains no induced subgraph isomorphic to P_4, the path on four vertices. Clearly the Erdős-Lovász double-critical conjecture can be proved in the affirmative if it can be shown that G contains no induced subgraph isomorphic to P_4. But this is not an easy task to accomplish. Conjecture 1 requires a much deeper investigation of Core(G) than what is presented here.

Conjecture 1. Core(G) *contains no [vertex] induced subgraph isomorphic to P_4.*

By Corollary 1 of [3], it is possible for Core (G) to be disconnected. Thus one need only consider the connected components of Core (G). Observe that the Erdős-Lovász double-critical conjecture is an immediate consequence of this Conjecture 1 if it can be proven to be true. For if Core $(G) = G$ and Core (G) contains no [vertex] induced subgraph isomorphic to P_4, then $\chi(G) = \omega(G)$ so that G contains a subgraph isomorphic to $K_{\chi(G)}$. Therefore, G would be isomorphic to a complete graph.

References

[1] J. Balogh, A. Kostochka, N. Prince, and M. Stiebitz, The Erdős-Lovász Tihany Conjecture for quasi-line graphs, to appear in *Discrete Mathematics.*

[2] P. Erdős, Problem 2, in *Theory of Graphs, Proc. Colloquium*, Tihany, Hungary (1966) 361.

[3] J. Lattanzio, Completely independent critical cliques, *Journal of Combinatorial Mathematics and Combinatorial Computing* **62** (2007) 165-170.

[4] J. Lattanzio, Doctoral dissertation, Department of Mathematics, University of Pittsburgh, 2001.

[5] J. Lattanzio, Edge double-critical graphs, Manuscript.

[6] T. Jensen and B. Toft, Graph Coloring Problems, Wiley-Interscience Series in Discrete Mathematics and Optimization, John Wiley & Sons, Inc., (1995) 105-106, 136-137.

[7] K-i. Kawarabayashi, A. Sune Pedersen, and B. Toft, Double-critical graphs and complete minors, available at http://adsabs.harvard.edu/abs/2008arXiv0810.3133K.

[8] D. Seinsche, On a property of the class of n-colorable graphs, *Journal of Combinatorial Theory* **16B** (1974) 191-193.

[9] M. Stiebitz, K_5 is the only double-critical 5-chromatic graph. *Discrete Mathematics* **64** (1987) 91-93.