Intermediate Minimal k-Rankings of Graphs

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Abstract

Given a graph G, a function $f : V(G) \to \{1, 2, ..., k\}$ is a k-ranking of G if $f(u) = f(v)$ implies every $u - v$ path contains a vertex w such that $f(w) > f(u)$. A k-ranking is *minimal* if the reduction of any label greater than 1 violates the described ranking property. The rank number of a graph, denoted $\chi_r(G)$, is the minimum k such that G has a minimal k-ranking. The *arank* number of a graph, denoted $\psi_r(G)$, is the maximum k such that G has a minimal k-ranking. It was asked by Laskar, Pillone, Eyabi, and Jacob if there is a family of graphs where minimal k-rankings exist for all $\chi_r(G) \leq k \leq \psi_r(G)$. We give an affirmative answer showing that all intermediate minimal k-rankings exist for paths and cycles. We also give a characterization of all complete multipartite graphs which have this intermediate ranking property and which do not.

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1 Introduction

A function $f : V(G) \to \{1, 2, ..., k\}$ is a k-ranking of G if $f(u) = f(v)$ implies every $u - v$ path contains a vertex w such that $f(w) > f(u)$. A k-ranking was defined by Ghoshal, Laskar, and Pillone [4] to be (locally) *minimal* if the reduction of any label greater than 1 violates the described ranking property. Another definition of a minimality is that a k-ranking f is *globally minimal* if for all $x \in V(G)$, $f(v) \leq g(v)$ for all rankings g. It was shown by Jamison [6] and Isaak, Jamison, and Narayan [5] that these two definitions of minimal rankings are equivalent. The rank number of a graph, denoted $\chi_r(G)$, is the minimum k such that G has a minimal k-ranking. The *arank* number of a graph, denoted $\psi_r(G)$, is the maximum k such that G has a minimal k-ranking. It was asked by Laskar, Pillone, Eyabi, and

Jacob [10] if there is a family of graphs where k-rankings exist for all $\chi_r(G) \leq k \leq \psi_r(G)$. We give an affirmative response to their question showing that all intermediate rankings exist for paths and cycles. Furthermore, we give a characterization of all complete multipartite graphs which have the intermediate ranking property and which do not.

If minimal k-rankings exist for all $\chi_r(G) < k < \psi_r(G)$ then G is said to have the *intermediate ranking property.* It is not too difficult to see that there exist graphs that do not have the intermediate ranking property. For example, the star graph $K_{1,n}$ has only two possible types of minimal k-rankings. In the first type, the center vertex is labeled 1, which forces all other vertices to have different labels. Hence $\psi_r(K_{1,n}) = n$. If the center vertex receives a label other than 1 then the vertices with degree 1 can all be labeled 1. The label of the center vertex can then be reduced to 2, which gives $\chi_r(K_{1,n}) = 2$. No other minimal k-rankings exist.

In our search for graphs with the intermediate ranking property we begin by reviewing known results for rank numbers and arank numbers. Rank numbers are known or many classes of graphs including paths , cycles, split graphs, and complete multipartite graphs, Möbius graphs, powers of paths and cycles, some grid graphs, some trees and unicyclic graphs, and cubic ladder graphs [3, 4, 11, 1, 12, 13, 14].

However less is known about arank numbers of graphs. It is known that if a graph has a vertex of degree $n-1$ then the arank number is n [4]. Here assign the label 1 to the vertex with degree $n-1$ and all other vertices are then forced to have different labels. Hence the result is immediate for complete graphs and wheels. The arank number is known only for a few other families of graphs: complete multipartite graphs, paths, and rook's graphs (the Cartesian product of $K_n \times K_n$), and within 1 for cycles [4, 8, 10, 7].

In this paper we include an unusual twist. It would appear that we would have to know the both the rank and arank numbers of a graph to determine if it has the intermediate ranking property or not. However we will show this to be false. Although we have not completely determined the arank number of a cycle, we can conclude that cycles do have the intermediate ranking property.

