On Mod(2)-Edge-Magic Graphs

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ABSTRACT. Let G be a (p,q)-graph where each edge of G is labeled by a number 1, 2,...,q without repetition. The vertex sum for a vertex v is the sum of the labels of edges that are incident to v. If the vertex sums equal to a constant (mod k) where $k \geq 2$, then G is said to be Mod(k)-edge-magic. In this paper we investigate graphs which are Mod(k)-edge-magic. When k =p, the corresponding Mod(p)-edge-magic graph is the edge-magic graph introduced by Lee (third author), Seah and Tan in [10]. In this work we investigate trees, unicyclic graphs and (p,p+1)-graphs which are Mod(2)-edge-magic.

Key words and phrases: Mod(k)-edge-magic, trees, unicyclic, (p,p+1)-graph.

AMS 2000 MSC : 05C78, 05C25

1. Introduction. All graphs in this paper are simple graphs with no loops or multiple edges.

Lee, Seah and Tan [10] introduced the following concept of edge-magic graphs

Definition 1.1 Let G be a (p,q)-graph in which the edges are labeled 1, 2,...,q without repetition. The vertex sum for a vertex v is the sum of the labels of the incident edges at v. If the vertex sums are constant, mod p, then G is said to be **edge-magic** (in short **EM**).

Example 1. Figure 1 shows a graph G with 6 vertices and 8 edges that is EM with different constant sums.

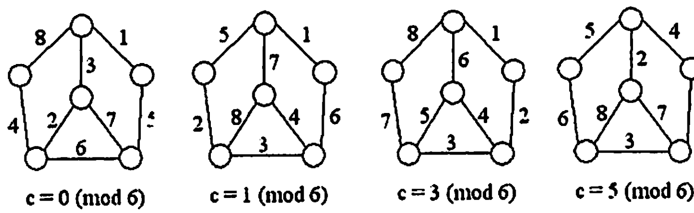


Figure 1.

Example 2. The following maximal outerplanar graphs with 6 vertices are EM.

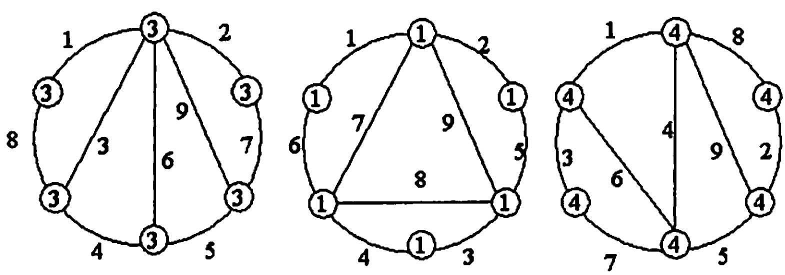


Figure 2.

A necessary condition for a (p,q)-graph to be edge-magic is $q(q+1)\equiv 0 \pmod p$. However, this condition is not sufficient. There are infinitely many connected graphs , such as trees and cycles, satisfy this condition that are not edge-magic.

Now we introduce the following concept.

Definition 1.2 Let $k \geq 2$ and G be a (p,q)-graph in which the edges are labeled 1, 2,...,q without repetition. The vertex sum for a vertex v is the sum of the labels of the incident edges at v. If the vertex sums are constant, mod k, then G is said to be Mod(k)-**edge-magic** (in short **Mod(k)-EM**).

A necessary condition for a graph to be edge-magic is given in the following Theorem

Theorem 1.1. If $p \equiv 0 \pmod k$ then a necessary condition for G to be Mod(k)-edge-magic is that $q(q+1) \equiv 0 \pmod k$.

Example 3. The path P_4 with 4 vertices is Mod(2)- EM, but not Mod(k)-EM for k=3,4.

Figure 3.

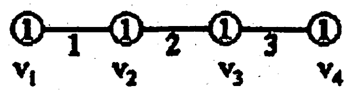
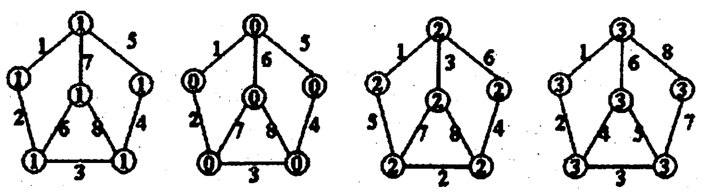


Figure 3.

Example 4. The graph G in Figure 4 is Mod(k)- EM, for k=2,3,4,6 but not 5.



Mod(2)-EM Mod(3)-EM Mod(4)-EM Mod(6)-EM

Figure 4.

2. (p, q)-Graphs which are Mod(k)-EM for all k.

Stewart [19,20] defined that a graph is **supermagic** if the edges are labeled 1,2,3,...q so that the vertex sums are constant. He showed that K_3, K_4, K_5 are not supermagic and when $n \equiv 0 \pmod 4$, K_n is not supermagic. For n>5, K_n is supermagic if and only if $n \not\equiv 0 \pmod 4$. For a generalization of this result see [6]. Hartsfield and Ringel ([3]) exhibited some new examples of supermagic graphs.

It is clear that

Theorem 2.1. If a (p,q)-graph G is supermagic then it is Mod(k)-EM for all k =2,3,…,p.

