Investigations on Strength Eight Balanced Arrays Using Some Classical Inequalities

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Dedicated to Professor Walter D. Wallis, on the occasion of his 68th birthday.

Abstract

In this paper, we derive some necessary existence conditions for balanced arrays (B-arrays) of strength eight and with two levels by making use of some classical inequalities such as Cauchy, Hölder, and Minkowski. We discuss the usefulness of these conditions in the study of the B-arrays, and also present some illustrative examples.

1 Introduction and Preliminaries

First, we list some basic concepts and definitions frequently used in the study of balanced arrays (B-arrays).

Definition. A *balanced array* (B-array) T with m rows (*constraints, factors*), N columns (*runs, treatment-combinations*), two *symbols* (say, 0 and 1; also called *levels*), and of *strength* $t = 8$ with *index set* $\underline{\mu}' = (\mu_0, \mu_1, \ldots, \mu_8)$ is merely a matrix T of size $(m \times N, m \geq 8)$ with elements 0 and 1 such that in every 8-rowed submatrix T^* of T (clearly, there are $\binom{m}{8}$ such submatrices), the following condition is satisfied: every column vector $\underline{\alpha}$ of weight i (the weight of a vector $\underline{\alpha}$ means the number of 1s in it; $0 \leq i \leq 8$) appears with the same frequency (say, μ_i). The B-array T is sometimes denoted by $\mathrm{BA}(m, N, t = 8, \underline{\mu}')$.

Remark: It is obvious that the number of columns N in T is known, once we are given $\underline{\mu}'$, and $N = \sum_{i=0}^{8} \binom{8}{i}\mu_i$.

Definition. A B-array is called an *orthogonal array* (O-array) if $\mu_i = \mu$, for each i. In this special case, $N = 2^8\mu = 256\mu$.

Thus, O-arrays are special cases of B-arrays.

Initially, the researchers confined themselves to O-arrays which were extensively used in information theory, coding theory, in industry, and the statisticians employed them in the construction of symmetrical as well as asymmetrical fractional factorial designs. Factorial designs play a very important role to study situations in which the final response is affected by numerous factors (i.e. experimental conditions), each factor being at different settings (i.e. levels). The total number of treatment-combinations becomes enormously large with the increase in the number of factors and their levels. For example, if the number m of factors is equal to six with each factor at two levels, then a complete factorial design would have $2^6 = 64$ treatment-combinations. A researcher would prefer to achieve his (her) objectives by running a cost-effective experiment. Fractional factorial designs (FFD) provide the necessary tools to the experimenter to conduct a cost-effective experiment. To this end, the concept of strength t of an O-array was introduced. An O-array T is said to be of strength t $(= 2\mu, \mu \geq 1)$ if one can estimate all the effects (up to and including μ-factor interactions) under the assumptions that all higher order interactions are negligible. T is said to be of strength t $(= 2\mu + 1, \mu \geq 1)$ if one can estimate all the effects up to and including 'μ' factors in the presence of $(\mu + 1)$-factor interactions (higher than $(\mu + 1)$-factor interactions being assumed to be negligible). It is well-known that O-arrays may not exist for each N. For example, for an O-array of strength t, the total number of treatment-combinations N has to be a multiple of 2^t (i.e. $N = \mu \cdot 2^t$, μ being the index of the O-array) which imposes a very severe limitation on the experimenter. Thus, the combinatorial structure on O-arrays was replaced by a weaker one, giving rise to B-arrays which appeared first in Chakravarti [2] (the idea given by Rao to Chakravarti). It is well-known that a B-array exists for each N. Thus, B-arrays would cover all experimental situations. In this paper, we will concern ourselves to experimental situations in which the researcher is interested to estimate all the effects up to and including 4-factor interactions when higher-order interactions are assumed to be negligible. Naturally, the value of t has to be equal to eight to resolve this problem. It is clear that B-arrays, as defined above, are generalizations of O-arrays. Furthermore, there are other combinatorial structures (such as balanced incomplete block designs, rectangular designs, doubly balanced designs, etc.) which are closely related to B-arrays. To gain further insight into the importance of these combinatorial arrays, the interested reader is referred to the list of references at the end (by no means an exhaustive list) of this paper, and also further references listed therein.