The focus of this paper will be to explore the intermediate ranking property for all classes of graphs where the rank numbers and arank numbers are known.

2 Preliminaries

We next give two elementary lemmas that will be used to establish that paths and cycles have the intermediate ranking property. The first lemma gives a sufficient condition for joining two rankings of paths to form a ranking of a larger path. The second lemma states that if the last vertex of a path has the largest label, the ranking of a path s vertices can be "wrapped" to form a ranking of a cycle on s vertices, where an edge is added between the first and last vertices of the path.

Lemma 1 *Let f be a minimal ranking of P_s on vertices $v_1, ..., v_s$ and let g be a minimal k-ranking of P_t on vertices $w_1, ..., w_t$. If $f(v_s) > f(v_i)$ for all $1 \leq i \leq s-1$ and $f(v_s) > f(w_i)$ for all $1 \leq i \leq t$ then the labeling $h(i) = f(i)$ for $1 \leq i \leq s$ and $h(i) = g(i)$ for $s+1 \leq i \leq t$ is a minimal k-ranking of $P_{s+t} = P_s \cup P_t \cup \{(v_s, w_1)\}$.*

Proof. The minimal ranking property is clearly preserved on each of the two small paths. Any path connecting two vertices in different parts contains a vertex with the highest label. ■

Lemma 2 *Let f be a minimal k-ranking of P_s on vertices $v_1, ..., v_s$ where $f(v_s) > f(v_i)$ for all $1 \leq i \leq s-1$. Then the labeling $h(i) = f(i)$ for $1 \leq i \leq s$ is a minimal k-ranking of the cycle C_s formed by adding an edge between vertices v_1 and v_s.*

Proof. We examine the ranking property for paths between different pairs of vertices in C_s. Any path connecting two vertices that does not contain the vertex v_s clearly preserves the ranking property. Any path connecting two vertices containing the vertex v_s preserves the ranking property since v_s has the largest label. ■

3 Intermediate k-rankings of paths

Bodlaender et al. [2] proved that $\chi_r(P_n) = \lfloor \log_2(n) \rfloor + 1$ and that optimal rankings of $P_n = v_1, v_2, \cdots, v_n$ can be constructed by labeling v_i with $\alpha + 1$ where 2^α is the largest power of 2 that divides i. We will refer to this particular ranking as the *standard ranking of a path.* Laskar and Pillone [9] investigated the arank number of a path and gave a construction for minimal k-rankings where the first vertex of the path is labeled $k-1$ and the m-th vertex is labeled k for some m dependent on k and the number of vertices in the path. Kostyuk, Narayan, and Williams [8] showed that this construction indeed produced arankings and that $\psi_r(P_n) = \lfloor \log_2 (n+1) \rfloor + \lfloor \log_2 (n + 1 - (2^{\lfloor \log_2 n \rfloor - 1})) \rfloor$.

Theorem 3 *Let P_n be a path on n vertices. There exist k-rankings for all $\chi_r(P_n) \leq k \leq \psi_r(P_n)$.*

Proof. When $k = \chi_r(P_n)$ or $\psi_r(P_n)$ the result is clear. We will consider $\chi_r(P_n) < k < \psi_r(P_n)$. Let P_m be the shortest path where $\psi_r(P_m) = k$. We construct a k-ranking of P_n as follows. We label the first m vertices using the ψ_r-ranking of P_m following the construction given by Laskar and Pillone [9], where the first vertex is labeled $k-1$ and the m-th vertex is labeled k. We then use the standard ranking of P_n for the remaining $n-m$ labels. For this to work we need to insure that all of the labels in the path P_{n-m} are less than or equal to $k-1$. Since $\chi_r(P_n) = \lfloor \log_2 n \rfloor + 1$ we have $k \geq \lfloor \log_2 n \rfloor + 2$. We claim that a path can contain $n-m$ labels that are at most $\lfloor \log_2 n \rfloor + 1$. To prove the claim we note that if we use the standard ranking of a path the number of labels can be $2^{\lfloor \log_2 n \rfloor} - 1 \geq 2^{\log_2 n} - 1 = n - 1 \geq n - m$. By Lemma 2, this labeling is a minimal k-ranking of P_n.We can apply Lemma 1 to conclude that this labeling is a minimal k-ranking of P_n. ∎

4 Intermediate k-rankings of cycles

Bruoth and Horňák [3] proved that $\chi_r(C_n) = \lfloor \log_2(n-1) \rfloor + 2$ and Kostyuk and Narayan [7] showed for any $n \geq 3$, $\psi_r(C_n)$ is equal to $\psi_r(P_n)$ or $\psi_r(P_n) + 1$.