Ho and Lee [6] extended the result of Stewart to regular complete k-partite graphs. Shiu, Lam and Cheng [14] considered a class of supermagic graphs which are disjoint union of $K_{3,3}$. A general construction of supermagic graphs is considered in [15].

Example 5. It is easy to see that the classical concept of a magic square of n^2 boxes corresponds to the fact that the complete bipartite graph K(n,n) is super magic if $n \geq 3$. Thus $K(n,n)$ is Mod(k)-EM for all $k \geq 2$. Figure 5 shows $K(3,3)$ is super magic.

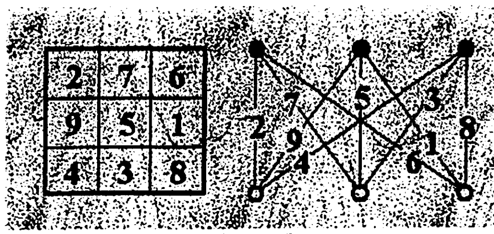


Figure 5.

In [7] supermagic regular complete multipartite graphs and supermagic cubes are characterized. In [11] and [1] supermagic labellings of the Möbius ladders and two special classes of 4-regular graphs are constructed. Some constructions of supermagic labellings of various classes of graphs are described in [1,6,7,21].

3. Mod(2)-EM graphs.

Theorem 3.1. A necessary condition for a (p,q)-graph G to be Mod(2)-EM is that $q(q+1) \equiv ps \pmod 2$, where s is the common vertex sum under a Mod(2)-EM labeling. Possible values, mod 2, for s are given in the following table:

	$p \equiv 0 \pmod 2$	$p \equiv 1 \pmod 2$
$q \equiv 0 \pmod 2$	0,1,	0,1
$q \equiv 1 \pmod 2$	0,1	0,1

We note here C_3 is not Mod(2)-EM. The more general result is

Theorem 3.2. The cycle C_{2k+1} is not Mod(2)-edge-magic for all $k \geq 1$.

Theorem 3.3. The cycle C_{2k} is Mod(2)-edge-magic for all $k \geq 2$.
Proof. Label the edges of the cycle by $1,2,3,\ldots,2k$ consecutively. It is obvious that each vertex has sum which is an odd integer. Hence the cycle C_{2k} is Mod(2)-edge-magic for all $k \geq 2$.♠

Corollary 3.4. The 2-regular graph $C(n_1,n_2,\ldots n_k)$ with k disjoint cycles $C(n_1),\ldots,C(n_k)$ is Mod(2)-edge-magic for any even $n_1,n_2,\ldots,n_k \geq 4$, and $k \geq 2$.

We can extend the above result to the following general
Theorem 3.5. For even integer $k > 2$. The k-regular graph G of even order is Mod(2)-edge-magic.

We see that P_2 is Mod(2)-EM, but $P_2 \cup P_2$ is not. There exists two non-Mod(2)-EM graphs G, H and their union is Mod(2)-EM. The reader can show that P_2xP_{2k}, St(2k-1) are not Mod(2)-EM for $k \geq 2$. However, we have

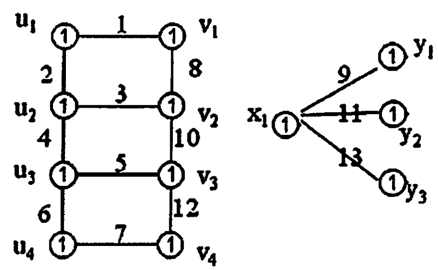
Theorem 3.6. The graph $P_2xP_{2k} \cup St(2k-1)$ is Mod(2)-EM for all $k \geq 2$.
Proof. Suppose $V(P_2xP_{2k}) = \{u_1,u_2,\dots,u_{2k}\} \cup \{v_1,v_2,\dots,v_{2k}\}$ and $E(P_2xP_{2k}) = \{(u_i,u_{i+1}): i=1,2,\dots,2k-1\} \cup \{(v_i,v_{i+1}): i=1,2,\dots,2k-1\} \cup \{(u_i,v_i): i=1,2,\dots,2k\}$.
$V(St(2k-1)) = \{x, y_1,y_2,\dots,y_{2k-1}\}$ and $E(St(2k-1)) = \{(x,y_i): i=1,2,\dots,2k-1\}$.

The graph $P_2xP_{2k} \cup St(2k-1)$ has 8k-3 edges.

We label the edges of $\{(u_i,u_{i+1}): i=1,2,\dots,2k-1\} \cup \{(v_i,v_{i+1}): i=1,2,\dots,2k-1\}$ by the even numbers $\{2,4,6,\dots,8k-4\}$ arbitrarily and $\{(u_i,v_i): i=1,2,\dots,2k\} \cup \{(x,y_i): i=1,2,\dots,2k-1\}$ by odd numbers $\{1,3,5,\dots,8k-3\}$.

The labeling is clearly a Mod(2)-EM.□

Example 6. Figure 6 shows that $P_2xP_4 \cup St(3)$ is Mod(2)-EM.



$G=P_2xP_4$ $H=St(3)$

Figure 6.

The following construction of graphs was introduced in [4].

