Total Coloring of Block-Cactus Graphs

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Abstract

In this paper, we present new results about the coloring of graphs. We generalize the notion of proper vertex-coloring introducing the concept of *range-coloring of order k*. The relation between range-coloring of order k and total coloring is presented: we show that for any graph G that has a range-coloring of order $\Delta(G)$ with t colors, there is a total coloring of G that uses $(t+1)$ colors. This result provides a framework to prove that some families of graphs satisfy the total coloring conjecture. We exemplify with the family of block-cactus graphs.

1 Introduction

Let $G(V,E)$ be a graph and C a finite set of colors. A *vertex-coloring* of a graph G is a function $c : V \to C$ such that $c(v) \neq c(w)$ whenever v

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and w are adjacent. The smallest k such that G has a *k-coloring* (i.e., a vertex-coloring $c : V \rightarrow \{1, \ldots, k\}$) is the *chromatic number* of G; it is denoted by $\chi(G)$. An *edge coloring* of $G(V, E)$ is a function $c : E \rightarrow C$ such that $c(e) \neq c(e')$ whenever e and e' are incident to the same vertex. The smallest k such that G has a *k-edge-coloring* (i.e., an edge coloring $c : E \rightarrow \{1, \ldots, k\}$) is the *chromatic index* of G; it is denoted by $\chi'(G)$. A *total coloring* of $G(V, E)$ is an assignment of colors to its vertices and edges so that adjacent or incident elements have distinct colors. The least number of colors sufficient for a total coloring of a graph is called its *total chromatic number* and it is denoted by $\chi_T(G)$. Clearly, for any graph G, $\Delta(G) + 1 \leq \chi_T(G)$, where $\Delta(G)$ is the maximum degree of G. The *total coloring conjecture* (*TCC*), posed by Vizing and Behzad independently [16], says:

$$\text{For any graph } G,\ \chi_T(G) \leq \Delta(G) + 2.$$

For some families, as dually chordal graphs [4], interval graphs [1] and spider graphs [13], the *TCC* has already been proved. The conjecture also holds for planar graphs with $\Delta(G) \neq 6$; the case of planar graphs with $\Delta(G) = 7$ is proved in [12] and the other ones were mentioned in [7, 8, 16]. Recently, it was proved that planar graphs with maximum degree 6 without 4-cycles satisfy the conjecture too [15].

In this paper, we first generalize the notion of vertex-coloring presenting the concept of *range-coloring of order k*. The relation between range-coloring of order k and total coloring is presented. We show that for any graph G that has a range-coloring of order $\Delta(G)$ with t colors, there is a total coloring of G that uses $(t + 1)$ colors. This result provides a new framework to prove that some families of graphs satisfy the total color conjecture as, for instance, the block-cactus graphs. For this family, linear algorithms to perform its range-coloring of order $\Delta(G)$ and its total coloring are presented, and the *TCC* is proved.

2 Basic concepts

Let $G(V, E)$ be a simple connected graph, $|V| = n$ and $|E| = m$. The *open neighborhood* of a vertex $v \in V$ is the set $N(v) = \{u \in V;\ uv \in E\}$, and the *closed neighborhood* is the set $N[v] = \{u \in V;\ uv \in E\} \cup \{v\}$. The *degree* of v is $d(v) = |N(v)|$.

A *cut-vertex* is a vertex that if removed (along with all edges incident with it) produces a graph with more connected components than the original graph. A *block* of a graph is a maximal connected subgraph containing no cut-vertices.

Theorem 1 [6] *The intersection of any two distinct blocks of a graph consists of at most one vertex.*

Let X denote the set of cut-vertices of G and Y the set of its blocks. The *block-cut-vertex graph* of G is the bipartite graph $H = (X \cup Y, F)$ in which there is an edge joining block B and cut-vertex v if and only if v is a vertex in B.

Theorem 2 [3] *The block-cut-vertex graph of a connected graph is a tree.*

3 Range-coloring of order k

Let $G(V, E)$ be a graph and a vertex-coloring $c : V \to C$ of G. The set of colors used by the neighbors of a vertex $v \in V$ is $c(N(v))$ and $|c(N(v))|$ is the number of colors appearing in the neighborhood of v.