Theorem 4 *Let $n \geq 3$ and C_n be a cycle on n vertices. Then there exist k-rankings for all $\chi_r(C_n) \leq k \leq \psi_r(C_n)$.*

Proof. When k equals$\chi_r(C_n)$ or $\psi_r(C_n)$ the result is clear. We will consider $\chi_r(C_n) < k < \psi_r(C_n)$. Let P_m be the shortest path where $\psi_r(P_m) = k$. We construct a k-ranking of C_n as follows. We label the first m vertices using the ψ_r-ranking of P_m. From the construction given in [8] we can build a labeling where the first vertex is labeled $k-1$ and the m-th vertex is labeled k. Then we append a segment consisting of the standard ranking of P_{n-m} at the end of the path to form a ranking of P_n. Finally we insert an edge joining the first and last vertices of the path creating the cycle C_n. For this to work we need to insure that all of the labels in the path P_{n-m} are less than or equal to $k-2$. Since $\chi_r(C_n) = \lceil \log_2 n \rceil + 1$, $k \geq \lceil \log_2 n \rceil + 2$. We claim that a path can contain $n-m$ labels that are at most $\lceil \log_2 n \rceil$. To prove the claim we note that if we use the standard ranking of a path the number of labels can be $2^{\lceil \log_2 n \rceil} - 1 \geq 2^{\log_2 n} - 1 = n - 1 \geq n - m$. By Lemma 2, this labeling is a minimal k-ranking of C_n. ■

We note that an interesting twist has occurred. Although the arank number of a cycle has not been completely settled, we have shown that cycles do have the intermediate ranking property. For example, it was shown by Kostyuk and Narayan [7] that $\psi(C_{30}) = 8$ or 9 but it is not known which is the answer. We highlight this fact in the following proposition.

Proposition 5 *The cycle C_{30} has the intermediate ranking property.*

Proof. We start by noting that $\chi(C_{30}) = 6$ and $\psi(C_{30}) = 8$ or 9. We know that a χ_r-ranking and a ψ_r-ranking must exist for all graphs. We display minimal k-rankings for $k = 6$,7, and 8 in the table below.

k	k-ranking of C_{30}
6	512134312161213121412131215121
7	612321454123217121312141213121
8	712134312156512134312181213121

We consider two cases. If $\psi(C_{30}) = 8$ then C_{30} has the intermediate ranking property since we have shown the existence of minimal k-rankings for $k = 6$,7, and 8. If $\psi(C_{30}) = 9$ then we know a minimal 9-ranking of C_{30} must exist, and then we have the other three rankings by construction. Although we do not know which case is true, both cases conclude with C_{30} having the intermediate ranking property. ■

5 Intermediate $k-$rankings for complete multipartite graphs

We mentioned in the introduction that stars do not have the intermediate ranking property. In this section we determine which complete multipartite graphs have the intermediate ranking property.

We begin with a few basic properties of minimal k-rankings of complete multipartite graphs. Clearly by the definition of k-ranking only one part may have a vertex labeled 1. Then by the minimality condition, if a part contains a vertex labeled 1, then all of the vertices in can be reduced to 1.

Next, we restate a lemma of Ghoshal, Laskar and Pillone [4].