Definition 3.1. Given graphs G and H with n and m vertices, respectively, the corona of G with respect to H is the graph G © H with vertex set V (G © H) = V (G) □ {n distinct copies of V (H) denoted $V(H_1)$, $V(H_2)$, . . ., $V(H_n)$ and edge set E(G © H) =E(G)□ {n distinct copies of E(H) denoted $E(H_1)$, $E(H_2)$, . . ., $E(H_n)$} □ {(u_i, v) : u_i □V (G), v □$V(H_i)$}.

Theorem 3.7. The corona of cycle C_n© K_1 is Mod(2)-edge-magic for all $n \geq 3$.

Theorem 3.8. The corona of cycle C_m© C_n is Mod(2)-edge-magic for all even $n,m \geq 4$.

We skip the proof, for the reader can check easily from the scheme of the following example.

Example 7. Figure 7 shows that C_4© C_6 is Mod(2)-EM.

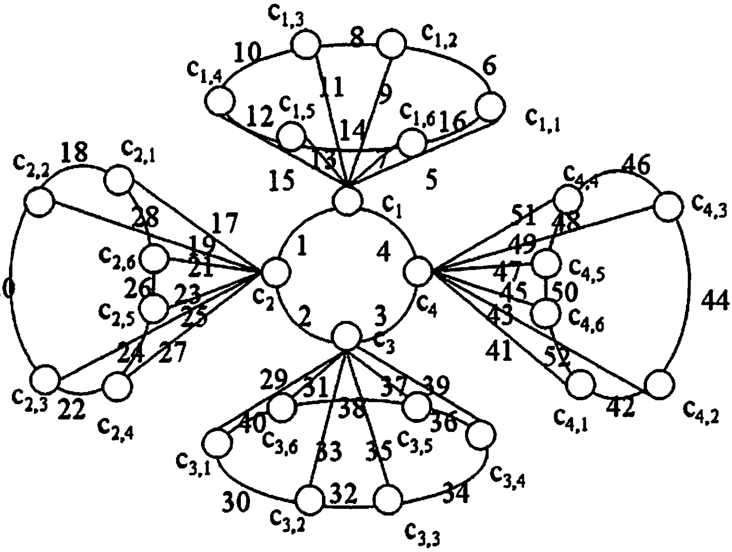
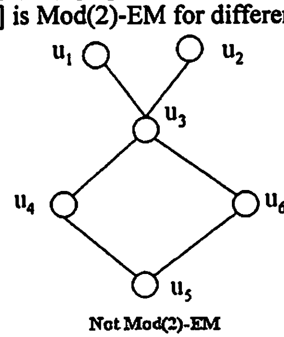


Figure 7.

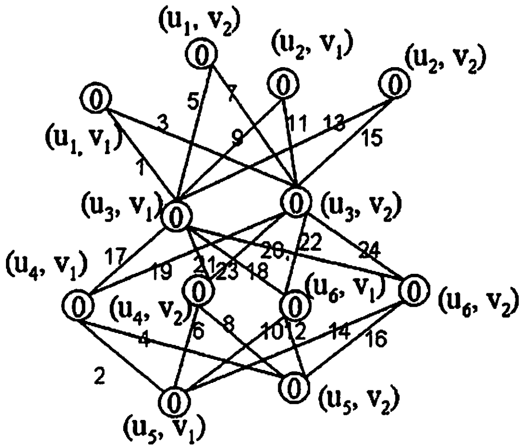
__**Definition 3.2**__.Given two graphs G and H. The composition of G with H, denote by $G[H]$, is the graph with vertex set V (G) xV (H) in which $(u_1; v_1)$ is adjacent to $(u_2; v_2)$ if and only if $u_1u_2 \in E(G)$ or $u_1 = u_2$ and $v_1v_2 \in E(H)$.

__**Theorem 3.9.**__ If G is a (p,q)-graph , with q is even, then $G[N_2]$ is Mod(2)-EM .
Proof. Let $V(N_2) = \{v_1, v_2\}$. Suppose q(G) =2n, we have $q(G[N_2])$ =2nx4 =8n.
Let $\Sigma = \{\{1,3,5,7\}, \{9,11,13,15\}.,,,,, \{8n-7,8n-5,8n-3,8n-1\}\}$
$\Omega = \{\{2,4,6,8\}, \{10,12,14,16\}.,,,,, \{8n-6,8n-4,8n-2,8n\}\}$.
Suppose f: $E(G) \to \Sigma \cup \Omega$ is a bijection, we define a mapping
$F: G[N_2] \to \{1,2,\ldots,8n\}$ as follows
If $f(u,w) = \{a_1,a_2,a_3,a_4\}$, then we label the edges of $G[N_2]$ by
$F((u, v_1)) = a_1,\ F((u, v_2)) = a_2,\ F((w, v_1)) = a_3,\ F((w, v_2)) = a_4.$
It is clear that $G[N_2]$ is Mod(2)-EM under F.□

__**Example 8.**__ The following (6,6)-graph G is not Mod(k)-EM for any k=2,3,4,5.However, $G[N_2]$ is Mod(2)-EM for different F_i, i=1,2.



Not Mod(2)-EM



$f((u_1,u_3))= \{1,3,5, 7\}$

$f((u_2,u_3))= \{9,11,13,15\}$

$f((u_3,u_4))= \{17,19,21,23\}$

$f((u_4,u_5))= \{2,4,6,8\}$

$f((u_5,u_6))= \{10,12,14,16\}$

$f((u_6,u_3))= \{18,20,22,24\}$

Mod(2)-EM

Figure 8.