A vertex-coloring $c : V \to C$ of G is called a *range-coloring of order* k, $1 \leq k \leq \Delta(G)$, if for all $v \in V$ such that $d(v) \leq k$, $|c(N(v))| = d(v)$; otherwise $|c(N(v))| \geq k$.

For now on, $\Delta(G)$ will be simply denoted by Δ.

Examples of range-coloring of orders 1, 2 and 3 are shown for the star graph $K_{1,4}$ in Figure 1.

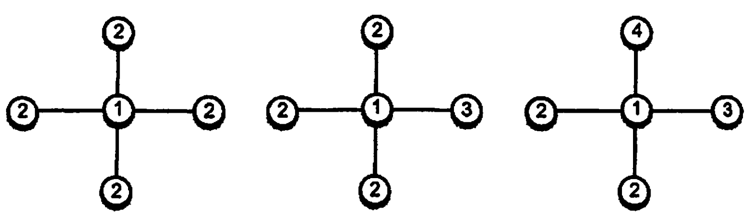

Figure 1: Range-colorings of a star graph

A graph G is *t-range-colorable of order k* if there exists a range-coloring of order k that uses t colors. Analogously, the *range chromatic number of order* k, denoted by $\chi_r^k(G)$, is the least value of t for which G can be t-range-colorable of order k. Note that the range-coloring generalizes some known vertex-colorings. The usual vertex-coloring of G is a range-coloring of order 1. The equivalence of the range-coloring of order Δ and the distant-2 coloring [2] was proved in [9].

The following lemma shows the determination of the range-chromatic number of order Δ of a cycle graph C_n.

Lemma 3 *For any cycle C_n,* $\chi_r^{\Delta}(C_n) = \begin{cases} 3, & n \bmod 3 = 0 \\ 5, & n = 5 \\ 4, & \textit{otherwise} \end{cases}$.

Proof: It is easy to verify that $\chi_r^{\Delta}(C_5) = 5$.

Consider C a finite set of colors and a vertex-coloring $\textit{color} : V \to C$.

In a cycle, $\Delta = 2$; so the range-coloring of order Δ needs at least three colors since, by definition, $|\textit{color}(N(v))|$ must be 2, for all $v \in V$.

Let $C_{3k} = \langle v_0, v_1, \ldots, v_{3k-1}, v_0 \rangle$ and $\textit{color} : V \to \{c_0, c_1, c_2\}$ be a vertex-coloring such that $\textit{color}(v_i) = c_{i \bmod 3}$, $i = 0, \ldots, (3k-1)$. Note that $\textit{color}(v_{i-1}) \neq \textit{color}(v_{i+1})$, $i = 1, \ldots, (3k-2)$; $\textit{color}(N(v_0)) = \{\textit{color}(v_1), \textit{color}(v_{3k-1})\} = \{c_1, c_2\}$, and $\textit{color}(N(v_{3k-1})) = \{\textit{color}(v_0), \textit{color}(v_{3k-2})\} = \{c_0, c_1\}$. Concluding, $\chi_r^{\Delta}(C_{3k}) = 3$.

Let C_{3k} be a colored cycle as above and consider a cycle $C_{3k+1} = C_{3k} \cup \{v\}$. Without loss of generality, $C_{3k+1} = \langle v_0, \ldots, v_i, v, v_j, \ldots, v_0 \rangle$ is colored with $\textit{color}(v_{i-1}) = c_0$, $\textit{color}(v_i) = c_1$, $\textit{color}(v_j) = c_2$ and $\textit{color}(v_{j+1}) = c_0$.

So,

$\textit{color}(v) \neq c_0$	otherwise $|\textit{color}(N(v_j))| = 1$;
$\textit{color}(v) \neq c_1$	otherwise $v_i v \in E$ with $\textit{color}(v_i) = \textit{color}(v)$;
$\textit{color}(v) \neq c_2$	otherwise $v v_j \in E$ with $\textit{color}(v) = \textit{color}(v_j)$.

Clearly, all possible assignments are forbidden and $\chi_r^{\Delta}(C_{3k+1}) = 4$.

