Loop Designs

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This article is dedicated to Walter Wallis on the occasion of his 66th birthday.

ABSTRACT. We show, for $k = 3, 4, 5$, that the necessary conditions are sufficient for the existence of graph designs which decompose $K_v(\lambda, j)$, the complete (multi)graph on v points with λ multiple edges for each pair of points and j loops at each vertex, into ordered blocks $(a_1, a_2, ..., a_{k-1}, a_1)$. Each block is the subgraph which contains both the set of unordered edges $\{a_i, a_j\}$, for each pair of consecutive edges in the ordered list, and also the loop at vertex a_1.

Key Words: BIBD, LD, loop design, graph design.

1. Introduction

The purpose of this note is to consider a new type of combinatorial design or graph design which we call a loop design. We use the notation LD(v, k, λ, j). From the graph point of view, we decompose $K_v(\lambda, j)$, the complete (multi)graph on v points with λ multiple edges for each pair of points and with j loops at each vertex, into ordered blocks $(a_1, a_2, ..., a_{k-1}, a_1)$ of size k. Each block is the subgraph which contains the unordered edges $\{a_s, a_j\}$, for each pair of consecutive edges in the list, and which contains the loop at vertex a_1. The block (a, b, c, d, a) contains the unordered edges $\{a, b\}$, $\{b, c\}$, $\{c, d\}$, $\{d, a\}$ and the loop $\{a, a\}$. Each block consists of two cycles (or loops) of lengths 1 and $k - 1$ which share a common vertex. For brevity, we denote blocks simply as *aba* or *abca*, etc.

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It may be noted that LDs have characteristics of other designs such as Mendelsohn designs, which apply the idea of cyclic triples of ordered pairs, and balanced ternary designs (BTDs) in which a point appears in a block 0, 1 or 2 times, or cycle designs in which a graph is decomposed into copies of C_k. In short, loop designs present an interesting variation from designs of current research interest. Undefined terms can be found in [2] or [4].

A balanced incomplete block design (or graph design), a BIBD with parameters (v, b, r, k, λ), is a decomposition of $K_v(\lambda, 0)$ into b copies of the complete graph K_k. Each copy of K_k is called a block, and here r is the replication number, the number of blocks in which each point appears. BIBDs play an important role in later sections, and we give the well-known [4] necessary conditions for existence of BIBDs:

$$vr = bk \text{ and } \lambda(v-1) = r(k-1).$$

The replication number of an LD will be defined shortly. In ternary designs, each point occurs as a singleton in ρ_1 blocks and ρ_2 times as a doubleton. The replication number r is thus given by $r = \rho_1 + 2\rho_2$ so that, in the ternary block $\{a, a, b, c, d\}$, for example, there are 9 (unordered) edges, $\{ab, ab, ac, ac, ad, ad, bc, bd.cd\}$. In ternary designs, blocks need not have doubletons, and the doubleton is usually not considered as a loop (although this latter point is a matter of convention only). The loop block *abcda*, however, has only 5 edges (counting the loop *aa*). In a cycle design block, there is no distinguished vertex to contain the loop.

We will define j_i to be the number of blocks in which, say, point x occurs as an interior point. This means there are $2j$ non-loop edges incident with x in blocks in which x is the endpoint and $2j_i$ edges incident with x in those blocks in which x is an interior point. It follows that $2j + 2j_i = \lambda(v-1)$, and this equation is independent of x. This shows that j_i is a constant for any point in an LD, and, therefore, it is proper to define a replication number for LDs by $r = j + j_i = \lambda(v-1)/2$.

The equations just derived also imply that $\lambda(v-1)$ is necessarily even, but this fact is also a consequence of the equation in part (a) Theorem 1, below, which gives a fundamental new necessary condition.

THEOREM 1. *For any LD(v, k, λ, j), it is necessary that*

(a) $j = \frac{\lambda(v-1)}{2(k-1)}$; *and*

(b) $j_i = j(k-2)$.