Lemma 6 *Let G be the complete multipartite graph $K_{n_1,n_2,\cdots,n_p}$. Then $\chi_r(G) = (\sum_1^p n_i) - \max\{n_1, n_2, ..., n_p\} + 1$ and $\psi_r(G) = (\sum_1^p n_i) - \min\{n_1, n_2, ..., n_p\} + 1$.*

Proof. A minimal k-ranking where $k = \chi_r(G)$ is obtained by labeling all vertices in the largest part with a 1 and labeling all other vertices in the other parts with 2, 3, ..., $(\sum_1^p n_i) - \max\{n_1, n_2, ..., n_p\} + 1$. A k-ranking where $k = \psi_r(G)$ is obtained by labeling all vertices in the smallest part with a 1 and labeling all other vertices in the other parts with 2, 3, ..., $(\sum_1^p n_i) - \min\{n_1, n_2, ..., n_p\}$ +1. Both rankings are minimal since the reduction of any label larger than one would result in the same label appearing in two different parts, which would violate the ranking property. ∎

The following theorem characterizes complete multipartite graphs into two classes according to whether the graph has the intermediate ranking property or not.

Theorem 7 *Let G be the complete multipartite graph $K_{n_1,n_2,\cdots,n_p}$ where $n_1 \geq n_2 \geq \cdots \geq n_p$. Then there exist k-rankings for all $\chi_r(G) \leq k \leq \psi_r(G)$ if and only if G contains at least one part of order s for all $|n_p| \leq s \leq |n_1|$.*

Proof. Assume G contains at least one part of order s for all $|n_p| \leq s \leq |n_1|$. Consider the subgraph $K_{m_1,m_2,\cdots,m_t}$ where $n_1 = m_1 \geq m_2 \geq \cdots \geq m_t = n_p$ and $|m_{i+1}| - |m_i| \leq 1$. For each i, $1 \leq i \leq \psi_r(G) - \chi_r(G)$, we can construct a minimal $(\chi_r(G) + i - 1)$-ranking of G as follows. Label all of the vertices in the part m_i with 1 and label all other vertices with $2, 3, ..., (\chi_r(G) + i - 1)$. Conversely, let j be the smallest integer such that $j \neq |n_\lambda|$ for any λ. Since there is no part of order j, it is impossible to have a minimal k-ranking of G with j vertices labeled 1. Hence there does not exist a minimal k-ranking where $k = \left(\sum_{i=1}^{\lambda-1} |n_i| + \sum_{i=\lambda+1}^{p} |n_i|\right)$. ∎

6 For which graphs does $\chi_r(G) = \psi_r(G)$?

The next logical question is: For which graphs G are the rank numbers and arank numbers equal? We can establish the following results using two bounds Ghoshal, Laskar, and Pillone [4] which we recall here.

Proposition 8 *Let G be a graph and let $\alpha(G)$ be the size of the largest independent set of G. Then $\chi_r(G) \leq n - \alpha(G)$.*

Proof. A k-ranking of G where $k = n - \alpha(G)$ can be obtained by labeling each vertex in the largest independent set with 1 and giving all other vertices distinct labels. We then label the other vertices and reduce if necessary to insure minimality. ∎

Proposition 9 *Let G be a graph. Then $\psi_r(G) \geq \Delta(G)$.*

Proof. Labeling the vertex of largest degree with 1 forces each of its neighbors to have different labels. We then label the other vertices and then reduce labels if necessary to insure minimality. This must result in a minimal k-ranking where $k \geq \Delta(G)$. ∎

Combining the above two propositions yields, $\chi_r(G) \leq n - \alpha(G) = \Delta(G) \leq \psi_r(G)$. Hence $n - \alpha(G) = \Delta(G)$ is a necessary condition for $\chi_r(G) = \psi_r(G)$. Applying this idea leads to the following graphs.

- K_n; $\chi_r(G) = \psi_r(G) = n$
- C_4; $\chi_r(C_4) = \psi_r(C_4) = 3$
- $L_3 = P_2 \times P_3$; $\chi_r(L_3) = \psi_r(L_3) = 4$

However it is very likely that others exist. We conclude by posing the following problem.

Problem 10 *Determine all graphs G where $\chi_r(G) = \psi_r(G)$.*

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