The same reasoning above can be applied to the cycle $C_{3k+2} = C_{3k} \cup \{u, v\}$, $k \geq 2$. It is already known that vertex u must be colored with a fourth color, c_3. In order to maintain a correct range-coloring, the sequence consisting of three vertices before and the three vertices after the vertex u in the cycle must be colored, at each side, with c_0, c_1 and c_2. So, vertex v receives also color c_3 but between vertices u and v it must exist at least three vertices, which is possible since $k \geq 2$. Finally, $\chi_r^{\Delta}(C_{3k+2}) = 4$. ∎

4 Relating range-coloring and total coloring

The next theorem relates range-coloring of order Δ with total coloring. It will be seen that it is possible, after performing a range-coloring of order Δ, to extend this coloring to the edges, using just one more color. This extension will allow us to prove the total color conjecture for some families of graphs.

Theorem 4 *Let $G(V, E)$ be a graph and $c : V \to \{1, 2, \ldots, t\}$ a range-coloring of order Δ of G. There is a total coloring c_T of G with at most $(t + 1)$ colors.*

Proof: Let $K_t(V', E')$ be the complete graph of order t and $c'_T : V' \cup E' \to \{1, 2, , \ldots, t + 1\}$ a total coloring of K_t. We extend the given range-coloring c of order Δ to the edges of G to obtain a total coloring c_T of G as following.

First, we define a function $f : V \to V'$, $f(v) = i$ such that $c(v) = c'_T(i)$. So, $c_T : V \cup E \to \{1, 2, , \ldots, t + 1\}$ is such that:

$$c_T(v) = c(v), \; v \in V,$$
$$c_T(vw) = c'_T(f(v)f(w)), \; vw \in E.$$

The coloring c_T is a total coloring of the graph G because, if two edges $vw, vu \in E$ are incident in v, the colors $c(v)$, $c(w)$ and $c(u)$ are all different, since that c is a range-coloring of order Δ. So, the vertices $f(v)$, $f(w)$, $f(u) \in V'$ are also different, and $c'_T(f(v)f(w)) \neq c'_T(f(v)f(u))$. Consequently, $c_T(vw) \neq c_T(vu)$. Reciprocally, if an edge $vw \in E$ is incident in $v \in V$, then $c'_T(f(v)) \neq c'_T(f(v)f(w))$. Then, $c_T(v) \neq c_T(vu)$. ∎

Let $G(V, E)$ be a graph with all vertices labeled by the colors determined by the range-coloring $c : V \to \{1, 2, \ldots, t\}$ of order Δ of G, and A a matrix of order $(t + 1)$ with a total coloring of a complete graph K_t. The proof of Theorem 4 yields a simple algorithm to obtain a total coloring $c_T : V \cup E \to \{1, 2, \ldots, t + 1\}$ of G.

```
Algorithm TotalColor;
begin
   for all v ∈ V do
      c_T(v) ← c(v);
   for all vw ∈ E do
      c_T(vw) ← A(c(v), c(w));
end
```

Corollary 4.1 *A graph G $(\Delta + 1)$-range-colorable of order Δ satisfies the total coloring conjecture.*

Thus, Theorem 4 provides a new framework to prove that some families of graphs satisfy the *TCC*.

5 Total coloring of block-cactus graphs

A graph $G(V, E)$ is called a *block-cactus graph* [11] if each block of G is a complete graph or a cycle. These graphs include two interesting subclasses, which appeared frequently in the literature: the *block graphs* (blocks are complete graphs) and the *cactus graphs* (blocks are cycles or K_2) .

An example of a block-cactus graph is presented in Figure 2.

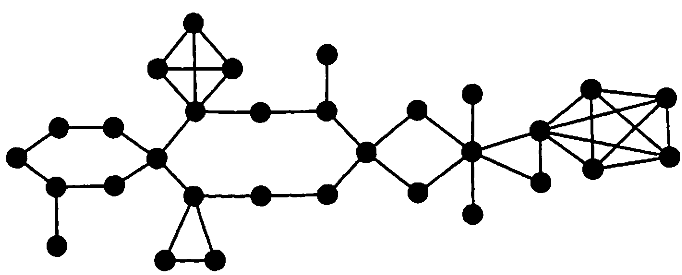

Figure 2: Block-cactus graph

Let G be a block-cactus graph and S the set of the cut-vertices of G. A block B is called *attached* to a cut-vertex $v \in S$ when v belongs to B. A block-cactus graph is *trivial* if it is composed by only one block.