PROOF. From the definition of LD, the number of blocks is the same as the number of loops. Part (a) follows from $\frac{\lambda}{k-1}\binom{v}{2} = jv$. The right hand side counts the number of loops (blocks), and the left hand side is the number of non-loop edges divided by the number of such edges per block. Part (b) follows from part (a) and the discussion of j_i just above. □

We next specialize to different values of k, the block size.

2. Loop Designs with $k = 3$.

When $k = 3$, an LD$(v, 3, \lambda, j)$ is essentially equivalent to a balanced ternary design in which each block has exactly one doubleton (see Section 3 of [**3**]).

EXAMPLE 1. *An LD*$(7, 3, 2, 3)$.

010	121	232	343	454	565	606
020	131	242	353	464	505	616
030	141	252	363	404	515	626

Table 1: The blocks of an LD$(7, 3, 2, 3)$

THEOREM 2. *There exist LD*$(2t+1, 3, 2, t)$ *for all* $t \geq 1$ *and LD*$(2t, 3, 4, 2t-1)$ *for all* $t \geq 2$.

PROOF. For $v = 2t + 1$, we use the cyclic construction illustrated in the example. If $v = 2t + 1$, the starter blocks are $0i0$ for $1 \leq i \leq t$. When v is even, the minimum λ is 4. An LD$(2t, 3, 4, 2t - 1)$ may be constructed by including blocks aba and bab for every distinct pair $\{a, b\}$ of points. □

If $k = 3$ in Theorem 1, then $j = \lambda(v-1)/4$. It follows that the preceding theorem gives a construction for the minimum index $\lambda = 2$ for odd values of v and $\lambda = 4$ for even values of v.

THEOREM 3. *The necessary conditions are sufficient for the existence of LD*$(v, 3, \lambda, j)$.

PROOF. By Theorem 1, λ and j are directly proportional. Higher values $\lambda = 2s$ (for odd v) and $\lambda = 4s$ (for even v) may be obtained using multiple copies of the blocks for the minimal case. □

3. Loop Designs with $k = 4$.

Theorem 1 implies that $j = \lambda(v - 1)/6$ is a necessary condition when $k = 4$. We show in this section that LD$(v, 4, \lambda, j)$ can be constructed from BIBD$(v, 3, \lambda)$. It is convenient therefore to divide the proof into the different cases for v mod 6. First, however, we derive our main construction for this section.

This requires a structure result of H. Agrawal on binary equi-replicate designs. A binary equi-replicate design is a collection of b blocks of size k (i.e. sets) over a v-set of elements such that each element appears in r blocks.

THEOREM 4. [1] *The elements of every binary equi-replicate design with* $bk = vr$ *and* $b = mv$*, can be arranged in a* k*-by-*b *array such that each column represents a block of the design and each row contains* m *copies of every element.*

THEOREM 5. *(Construction 1) Suppose a BIBD*$(v, 3, \lambda)$ *satisfies* $b = mv$ *for some* m*. Then there exists an LD*$(v, 4, \lambda, m)$*.*

PROOF. The hypothesis $b = mv$ guarantees that Agrawal's theorem applies (and that $r = mk$). Let A denote the Agrawal array and suppose the block abc of the BIBD is a column in the array A ordered with point a in row 1, b in row 2, and c in row 3. Replace this block with the ordered block $abca$ in the LD. The number of edges of the BIBD is the same as the number of non-loop edges for the LD since each has 3 per block. Since each point appears in row 1 of A exactly m times, there are m loops per point. □

When $k = 3$, the index λ was necessarily even. When $k = 4$, $\lambda = 1$ is possible.

THEOREM 6. *(a) There exist LD*$(6t + 1, 4, 1, t)$ *for all* $t \geq 1$.

(b) There exist LD$(6t + 2, 4, 6, 6t + 1)$ *for all* $t \geq 1$.