It is obvious the construction of a B-array T amounts to constructing a matrix T under a combinatorial constraint imposed on its t-rowed submatrices. The matrix T, when considered as a B-array, represents a factorial design of 2^m Series (m factors, each at 2 levels) in which the rows of T correspond to factors, columns to treatment-combinations (also called runs), and the symbols 0 and 1 to the levels of each factor. In general, for a given N, we may have more than one B-array. It is also clear that for a given $\underline{\mu}'(\mu)$ and m, a B-array (O-array) may not exist. The problem of constructing these arrays, for a given $\underline{\mu}'(\mu)$, with the maximum value of m is very important–both in combinatorics and statistical design of experiments. This problem, for O-arrays, has been studied (among others) by Bose and Bush [1], Rao [11, 12], Seiden and Zemach [14], Yamamoto et. al [16], etc. while the corresponding problem for B-arrays has been investigated (among others) by Cheng [3], Chopra [4], Chopra and Dios [5], Chopra and Bsharat [6], Longyear [8], Rafter and Seiden [10], Saha et. al [13], etc. In this paper, we consider B-arrays with $t = 8$, and derive a set of necessary existence conditions involving the parameters $\mu_0, \mu_1, \ldots, \mu_8$ and m using some of the classical inequalities such as Cauchy, Hölder, and Minkowski. We present some illustrative examples of obtaining, for a given $\underline{\mu}'$, the maximum number of constraints m for which the B-array may possibly exist.

2 Main Results with Applications

We first state some results on B-arrays with $t = 8$ for later use.

Lemma 1. *For $t = 8$, a B-array T with $m = 8$ always exists.*

Lemma 2. *A B-array T with index set $\underline{\mu}' = (\mu_0, \mu_1, \ldots, \mu_8)$ is also of strength t', where $0 \le t' \le 8$. Considered as an array of strength t', its index set $\underline{\mu}'(t')$ is given by $(A_{j,t'}; j = 0, 1, 2, \ldots, t')$, where $A_{j,t'}$ is a linear function of the μ_i's given by $A_{j,t'} = \sum_{i=0}^{8-t'} \binom{8-t'}{i} \mu_{i+j}$, $j = 0, 1, 2, \ldots, t'$.*

Remark: It is clear that $t' = 0$ corresponds to N (the total number of columns), while $t' = 8$ corresponds to the index set $\underline{\mu}'$.

The next result will connect the moments of the weights of the column vectors of the array T with its parameters $\mu_0, \mu_1, \ldots, \mu_8$ and m.

Lemma 3. *Let x_j $(0 \le j \le m)$ be the frequency of the columns of weight j constituting a B-array T with m constraints and having $\underline{\mu}'$ as its index set. For*

T to exist, the following results must hold:

$$L_k = \sum_{j=0}^{m} j^k x_j = \sum_{l=1}^{k-1} (-1)^{l+k-1} a_{l,k} L_l + m_k A_{k,k}, \text{ for } 2 \le k \le 8, \text{ with} \quad (2.1)$$

$$L_0 = \sum_{j=0}^{m} x_j = N,$$

$$L_1 = \sum j x_j = m_1 A_{1,1}, \text{ where}$$

$$m_r = m(m-1)(m-2)\cdots(m-r+1).$$

Remark: It is not difficult to observe that the nine equalities in (2.1) represent the moments L_k ($0 \le k \le 8$) of order k of the weights of the column vectors of T in terms of the parameters of T. The constants $a_{l,k}$ in (2.1) appear in the process of deriving (2.1) and are known. Clearly, any moment of order k is a linear function of moments of lower order.

Proof. (Outline). (2.1) can be derived easily by counting vectors of weight t' ($0 \le t' \le 8$) in two ways–through columns and through rows using the fact that T is also of strength t'. □

Remark: For computational ease, we provide the values of $a_{l,k}$ which appear in (2.1). The values of $a_{l,k}$ (listed in order, starting with $l = 1$ and ending with $l = k-1$) are: ($k = 2; a_{1,2} = 1$), ($k = 3; 2, 3$), ($k = 4; 6, 11, 6$), ($k = 5; 24, 50, 35, 10$), ($k = 6; 120, 274, 225, 85, 15$), ($k = 7; 720, 1764, 1624, 735, 175, 21$), and ($k = 8; 5040, 13068, 13132, 6769, 1960, 322, 28$).