__Corollary 3.10.__ Every finite graph G is an induced subgraph of a Mod(2)-EM graph.

4. Mod(2)-Edge-Magic Trees.

In this section, we give a condition for trees to be Mod(2)-EM. First we recall the following definition.

__Definition 4.1__. Given a graph G=(V,E), a matching *M* in *G* is a set of pairwise non-adjacent edges; that is, no two edges share a common vertex. A __perfect matching__ is a matching which matches all vertices of the graph. That is, every vertex of the graph is incident to exactly one edge of the matching.

__Theorem 4.1.__ A tree T is Mod(2)-EM if it has a perfect matching.
Proof. If T has a perfect matching then its order is even, say p =2k. So its has k edges in the perfect matching P. We label all the edges in P by {1,3,5,...,2k-1} and the remaing edges in T by {2,4,6,...,2k-2}. It is obvious that each vertex has a sum which is an odd integer. Hence the tree T is Mod(2)-edge-magic.■

__Example 9.__ Figure 9 shows a tree which is Mod(2)-EM and has a perfect matching.

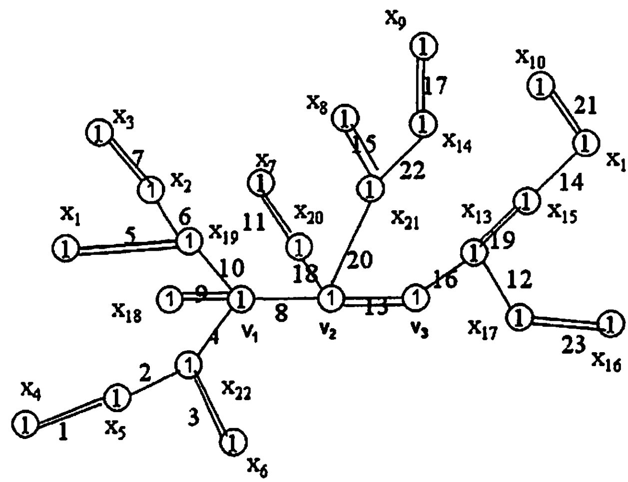


Figure 9.

Corollary 4.2. The path P_{2k} is Mod(2)-edge-magic for all $k \geq 1$.

Corollary 4.3. The corona of path P_n© K_1 is Mod(2)-edge-magic for all $n \geq 2$.

Example 10. Figure 9 shows that P_n© K_1 is Mod(2)-edge-magic for n=3 and 4.

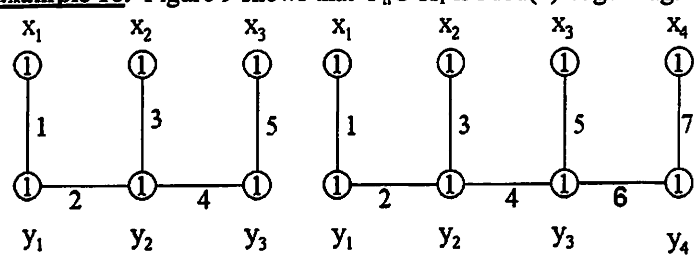


Figure 10.

Remark. One may wonder if the result of Theorem 4.1 can be extended to a "forest". We note that the forest $2K_2$ has a perfect matching but it is not Mod(2)-EM.

5. Mod(2)-Edge-Magic Unicyclic Graphs.

As the proof of Theorem 4.1., we can show that

Theorem 5.1. A unicyclic graph G is Mod(2)-EM if it has a perfect matching.

Let Gph(*) be the class of all graphs (G,u) with a distinguished vertex u. For (G_1,u_1), (G_2,u_2) in Gph(*), the amalgamation of (G_1,u_1), (G_2,u_2) is the graph obtained by disjoint union of G_1, G_2 and identify two vertices u_1 and u_2. We denote the resulting graph by Amal((G_1,u_1) (G_2,u_2)). This construction can be extended to the amalgamation of an arbitrary number of graphs.Let G=(V,E) and S⊆ V. If ϕ : S→ Gph(*), then we can form G amalgamate the graphs $\{\phi(s) = (G_s,u_s) : (G_s,u_s)$ in Gph(*)} by forms the disjoint union of G and $\{ \phi(s) = (G_s,u_s) : (G_s,u_s)$ in Gph(*)}by identify s with u_s. We denote the resulting graph by Amal(G, S; ϕ).

Example 11 . The following two unicyclic graphs are of order 6. One is not Mod(2)-EM and the other is Mod(2)-EM.

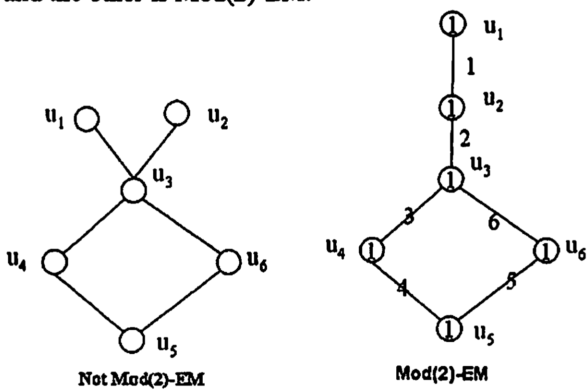


Figure 11.