(c) There exist LD$(6t + 3, 4, 3, 3t + 1)$ *for all* $t \geq 1$.

(d) There exist LD$(6t + 4, 4, 2, 2t + 1)$ *for all* $t \geq 1$.

(e) There exist LD$(6t + 5, 4, 3, 3t + 2)$ *for all* $t \geq 1$.

(f) There exists an LD$(6t, 4, 6, 6t - 1)$ *for all* $t \geq 1$.

PROOF. The proofs of each part follow from the fact that $b = \lambda v(v - 1)/6$ for a BIBD(v, k, λ). (a) There exist BIBD$(6t+1, 3, 1)$ and in this case, $b = t(6t + 1)$. Thus, we may apply Agrawal's construction with $m = t$. Now this part follows by Construction 1. (b) For $v = 6t + 2$, we have $j = \lambda(6t+1)/6$. It is clear that $\lambda = 6$ is the smallest integer such that j is an integer. Now, since BIBD$(6t+2, 3, 6)$ exist, and as $b = (6t+2)(6t+1) = mv$ for $m = v - 1$, this part follows by Construction 1. (c) For $v = 6t + 3$, we have $j = \lambda(6t + 2)/6$. Thus, $\lambda = 3$ is the least possible index. From the existence of BIBD$(6t + 3, 3, 3)$, we have $b = (6t + 3)(63t + 1) = mv$ for $m = 3t + 1$. This part follows by Construction 1. The rest is similar. □

In the preceeding theorem the index λ is the least possible index for the LD in each of the various cases. However, we note as a curiosity, for $v = 6t + 3$ and $v = 6t$, the smallest index λ for which there exists an LD is not the smallest index for which there exists a BIBD(v, 3, μ).

The vital connection between minimal indices of LDs and related BIBDs is far more involved for larger block size, as we will see in the next section.

For every possible value of v we have constructed a minimal index example, and since for any LD (by Theorem 1) the index is necessarily a multiple of the minimal one, the existence problem for LDs with $k = 4$ is solved, and we have proved the following:

THEOREM 7. *The necessary conditions are sufficient for the existence of LD*$(v, 4, \lambda, j)$.

4. Loop Designs with $k = 5$.

For convenience, we note that, in this section, $j = \lambda(v-1)/8$. We begin with several useful examples:

EXAMPLE 2. *An LD*$(4, 5, 8, 3)$ *with* 12 *blocks. The columns are blocks, and the index is minimal.*

1	1	1	2	2	2	3	3	3	4	4	4
2	2	3	3	3	4	4	4	1	1	1	2
3	4	2	4	1	3	1	2	4	2	3	1
4	3	4	1	4	1	2	1	2	3	2	3
1	1	1	2	2	2	3	3	3	4	4	4

Table 2: The blocks of an LD(4, 5, 8, 3).

EXAMPLE 3. *An LD*$(5, 5, 2, 1)$ *is shown with columns as blocks. If the bottom row is deleted, the four columns give a BIBD*$(5, 4, 3)$.

a	b	c	d	e
b	c	d	e	a
d	e	a	b	c
c	d	e	a	b
a	b	c	d	e

Table 3: An LD(5, 5, 2, 1).

EXAMPLE 4. *A cyclic LD*$(9, 5, 1, 1)$ *is generated modulo* 9 *by the starter block* $(0, 2, 5, 4, 0)$. *This is the smallest possible example with index* $\lambda = 1$.

EXAMPLE 5. *An LD*$(6, 5, 8, 5)$ *is generated cyclically modulo* 5 *using starter blocks* $\infty 213 \infty$, $1 \infty 341$, $3 \infty 123$, $0 \infty 420$, 12341, 02410.

EXAMPLE 6. *An LD*$(8,5,8,7)$ *with starter blocks expanded modulo 7:* 02140, 01420, 04210, 02140, $5\infty635$, $6\infty356$, $3\infty563$, $\infty356\infty$.