Our goal here is to prove that block-cactus graphs satisfy the *TCC*. First, we present an algorithm for the determination of a range-coloring of order Δ of a non trivial block-cactus graph G with $\Delta \geq 4$ using $(\Delta + 1)$ colors. The next step is to extend this coloring to the edges using Theorem 4, obtaining a total coloring with, at most, $(\Delta + 2)$ colors. Note that block-cactus graphs with $\Delta < 4$ are planar and it is already known that these graphs satisfy the *TCC*. This means that if the block-cactus graph is planar the conjecture holds even if $\Delta = 6$.

The main idea of the algorithm is to greedily color the cut-vertices of G together with the vertices of the blocks attached to them. It begins by selecting a cut-vertex v such that $d(v) = \Delta$. Vertex v is colored and inserted in set S', which stores the colored cut-vertices not yet processed (we consider a cut-vertex v *processed* when all the vertices of the blocks B_i attached to v are colored). After that, at each iteration, a cut-vertex w is processed and removed from S'. Two situations can happen:

1. Vertex w is the first cut-vertex to be processed and none of its adjacent vertices is colored. In this case, all vertices belonging to $N(w)$ must be colored, spending Δ colors different from the color of w. As the set of colors C has cardinality $(\Delta + 1)$, all its colors are used.

2. Vertex w is not the first cut-vertex to be processed. So, vertex w is already colored because it belongs to a block B, attached to another cut-vertex already processed. It is important to note that B is the only one attached to w with all vertices colored. This is easy to prove considering the block-cut-vertex graph of G, since, by Theorem 2, there is only one path from any cut-vertex already processed to vertex w.

   At this point, some vertices of $N(w)$ are already colored, depending on the nature of block B. If block B is an edge, only one vertex of $N(w)$ is colored; if it is a cycle, two vertices of $N(w)$ are colored, otherwise, $B = K_i$ and obviously there are i vertices belonging to $N(w)$ colored with i different colors. We know that $|C| = (\Delta + 1)$ and $\Delta \geq 4$; so, there is at least $(|C| - i + 1)$ colors available to be used when coloring the remaining adjacent vertices of $N(w)$. As $d(w) \leq \Delta$, the assignment is always possible.

In any case, after coloring $N(w)$, the blocks attached to w must be analysed. If the blocks are complete graphs, all vertices are already colored; the vertices remaining in the other possible cycles must be colored following Lemma 3. Any vertex, after being colored, must be tested whether or not it belongs to the set of cut-vertices S; in affirmative case it is inserted on another set S'. Vertex w can now be considered *processed*.

It is interesting to note that, at each iteration of the algorithm, a connected subgraph of G is colored.

The detailed algorithm follows. The data structures considered as input are:

- A block-cactus graph G; the set of adjacency lists $N(v)$, the degree $d(v)$ of each vertex $v \in V$.
- The set S of cut-vertices of G.
- For each vertex $v \in S$, the set *Badj*$[v]$ stores the indices of the blocks attached to v.
- The vertices of each block B of G.

Two simple procedures will be called by the algorithm:

- $ColorVertex(v, C)$ colors the vertex v with some color belonging to set C and tests whether or not it is a cut-vertex; in the affirmative case, the vertex v must be included in the set S'.
- $ColorCycle(B, v)$ colors the vertices not yet colored of each cycle B attached to v following directly the constructive proof of Lemma 3.

```
Algorithm RangeColorBlockCactus;
begin
   C ← {1,2,...,Δ+1};          % C: set of colors
   S' ← ∅;
   Choose v ∈ S such that d(v) = Δ(G);
   ColorVertex(v, C);
   while S' not empty do
      Choose w ∈ S';
      Ctemp ← C − {c(w)};      % Ctemp: set of available colors
      for all u ∈ N(w) do
         if u is colored then
            Ctemp ← Ctemp − {c(u)};
      for all u ∈ N(w) do
         if u not colored then
            ColorVertex(u, Ctemp);
            Ctemp ← Ctemp − {c(u)};
      for all B ∈ Badj[w] do
         if |B| > 3 and B is a cycle then
            ColorCycle(B, w);
      S' ← S' − {w};
end
```

Lemma 5 *Algorithm RangeColorBlockCactus computes an exact range-coloring of order* Δ *of a block-cactus graph in time complexity* $O(m)$.

Proof: The input of the algorithm is a non trivial block-cactus graph G with $\Delta \geq 4$.