Corollary 5.2. If C_n is a cycle with vertex set $\{c_1,c_2,\ldots,c_n\}$ and P_m is a path with vertex set $\{v_1,v_2,\ldots,v_m\}$, then Amal((C_{2n+1}, c_1),(P_{2m}, v_1)) is Mod(2)-EM.

Example 12.

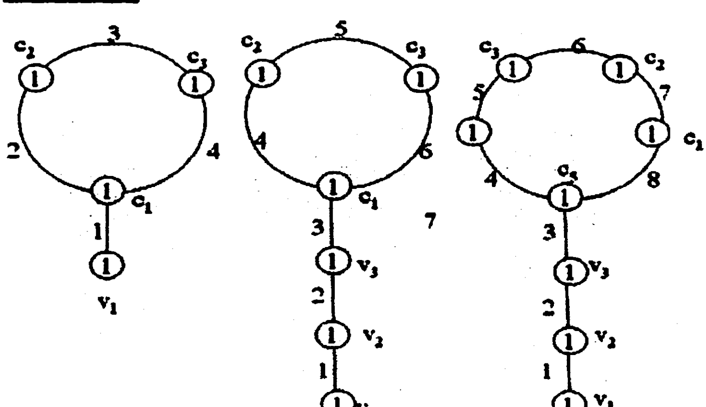


Figure 12.

Corollary 5.3. The unicyclic graph Amal((C_{2n}, c_1, c_2),{(P_{2m}, v_1), (P_{2t}, v_1)}) is Mod(2)-EM.

Example 13.

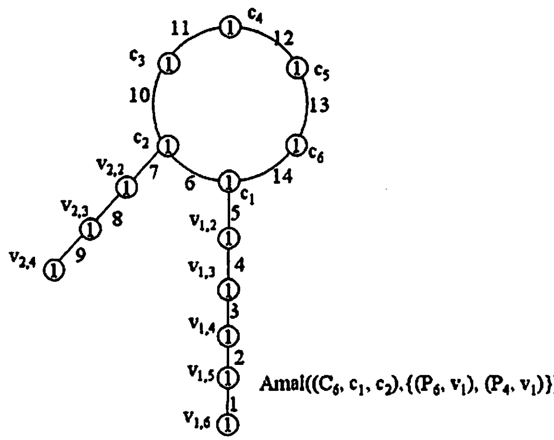


Figure 13.

Corollary 5.4. The graph Amal(($SP(2k+1, 2s),v_1$), (C_{2n+1},c_1)) is Mod(2)-EM for any k ,$s \geq 1$ and $n \geq 1$.

Example 14.

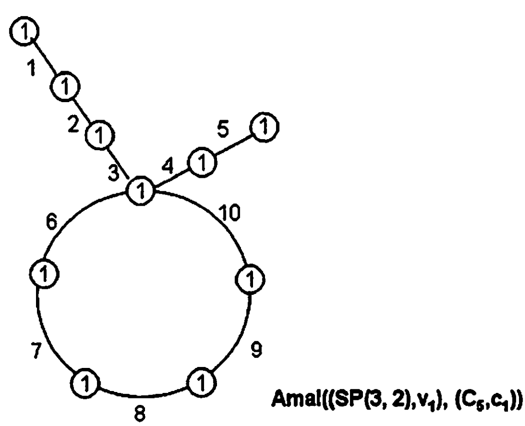
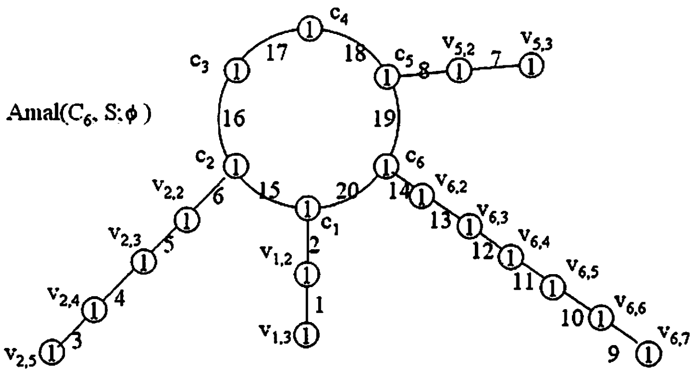


Figure 14.

Corollary 5.5. Suppose $S \subseteq V(C_{2n})$ and $\phi : S \rightarrow Gph(*)$ where $\phi(s)= (P_{2a+1},v_1)$, for some a. Then the graph Amal (C_{2n},S, ϕ) is Mod(2)-EM.

We illustrate the above result by the following example.

Example 15.



$S = \{c_1, c_2, c_5, c_6\}, \phi(c_1) = P_3, \phi(c_2) = P_5, \phi(c_5) = P_3, \phi(c_6) = P_7.$

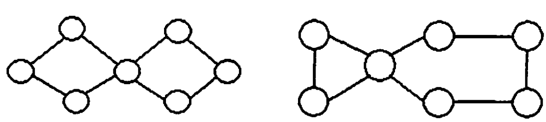
Figure 15.

6. (p,p+1)-graphs which are Mod(2)-EM.

We consider five special classes of Mod(2)-EM (p,p+1)-graphs in this section.