EXAMPLE 7. *An LD*$(10,5,8,9)$ *with starter blocks expanded modulo 9: use* 6 *copies of* $(0,4,5,3,0)$ *and one copy each of* $(\infty,1,0,2,\infty)$, $(1,\infty,2,0,1)$, $(4,\infty,3,0,4)$, *and* $(3,\infty,4,0,3)$.

In the next three subsections below, we show the necessary conditions are sufficient for existence of loop designs with $k = 5$ and with $v = 24t + s$. For $s = 7, 13, 19$ we use a doubling construction and Agrawal's theorem (Section 4.1). In Section 4.3, for all other odd v, we construct cyclic designs. Minimal index LDs for all even v come from the examples above and the Latin square construction in Section 4.2. These will prove:

THEOREM 8. *The necessary conditions are sufficient for the existence of LD*$(v,5,\lambda,j)$.

Some values of v mod 24 allow for alternative constructions, and, to simplify the argument for the main result (Theorem 8), we have deferred discussion of such possibilities to Section 5 where there are two additional constructions which are of interest in their own right.

4.1. A Doubling Constructions for k = 5. Agrawal's Theorem can be used in a powerful way for $k = 5$. For the convenience of the reader we recall the necessary and sufficient conditions for existence of BIBD$(v,4,\mu)$ for a minimal μ [2].

v	minimal μ
1, 4 (mod 12)	1
7, 10 (mod 12)	2
0, 5, 8, 9 (mod 12)	3
2, 3, 6, 11 (mod 12)	6

Table 4: Minimal index BIBD$(v,4,\mu)$.

THEOREM 9. *(Construction 2) If there exists a BIBD*$(v,4,\lambda)$ *with replication number* $r = 4m$ *for some* m, *then there exists an LD*$(v,5,2\lambda,3m)$.

PROOF. Let X be a BIBD$(v,4,\lambda)$ with $r = 4m$. In this case, $r = \lambda(v-1)/3 = 4m$ and the number of blocks $b = \lambda v(v-1)/12 = vm$, and so Agrawal's theorem may be applied. For each block of X, with w in the first row of the Agrawal array, say for block $\{w,x,y,z\}$, create the following 3 blocks for the LD: $wxyzw$, $wyzxw$, $wzxyw$. Observe that wx was an edge in the block of X, but wx is an edge in two corresponding blocks of the LD, and similarly for y and z. The rest is clear. □

THEOREM 10. *There exist* $LD(v,5,4,j)$ *whenever* $v \equiv 7, 19 \pmod{24}$ *and* $LD(v,5,2,j)$ *for* $v \equiv 13 \pmod{24}$. *In both cases the index is minimal.*

PROOF. The result will follow immediately from Table 4 and Construction 2. Note that, for a BIBD$(v,4,\lambda)$, the number of blocks is $b = \lambda v(v-1)/12$. For example, when $v = 24t+7$, and $\lambda = 2$ (see Table 4), $b = mv$ for $m = 4t+1$ and Agrawal's theorem applies. The other two cases are similar and we omit the details. □

4.2. The Latin Square Construction. A Latin square (LS) of order n is a square n-by-n array of cells, each of which contains some element from a set of size n. It is required that every row and every column contains each element exactly once. In what follows we assume that $\{0,1,2,...,n-1\} = N$ are the n elements in any Latin square and that these numbers are used to index the rows and columns. The (i,j) entry of any LS will usually be denoted $L(i \circ j)$. Two Latin squares are mutually orthogonal if the squares, when superimposed, contain each of the n^2 pairs of elements exactly once. These are called MOLS. An LS is idempotent if $L(i \circ i) = i$ for every $i \in N$.