Next, we state the three classical inequalities (Cauchy, Hölder, and Minkowski) which will be used in this paper.

Cauchy's Inequality: If $a = (a_1, a_2, \ldots, a_n)$ and $b = (b_1, b_2, \ldots, b_n)$ are sequences of real numbers, then

$$\left[\sum_{k=1}^{n} a_k b_k\right]^2 \le \left[\sum_{k=1}^{n} a_k^2\right]\left[\sum_{k=1}^{n} b_k^2\right]. \quad (2.2)$$

Minkowski's Inequality: For $a_i, b_i \ge 0$ and $p > 1$, we have

$$\left[\sum_{i=1}^{n} (a_i + b_i)^p\right]^{\frac{1}{p}} \le \left[\sum_{i=1}^{n} a_i^p\right]^{\frac{1}{p}} + \left[\sum_{i=1}^{n} b_i^p\right]^{\frac{1}{p}}. \quad (2.3)$$

Hölder's Inequality: For $a_k, b_k \geq 0$, $\frac{1}{p} + \frac{1}{q} = 1$ with $p > 1$, we have

$$\sum_{k=1}^{n} a_k^{\frac{1}{p}} b_k^{\frac{1}{q}} \leq \left[\sum_{k=1}^{n} a_k\right]^{\frac{1}{p}} \left[\sum_{k=1}^{n} b_k\right]^{\frac{1}{q}}. \tag{2.4}$$

Theorem 1. *Suppose that T is a B-array with m rows and with index set $\underline{\mu}' = (\mu_0, \mu_1, \ldots, \mu_8)$. For T to exist, the following must be satisfied:*

$$L_4^2 \leq L_7 L_1. \tag{2.5}$$

$$L_4^2 \leq L_8 L_0. \tag{2.6}$$

$$L_5^2 \leq L_7 L_3. \tag{2.7}$$

$$L_5^2 \leq L_8 L_2. \tag{2.8}$$

$$L_6^2 \leq L_7 L_5. \tag{2.9}$$

$$L_6^2 \leq L_8 L_4. \tag{2.10}$$

$$L_7^2 \leq L_8 L_6. \tag{2.11}$$

Proof. We prove (2.5) and for the rest of the inequalities, we provide appropriate substitutions in Cauchy's Inequality. To derive (2.5), we set $a_k = j^{\frac{7}{2}}\sqrt{x_j}$ and $b_k = \sqrt{jx_j}$ in (2.2), and we obtain $(\sum j^4 x_j)^2 \leq (\sum j^7 x_j)(\sum j x_j)$, which leads to $L_4^2 \leq L_7 L_1$. For (2.6) through (2.11), we use the following substitutions in (2.2) respectively: $(a_k = j^4\sqrt{x_j}, b_k = \sqrt{x_j})$; $(a_k = j^{\frac{7}{2}}\sqrt{x_j}, b_k = j^{\frac{3}{2}}\sqrt{x_j})$; $(a_k = j^4\sqrt{x_j}, b_k = j\sqrt{x_j})$; $(a_k = j^{\frac{7}{2}}\sqrt{x_j}, b_k = j^{\frac{5}{2}}\sqrt{x_j})$; $(a_k = j^4\sqrt{x_j}, b_k = j^2\sqrt{x_j})$; and $(a_k = j^4\sqrt{x_j}, b_k = j^3\sqrt{x_j})$. □

Theorem 2. *For a B-array T with m rows and index set $\underline{\mu}' = (\mu_0, \mu_1, \ldots, \mu_8)$ to exist, the following inequalities must hold:*

$$L_2 + L_4 \leq \sqrt[3]{L_0 L_6}[L_0^{\frac{1}{3}} + L_6^{\frac{1}{3}}]. \tag{2.12}$$

$$L_3 + L_4 \leq \sqrt[3]{L_2 L_5}[L_2^{\frac{1}{3}} + L_5^{\frac{1}{3}}]. \tag{2.13}$$

$$L_4 + L_5 \leq \sqrt[3]{L_3 L_6}[L_3^{\frac{1}{3}} + L_6^{\frac{1}{3}}]. \tag{2.14}$$

$$2L_5 + 3L_6 + 2L_7 \leq L_4^{\frac{1}{4}} L_8^{\frac{1}{4}}[2L_4^{\frac{1}{2}} + 3L_4^{\frac{1}{4}} L_8^{\frac{1}{4}} + 2L_8^{\frac{1}{2}}]. \tag{2.15}$$

$$L_2 + 2L_3 + 2L_4 + L_5 \leq L_1^{\frac{1}{5}} L_6^{\frac{1}{5}}[L_1^{\frac{3}{5}} + 2L_1^{\frac{2}{5}} L_6^{\frac{1}{5}} + 2L_1^{\frac{1}{5}} L_6^{\frac{2}{5}} + L_6^{\frac{3}{5}}]. \tag{2.16}$$