(A) One-point Union of cycles.

Let G, H be two graphs with $u \in V(G)$ and $v \in V(H)$ respectively. The amalgamation of (G,u) with (H,v) is the graph obtained by forming the disjoint union of G and H and then identifying u and v. The construction is called the **one-point union**. We will use Amal(G,H,{u,v}) to denote the amalgamation of (G,u) and (H, v). The following graphs C(4,4) and C(3,5) are the Amal(C_4,C_4,{u,u}) and Amal(C_3,C_5,{u,v}) respectively (Figure 16).



C(4,4) **C(3,5)**

Figure 16.

Theorem 6.1. One-point union C(n,n+1) is Mod(2)-EM for any $n \geq 3$.

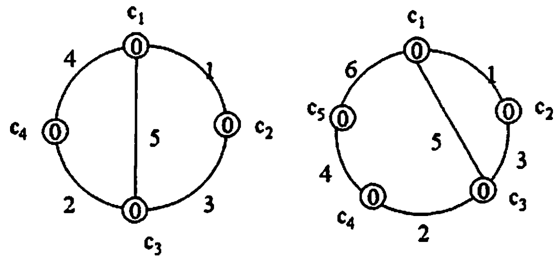
Proof. If n is odd, we label the edges of C_n component by 2,4,6,...,2n and the edges of C_{n+1} component by 1,3,5,...,2n+1. It is clear that C(n,n+1) is Mod(2)-EM.

If n is even, we label the edges of C_n component by 2,4,6,...,2n and the edges of C_{n+1} component by 1,3,5,...,2n+1. It is clear that C(n,n+1) is Mod(2)-EM.□

(B)Cycle with a chord.

Notation. For a cycle C_n with vertex set $\{c_1,c_2,\dots,c_n\}$, we denote by $C_n(t)$ the cycle with a chord (c_1,c_t).

Theorem 6.2. The cycle with a chord $C_n(t)$ is Mod(2)-EM if and only if
(1) $t = s+1$ if $n = 2s$
(2) $t = s+1$ if $n = 2s+1$
Proof. We see that the graphs $C_n(t)$ for $(n,t) = (4,3)$ and $(5,3)$ are Mod(2)-EM.



$C_4(3)$ $C_5(3)$

Figure 17.

If n is even, and $n=2s > 4$, $C_n(t)$ has $2s+1$ edges. If the graph is Mod(2)-EM, then we see that among $\{1,2,3,\ldots,2s+1\}$ numbers we have $s+1$ odd numbers $\{1,3,5,\ldots,2s+1\}$, these numbers should label the edges of $C_n(t)$ consists of $\{(c_1,c_2),(c_2,c_3)\ldots,(c_t,c_1)\}$. Thus t must equal s+1. The remaining numbers $\{2,4,\ldots,2s\}$ will be assigned to the other edges of $C_n(t)$. The similar argument can apply for n is odd.□

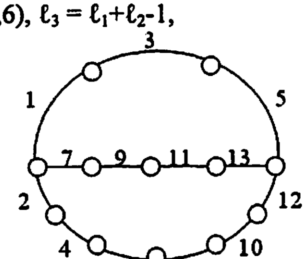
(C) The theta graph $\Theta(\ell_1,\ell_2,\ell_3)$.

The theta graph $\Theta(\ell_1,\ell_2,\ell_3)$, which consists of three paths of length ℓ_1,ℓ_2,ℓ_3 joined at their endpoints u and v. It has $p = \ell_1+\ell_2+\ell_3-1$ vertices.

Theorem 6.3. For any $\ell_1,\ell_2 \geq 1$, the theta graph $\Theta(\ell_1,\ell_2,\ell_3)$ is Mod(2)-EM, if
(1) $\ell_3 = \ell_1+\ell_2-1$,
(2) $\ell_3 = \ell_1+\ell_2$.

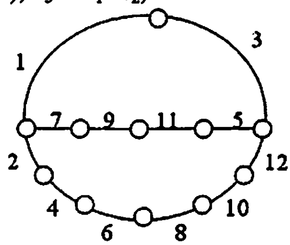
We skip the proof, but illustrate the labeling scheme by the following examples.

Example 16.
(1) $(\ell_1,\ell_2,\ell_3) = (3,4,6)$, $\ell_3 = \ell_1+\ell_2-1$,



$\Theta(3,4,6)$

(2) (ℓ_1,ℓ_2,ℓ_3) =(2,4,6), $\ell_3 = \ell_1+\ell_2$,

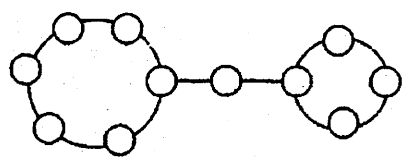


⊗(2,4,6)

Figure 18.

(D)The dumbbell graph DB (ℓ_1,ℓ_2,ℓ_3).

The dumbbell graph DB (ℓ_1,ℓ_2,ℓ_3), consists of two cycles of length $\ell_1$and ℓ_2, connected by a path of length $\ell_3$at its endpoints u and v, and has $p = \ell_1+\ell_2+\ell_3 -1$ vertices .