We construct a set of blocks of size $k = 4$ using two idempotent MOLS, say L_1 and L_2, on the set $\{0,1,2,...,n-1\} = N$ for $n \geq 12$. Suppose $a, b \in N$ and $a \neq b$. Then form the block $\{a, b, L_1(a \circ b), L_2(a \circ b)\}$. Since the MOLS are idempotent, neither $L_1(a \circ b)$ nor $L_2(a \circ b)$ equals a or b, and $L_1(a \circ b) \neq L_2(a \circ b)$ by the orthogonality. If we form such a block for every ordered pair of elements in N, we note we have also formed, for instance, the block $\{b, a, L_1(b \circ a), L_2(b \circ a)\}$. Consider those blocks in which point a occurs in the first position. Point b also occurs those blocks, obviously once in the second position, but also in positions 3 and 4 in other blocks. This is because Latin squares (or quasigroups) have a unique solvability property. Thus, with point a in position 1, as all the other points of N appear in position 2 with a in some block, there is a block say $\{a, x, L_1(a \circ x), L_2(a \circ x)\}$ which will have $L_1(a \circ x) = b$, and another which will have $L_2(a \circ y) = b$. Thus, a and b appear together 3 times in blocks in which a is in the first position. The same argument, applied to positions 2, 3, and 4, demonstrates that a and b appear together 12 times in blocks. Thus, the unordered blocks created give a BIBD$(v,4,12)$.

Now we alter each unordered BIBD block of size 4 to an ordered LD block of size 5 as follows: for all $a \neq b \in N$, construct the block $(a, b, L_1(a \circ b), L_2(a \circ b), a)$.

Notice that point a and point $L_1(a \circ b)$ are not adjacent in the LD block and are not counted in the index. This means that, when $L_1(a \circ x) = b$, the points a and b occur together one time less than in the BIBD. Also, when

a is in positions 2, 3, and 4 in other blocks, points a and b occur one time less. Thus, the total number of times a and b appear together in LD blocks has dropped from 12 to 8. This is clearly true for every pair of points, and the ordered blocks constructed give an LD$(v, 5, 8, v-1)$.

THEOREM 11. *(Construction 3) There is a minimal index LD$(v, 5, 8, v-1)$ for all even $v \geq 4$.*

PROOF. For $v = 4, 6, 8, 10$ see Examples 2,5,6,7. It is well-known that there are 3 MOLS for each order $v > 10$, and that from these one can get 2 idempotent MOLS. Thus, the existence of our LDs follows for all v for which there exist two idempotent MOLS; in particular they exist for $11 \leq v = 2t$, and the index $\lambda = 8$ is minimal for these values. □

4.3. A Striking Property of Differences. The only examples for $k = 5$ and with index 1 occur for $v = 8t + 1$.

THEOREM 12. *There exist LD$(8t + 1, 5, 1, j)$ for all $t \geq 1$.*

PROOF. Suppose $v = 8t+1$. Use t starter blocks $(0, 4s, 8s-3, 4s-1, 0)$ where $1 \leq s \leq t$. □

The difference family given above has a striking property. The differences between adjacent elements in $(0, 4, 5, 3, 0)$ are, respectively, left to right: $4, 1, 2, 3$. When $s = 2$, the differences are $8, 5, 6, 7$, and so on. In general, the differences are $4s - 3$, $4s - 2$, $4s - 1$, and $4s$. One can exploit this regularity of the differences to obtain difference families for other values of v if, omitting or altering the last starter block(s), one can obtain the other needed differences.

EXAMPLE 8. *An LD$(11, 5, 4, 5)$, with minimal index 4. The design is cyclic, generated mod 11 by the starter blocks:* 04530, 04530, 05940, 05610, 02530.