Proof. We use Minkowski's Inequality here. For (2.12) to (2.14), we take $p = 3$ and raise both sides to the third power to obtain

$$\sum a_i^2 b_i + \sum a_i b_i^2 \leq \sqrt[3]{\sum a_i^3 \sum b_i^3}\left[\left(\sum a_i^3\right)^{\frac{1}{3}} + \left(\sum b_i^3\right)^{\frac{1}{3}}\right].$$

Setting $a_i = x_j^{\frac{1}{3}}$ and $b_i = j^2 x_j^{\frac{1}{3}}$, we obtain (2.12). For (2.13) and (2.14), we set $(a_i = j^{\frac{2}{3}} x_j^{\frac{1}{3}}, b_i = j^{\frac{5}{3}} x_j^{\frac{1}{3}})$ and $(a_i = j x_j^{\frac{1}{3}}, b_i = j^2 x_j^{\frac{1}{3}})$, respectively. For (2.15), take $p = 4$, raise both sides of Minkowski's Inequality to the fourth power, and set $(a_i = jx_j^{\frac{1}{4}}, b_i = j^2 x_j^{\frac{1}{4}})$. For (2.16), set $p = 5$, raise both sides of Minkowski's Inequality to the fifth power, and set $(a_i = \sqrt[5]{jx_j}, b_i = \sqrt[5]{j^6 x_j})$. □

Theorem 3. *Consider a B-array T with m rows, of strength 8, and with index set $\underline{\mu}'$. For T to exist, the following conditions must hold:*

$$L_3^3 \le L_1 L_4^2. \tag{2.17}$$

$$L_6^3 \le L_2 L_8^2. \tag{2.18}$$

$$L_5^4 \le L_2 L_6^3. \tag{2.19}$$

$$L_4^4 \le L_1 L_5^3. \tag{2.20}$$

$$L_7^4 \le L_4 L_8^3. \tag{2.21}$$

$$L_2^4 \le L_5 L_1^3. \tag{2.22}$$

$$L_7^5 \le L_3 L_8^4. \tag{2.23}$$

$$L_5^5 \le L_1 L_6^4. \tag{2.24}$$

$$L_6^5 \le L_2 L_7^4. \tag{2.25}$$

$$L_6^6 \le L_1 L_7^5. \tag{2.26}$$

$$L_7^6 \le L_2 L_8^5. \tag{2.27}$$

Proof. We use Hölder's Inequality here. To obtain (2.17) and (2.18), we take $p = 3$, $q = \frac{3}{2}$, $a_k = jx_j$, and $b_k = j^4 x_j$ in (2.4). This yields

$$\sum \sqrt[3]{j^9 x_j^3} \le \left(\sum jx_j\right)^{\frac{1}{3}} \left(\sum j^4 x_j\right)^{\frac{2}{3}},$$

which is $L_3 \le L_1^{\frac{1}{3}} L_4^{\frac{2}{3}}$. Cubing both sides of this inequality gives (2.17). In similar fashion, the substitutions $a_k = j^2 x_j$ and $b_k = j^8 x_j$ give (2.18). For (2.19) through (2.22), take $p = 4$, $q = \frac{4}{3}$ and the following substitutions, respectively: $(a_k = j^2\sqrt{x_j}, b_k = j^6\sqrt{x_j})$, $(a_k = jx_j, b_k = j^5 x_j)$, $(a_k = j^4 x_j, b_k = j^8 x_j)$, and $(a_k = j^5 x_j, b_k = jx_j)$. For (2.23) through (2.25), take $p = 5$, $q = \frac{5}{4}$ and the following substitutions, respectively: $(a_k = j^3 x_j, b_k = j^8 x_j)$, $(a_k = jx_j, b_k = j^6 x_j)$, and $(a_k = j^2 x_j, b_k = j^7 x_j)$. Finally for (2.26) and (2.27), take $p = 6$, $q = \frac{6}{5}$ and the following substitutions, respectively: $(a_k = jx_j, b_k = j^7 x_j)$ and $(a_k = j^2 x_j, b_k = j^8 x_j)$. □