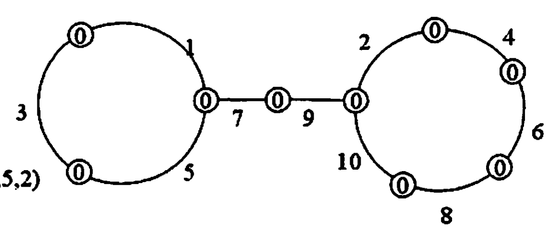
DB(6,4,2)

Figure 19.

Theorem 6.4. The dumbbell graph DB (ℓ_1,ℓ_2,ℓ_3) is Mod(2)-EM for
(1) $\ell_2 = \ell_1+ \ell_3$,
(2) $\ell_3 = \ell_1+ \ell_2 -1$,
(3) $\ell_3 = \ell_1+ \ell_2$.

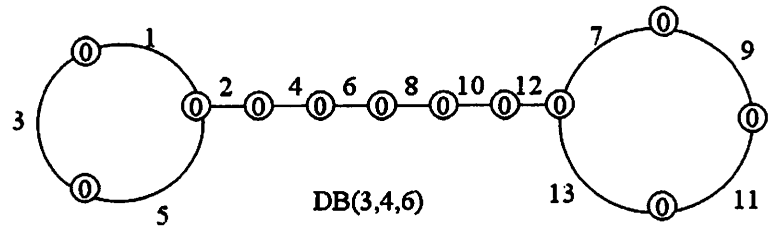
We skip the proof, but illustrate the idea by examples.

Example 17.
(1) (ℓ_1,ℓ_2,ℓ_3) =(3,5,2)



DB(3,5,2)

(2) (ℓ_1,ℓ_2,ℓ_3) =(3,4,6), $\ell3 = \ell1+ \ell2 -1$,



(3) $(\ell_1,\ell_2,\ell_3)=(3,4,7)$, $\ell 3 = \ell 1 + \ell 2$,

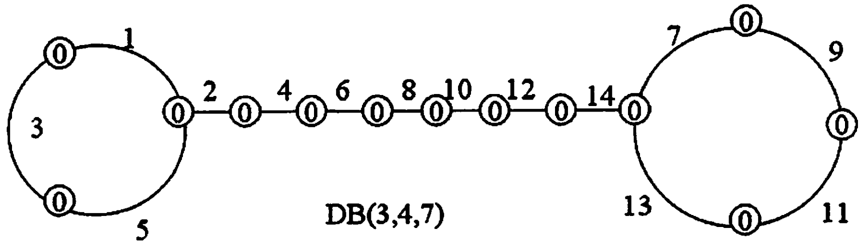


Figure 20.

In section 4 and 5, we show that a tree or uncyclic graph that has a perfect matching is always Mod(2)-EM. However, this result cannot be extended to (p,p+1)-graphs. We have the following example which shows such an extension is impossible.

__Example 18__. The following (10,11)-graph has a perfect matching $\{(x_1,v_1), (x_2,v_2), (v_3,v_5), (x_4,v_4), (v_6,v_7)\}$. If it is Mod(2)-EM , the perfect matching must be labeled by even numbers, then vertices v_6,v_7 will have vertex label 1 and x_1,x_2 will have vertex label 0, which is impossible.

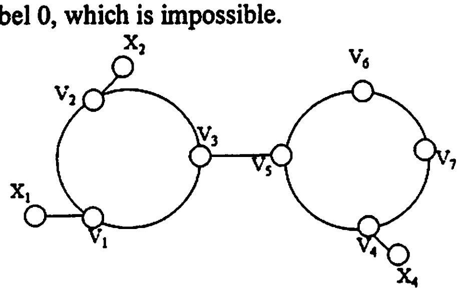


Figure 21.

__(E)A special class of (p,p+1)- graphs.__

However, we construct infinitely many Mod(2)-EM graphs by extending specific subclasses of (p,p+1) graphs.

__Theorem 6.4.__ Let G be any (p,p+1) graph without vertex of degree 1 with vertex set $V(G) = \{v_1,v_2,\ldots,v_p\}$, we can construct a (2p+2, 2p+3)-graph G# which is Mod(2)-EM and contain G as an induced subgraph.

Proof. Let $V(G\#) = V(G) \cup \{x_1,x_2,\ldots,x_p,x_{p+1},x_{p+2}\}$ and $E(G\#) = E(G) \cup \{(x_1, v_1),(x_2, v_2),\ldots,(x_i, v_i)\ldots,\ldots (x_{p-1}, v_{p-1}),(x_p, v_p),(x_{p+1}, v_p),(x_{p+2}, v_p)\}$. Thus G# is constructed by append an edge toeach vertex of G, except the last vertex is appended with three edges.

Now we label the p+1 edges of G by :2,4,6,...,2p+2. All the p+2 appended edges by :1,3,5,...,2p+3.

Clearly, the labeling is Mod(2)-EM.□

Example 19.. The following (11,12)-graph G and its extension G# which is a Mod(2)- EM (24,25)-graph.