EXAMPLE 9. *An LD$(27, 5, 4, 13)$. The design is cyclic, generated modulo 27, and the blocks are:* $(0, 4, 5, 3, 0) \times 4$, $(0, 8, 13, 7, 0) \times 4$, $(0, 12, 21, 11, 0) \times 2$, $(0, 9, 22, 14, 0)$, $(0, 13, 25, 12, 0)$, $(0, 10, 21, 11, 0)$, *where $b \times i$ means i-copies of block b.*

THEOREM 13. *There exist LD$(24t + 3, 5, 4, 12t + 1)$ for all $t \geq 1$.*

PROOF. Use the starter blocks $(0, 4s, 8s - 3, 4s - 1, 0) \times 4$ for $1 \leq s \leq 3t - 1$, and $(0, 12t, 24t - 3, 12t - 1, 0) \times 2$. Also, use one copy each of block $(0, 12t-3, 24t-2, 12t+1, 0)$, with differences $12t-3$ and $12t+1$ twice each; block $(0, 12t+1, 24t+1, 12t, 0)$ with differences $12t+1$ and $12t$ twice each;

and block $(0, 12t-2, 24t-3, 12t-1, 0)$ with differences $12t-2$, $12t-1$ twice each. □

THEOREM 14. *There exist* $LD(24t+5, 5, 2, 6t+1)$ *for all* $t \geq 0$.

PROOF. Table 3 shows the case for $v = 5$. We assume $v \geq 29$. Use the starter blocks $(0, 4s, 8s-3, 4s-1, 0) \times 2$ for $1 \leq s \leq 3t$, and $(0, 12t+1, 24t+3, 12t+2, 0)$. □

THEOREM 15. *There exist* $LD(24t+11, 5, 4, 12t+5)$ *for all* $t \geq 0$.

PROOF. Use blocks $(0, 4s, 8s-3, 4s-1, 0) \times 4$ for $1 \leq s \leq 3t$, $(0, 12t+4, 24t+5, 12t+3, 0) \times 2$, $(0, 12t+5, 24t+9, 12t+4, 0)$, $(0, 12t+5, 24t+6, 12t+1, 0)$, and $(0, 12t+2, 24t+5, 12t+3, 0)$. □

THEOREM 16. *There exists* $LD(24t+15, 5, 4, 12t+7)$ *for all* $t \geq 0$.

PROOF. Use the starter blocks $(0, 4s, 8s-3, 4s-1, 0) \times 4$, for $1 \leq s \leq 3t+1$, and $(0, 12t+8, 24t+13, 12t+7, 0) \times 2$. In this last block, the differences $12t+7$ and $12t+8$ both occur, but they are additive inverses mod v. The other two differences mod v in this block (which we use twice) are $12t+5$ and $12t+6$. It thus only remains to use $12t+5$ and $12t+6$ as differences twice more each, and we do this with block $(0, 12t+5, 24t+11, 12t+6, 0)$. □

THEOREM 17. *There exist* $LD(24t+21, 5, 2, 6t+5)$ *for all* $t \geq 0$.

PROOF. Use starter blocks $(0, 4s, 8s-3, 4s-1, 0) \times 2$ for $1 \leq s \leq 3t+2$, and $(0, 12t+10, 24t+19, 12t+9, 0)$. □

THEOREM 18. *There exist* $LD(24t+23, 5, 4, 12t+11)$ *for all* $t \geq 0$.

PROOF. Use starter blocks $(0, 4s, 8s-3, 4s-1, 0) \times 4$ for $1 \leq s \leq 3t+2$. Use $(0, 12t+12, 24t+21, 12t+11, 0) \times 2$ and one copy of $(0, 12t+9, 24t+19, 12t+10, 0)$. □

Since we have shown $\mathrm{LD}(v, 5, \lambda, j)$ exist for all v and with minimal index λ, we have proved Theorem 8.

5. Additional Remarks

To clarify the earlier exposition, we omit discussion of larger k and present some other constructions which may be useful for another purpose.