By the definition of range-coloring of order Δ, the coloring of the neighborhood of each vertex v uses $d(v)$ colors. The algorithm colors a cut-vertex v at each iteration; the only forbidden colors are exactly the ones already used in some of the vertices belonging to $N(v)$. So, the algorithm uses, for cut-vertices, at most $(\Delta+1)$ colors. As each cut-vertex is colored once, the coloring of all cut-vertices and their adjacent vertices takes, in the worst case, $O(m)$.

Procedure *ColorCycle* colors the remaining vertices that belong to a cycle following the constructive proof of Lemma 3. By the lemma, a cycle is colored with at most five colors. For a non trivial block-cactus graph, in the worst case $\Delta = 4$, so the procedure uses $\Delta + 1$ colors. Let $B = \langle v_0, v_1, \ldots, v_k \rangle$ be the cycle to be colored by the procedure such that v_0 is the cut-vertex v. Vertices v_0, v_1 and v_k are already colored; v_1 and v_k are adjacent to the vertex v_0. For each vertex w colored by the procedure, at most three colors must be analyzed, so it takes constant time complexity to color w.

Thus, the algorithm performs in $O(m)$. ■

The procedure *TotalColor* can be used in order to obtain a total coloring of a block-cactus graph. Its time complexity is $O(n^2)$, however, it is possible to build a more efficient implementation of the algorithm. It is not difficult to note that the lines of the latin square that give us a total coloring of a complete graph are simple rearrangements of the first line. For instance, the first and the second lines of a possible latin square of order seven are 1 5 2 6 3 7 4 and 5 2 6 3 7 4 1, respectively. So, only one array A is needed in the implementation. Observe that if Δ is odd the dimension d of A is t, otherwise $d = t + 1$.

Procedure *ColorEdge*(d, v, w, tc)
begin
  $i \leftarrow c(v)$;
  $j \leftarrow c(w)$;
  $index \leftarrow j + i - 1$;
  if $index > d$ **then** $index \leftarrow index - d$;
  $tc \leftarrow A(index)$;
end

Finally, we summarize our results about block-cactus graphs.

Lemma 6 *A block-cactus graph satisfies the total coloring conjecture.*

Proof: A block-cactus graph G with $\Delta < 4$ is planar, So, it obeys the *TCC*. If G has $\Delta \geq 4$, by Theorem 4, there is a total coloring using at most $(\Delta + 2)$ colors. In both cases, $\chi_T(G) \leq \Delta + 2$. ■

6 Conclusion

In this paper, it was shown the relation between range-coloring of order k and total coloring: if a graph G has a range-coloring of order $\Delta(G)$ with t colors, there is a total coloring of G that uses $(t+1)$ colors. This result allowed to prove that the *TCC* is valid for the block-cactus graphs. It is interesting to note that this family actually includes two subfamilies: block graphs and cactus graphs. It can be easily observed that both have disjoints cuts and both can be represented by a unique tree structure. It is possible that families with similar properties can use the same reasoning in order to prove the conjecture.

We exemplify it with the family of *path-complete graphs*, considered by Harary [5] in 1962. Soltès [14] proved that these graphs are the sole connected graphs with n vertices, m edges and maximum average distance between pairs of vertices. The following definition can be found in [10].

Let $m, n, p, t \in N$, with $1 \leq t \leq n-2$ and $1 \leq p \leq n-t-1$. A graph with n vertices and m edges such that

$$\frac{(n-t)(n-t-1)}{2} + t \leq m \leq \frac{(n-t)(n-t-1)}{2} + n - 2$$

is called a (n, p, t) *path-complete graph*, denoted $PC_{n,p,t}$, if and only if

1. the maximal clique of $PC_{n,p,t}$ is K_{n-t};
2. $PC_{n,p,t}$ has a t-path $P_{t+1} = [v_0, v_1, \dots v_t]$ such that $v_0 \in K_{n-t} \cap P_{t+1}$ and v_1 is joined to K_{n-t} by p edges;
3. there are no other edges.

Path-complete graphs can be partioned in two subgraphs: one block consisting of two cliques that have p vertices in common and a path. So, it is easy to adjust the algorithm *RangeColorBlock-Cactus* in order to obtain a $(\Delta(G)+1)$-range-color for the vertices of graphs belonging to this family and consequently to obtain a total coloring with $\Delta(G)+2$ colors.

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