3 Comments and Illustrative Examples

Note 1: It is quite clear that all the inequalities given above involve functions (in many cases, polynomial functions) of m, $\mu_0, \mu_1, \ldots, \mu_8$. These are necessary conditions for the existence of balanced arrays. Given a B-array T with m and $\underline{\mu}'$, each of these conditions must be satisfied for T to exist. However, we must point out that T may still not exist, even if all of the inequalities are satisfied.

Nonetheless, it is obvious that for $m = 8$ and any $\underline{\mu}'$, all of these conditions must be satisfied (see Lemma 1). The problem of finding the maximum m for an array T (with a given $\underline{\mu}'$) can be addressed by these inequalities. Since $\underline{\mu}'$ is given, both sides of the inequality are functions of m only. Thus, we are able to determine the maximum m, given $\underline{\mu}'$, for which a B-array may possibly exist. Using a specific condition and starting with $m = 9$, if the first contradiction occurs at $m = k$ (say, $k \geq 9$), then $\max(m) = k - 1$.

Note 2: Next, we present some illustrative examples to demonstrate that no condition is uniformly better than every other condition. We make some comparisons of these conditions using B-arrays with given $\underline{\mu}'$, and also compare these new results with some of the old ones.

Examples:

1. Take $\underline{\mu}' = (1,1,1,1,1,7,2,1,1)$. Using (2.13), we found $m \leq 10$, where as the rest of the conditions gave us $m \geq 11$ (e.g. (2.6) gave us $m \leq 12$, while (2.15) gave us $m \leq 1000$). So, condition (2.13) is the best one for this array.
2. Let us now consider $\underline{\mu}' = (4,6,6,6,6,2,6,6,4)$. For this, (2.13) gives us a very large m (exceeding 1000), but (2.11) gives us the best result of $m \leq 17$ when compared with all the other conditions.
3. Take $\underline{\mu}' = (9,7,5,3,2,3,9,5,7)$. Here, the best value of m (≤ 25) comes from using (2.11); (2.15) gives $m \leq 43$ while (2.13) gives a very large m (≤ 1000).
4. Take $\underline{\mu}' = (1,1,1,1,1,4,6,3,1)$. The best m (≤ 11) is obtained by using (2.15); while (2.6), (2.11), and (2.13) respectively give us $m \leq 13$, $m \leq 12$, and $m \leq 13$.
5. The following two inequalities are found in Chopra and Bsharat [6]. We compare these particular inequalities with the new ones.

   (a) $L_5 + 2L_3 \leq 2\sqrt[3]{L_7 L_1^2} + \sqrt[3]{L_7^2 L_1}$.

   (b) $L_6 + 2L_4 \leq 2\sqrt[3]{L_8 L_2^2} + \sqrt[3]{L_8^2 L_2}$.

   For $\underline{\mu}' = (1,1,1,1,1,6,3,1,1)$, (a) gives $m \leq 11$ and (b) gives $m \leq 12$. Thus, (a) is the better of the two in this particular case. For $\underline{\mu}' = (1,1,3,1,1,6,6,2,2)$, we get $m \leq 13$ using (a), and $m \leq 12$ using (b). So for this particular $\underline{\mu}'$, (b) is the better of the two. Using our conditions on the array $\underline{\mu}' = (1,1,3,1,1,6,6,2,2)$, we observed that (2.13) gives $m \leq 12$, while (2.5) gives $m \leq 11$ for the array $\underline{\mu}' = (1,1,1,1,1,6,3,1,1)$. However, conditions (a) and (b) do not fare well for other arrays, when compared with our conditions. For example, if we take $\underline{\mu}' = (1,3,3,2,2,2,3,3,3)$, we get $m \leq 33$ using (2.9). This is the best result using our conditions. However in this particular case, conditions (a) and (b) give us $m \leq 99$ and $m \leq 53$, respectively.

The study of the existence and construction of these combinatorial arrays is very complex, and the results presented here do tend to provide some partial answers to some of the problems.

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