5.1. Some Useful Constructions. The first is a second doubling construction.

THEOREM 19. *(Construction 4) If there exists a BIBD$(v, 5, \lambda)$, then there exists an LD$(v, 5, 2\lambda, j)$ where $j = \lambda(v-1)/4$.*

PROOF. For each block in the BIBD, create 5 blocks as in Example 3. □

THEOREM 20. *(a) Suppose $v = 10n + 1$ or $v = 10n + 5$ for $n \geq 1$ but $v \neq 15$. Then there exist LD$(v, 5, 4, j)$. (b) If $v = 20n + 1$ or $20n + 5$, then LD$(v, 5, 2, j)$ exist. (c) If $v \equiv 0, 1 \pmod 5$, then LD$(v, 5, 8, j)$ exist.*

PROOF. BIBD(v, 5, 2) exist for $v \equiv 1, 5 \pmod{10}$ except for $v = 15$. Note BIBD$(v, 5, 1)$ exist for $v \equiv 1, 5 \pmod{20}$. BIBD$(v, 5, 4)$ exist for $v \equiv 0, 5 \pmod 5$ (see p.72 of [**2**]). □

A BIBD(v, k, λ) is near-resolvable if the blocks of the design can be partitioned into near-parallel classes so that one and only one point is missing in each class. There exist near-resolvable BIBD$(v, 3, 2)$ for every v $\equiv 1 \pmod 3$ (see p. 128 of [**2**]).

THEOREM 21. *(Construction 5) Suppose $v = 1 + 3s$ for some s. Then there exist LD$(v, 5, 8, v-1)$.*

PROOF. Suppose C_x is a near-parallel class of a near-resolvable BIBD(v, 3, 2) with point x missing.. If abc is any block in that class, create blocks $xabcx$, $xbcax$, and $xcabx$ for the LD. Do this for each class. Points x and a appear 2 times in BIBD blocks in some 2 classes, say C_u and C_v, but not C_x and C_a. Using C_u and C_v, points x and a appear together 4 times in newly created LD blocks. They appear twice more in blocks created from each of C_x and C_a, and thus appear 8 times in LD blocks. □

This was useful for certain even values of v, but Constructions 2 and 3 dealt with the relevant cases.

A pair-wise balanced design, a PB(v, K), of order v with block sizes from K is a pair $(V, \mathcal{B})$ where V is a set of size v and $\mathcal{B}$ is a collection of subsets of V, and these satisfy the properties: (1) If $B \in \mathcal{B}$ then $|B| \in K$; (2) Every pair of distinct elements of V occurs in exactly one block of $\mathcal{B}$.

For $K = \{5, 7\}$, there exist PBD(v, K) for all odd $v = 2s + 1 \geq 641$. There are 13 values of v ("definite exceptions") for which there is no such PBD and about 80 possible exceptions with $9 \leq v \leq 639$ (see Chap. IV.3 of [**2**]). It follows that:

THEOREM 22. *(Construction 6) There exist LD*$(v,5,4,j)$ *for all odd* $v \geq 5$ *except for exceptional values in* [**2**].

PROOF. Suppose v is odd and X is a PBD$(v,\{5,7\})$. For any block of size 5 in X, construct two copies of the blocks indicated in Table 3. For any block of size 7, use one copy of the blocks constructed as in Theorem 10 for LD$(7,5,4,3)$. This constructs an LD$(v,5,4,j)$. This construction is minimal for $v = 4t+3$. □

5.2. Loop Designs with $k = 6$. We include this brief subsection only to illustrate that the techniques used earlier can be applied to loop designs with larger indices. In particular, we give a doubling construction like those in previous sections.

EXAMPLE 10. *An LD*$(6,6,2,1)$. *Columns are blocks.*

1	2	3	4	5	6
2	5	1	1	3	3
3	3	6	5	2	1
4	6	5	2	1	4
5	4	4	6	6	2
1	2	3	4	5	6

Table 6: The blocks for an LD$(6,6,2,1)$.

THEOREM 23. *If there exists a BIBD*$(v,6,\lambda)$, *then there exists an LD*$(v,6,2\lambda,j)$.

PROOF. Expand every block of the BIBD using Table 6. □

EXAMPLE 11. *There exists an LD*$(11,6,1,1)$ *using the starter block* $(0,5,3,7,10,0)$ *modulo* 11.

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