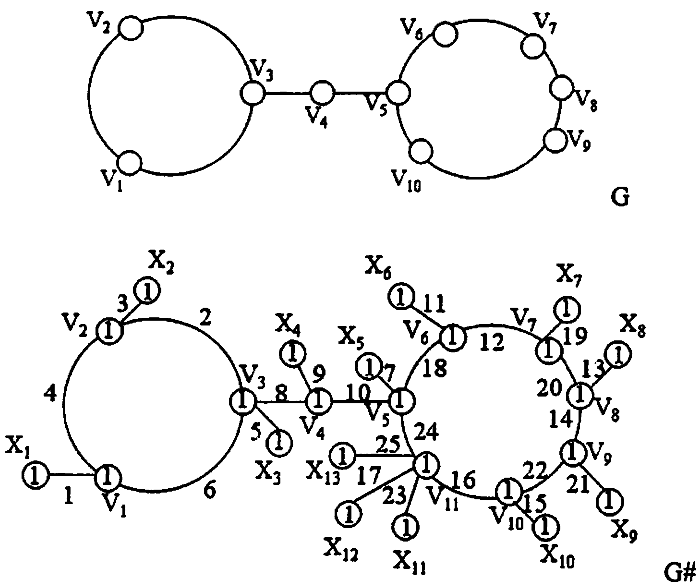


Figure 22.

7. Some Unsolved Problems.

We propose the following problems for future research.

Problem 1. Can we find a (p,q)-graph G which is Mod(k)-EM for k=2,3,...,p, but not super magic?

In 1993 the third author proposed the following conjecture [5] :

Conjecture: Every cubic simple graph of order p ≡2 (mod 4) is edge-magic (over Zp).

Problem 2. Characterize Mod(2)-EM 3-regular graphs.

Problem 3. Characterize Mod(2)-EM (p.p+1)-graphs.

Acknowledgement. We express our thanks to Prof. Harris Kwong and Prof. Andrew Chung Lee for their careful checking and comments on this paper.

References

[1] M. Bača and I. Holländer and Ko-Wei Lih, Two classes of super-magic graphs,J. Combin. Math. Combin. Comput. **23** (1997), 113–120.

[2] D. Chopra, H.Kwong and Sin-Min Lee, On edge-magic (p, 3p – 1)-graphs *Congressus Numerantium* **179** (2006), 49-63.

[3] J.A. Gallian, A dynamic survey of graph labeling, *The Electronic J. of Combin.* (2007), # DS6, 1-180.

[4] R. Frucht and F. Harary, On the coronas of two graphs, *Aequationes Math.* **4** (1970), 322-324.

[5] N. Hartsfield and G. Ringel, Supermagic and Antimagic Graphs, *Journal of Recreational Mathematics* **21** (1989), 107-115.

[6] Y.S. Ho and Sin-Min Lee, An initial result of supermagicness of complete k-partite graphs, *Journal of Combinatorial Mathematics and Combinatorial Computin* **39** (2001), 3-17

[7] J. Ivanˇco, On supermagic regular graphs, *Mathematica Bohemica* **125** (2000), 99–114.

[8] Sin-Min Lee, M. Kitagaki, J. Young, and W. Kocay, On edge-graceful and edge-magic maximal outerplanar graphs, J. *Combin. Math. Combin. Comput.* **59** (2006), 119-129.

[9] Sin-Min Lee, W.M. Pigg and T.J. Cox, On edge-magic cubic graphs conjecture, *Congressus Numerantium* **105** (1994), 214-222.

[10] Sin-Min Lee, Eric Seah and S.K. Tan, On edge-magic graphs, *Congressus Numerantium* **86** (1992), 179-191.

[11] Sin-Min Lee, E. Seah, Siu-Ming Tong, On the edge-magic and edge-graceful total graphs conjecture, *Congressus Numerantium* **141** (1999), 37-48.

[12] Sin-Min Lee, Ling.Wang and Yihui Wen, On The Edge-magic Cubic Graphs and Multigraph, *Congressus Numerantium* **165** (2003), 145-160.

[13] Karl Schaffer and Sin Min Lee, Edge-graceful and edge-magic labellings of Cartesian products of graphs, *Congressus Numerantium* **141** (1999), 119-134.

[14] W.C. Shiu, P.C.B. Lam and H.L. Cheng, Supermagic labeling of sKn,n, *Congressus Numerantium*, **146** (2000) ,119-124.

[15] W.C. Shiu, P.C.B. Lam and Sin-Min Lee, On a construction of supermagic graphs, *Journal of Combinatorial Mathematics and Combinatorial Computing* **42** (2002),147-160.

[16] W.C. Shiu, P.C.B. Lam and Sin-Min Lee, Edge-magicness of the composition of a cycle with a null graph, *Congressus Numerantium,* **132** (1998), 9-18.

[17] W.C. Shiu, F.C.B. Lam and Sin-Min Lee, Edge - magic index sets of (p, p-1)-graphs, *Electronic Notes in Discrete Mathematics* **11** (2002).

[18] W.C. Shiu and Sin-Min Lee, Some edge-magic cubic graphs, *Journal of Combinatorial Mathematics and Combinatorial Computing*, **40** (2002), 115-127.

[19] B.M. Stewart, Magic graphs, *Canadian Journal of Mathematics,* **18** (1966), 1031-1059.

[20] B.M. Stewart, Supermagic complete graphs, *Canadian Journal of Mathematics,* **19** (1967), 427-438.

[21] Yihui Wen, Sin-Min Lee, and Hugo Sun, On the Supermagic Edge-splitting Extension of Graphs, *Ars Combinatoria* 79 (2006), 115-128